

1. (10 pts) Evaluate the following integral

$$\int \frac{\ln(x)}{\sqrt{x}} dx$$

Use by-parts

$$u = \ln(x)$$

$$v' = x^{-1/2}$$

$$u' = \frac{1}{x}$$

$$v = \frac{x^{1/2}}{1/2} = 2x^{1/2}$$

$$\int \frac{\ln(x)}{\sqrt{x}} dx = 2\sqrt{x} \ln(x) - \int \frac{1}{x} \cdot 2x^{1/2} dx$$

$$= 2\sqrt{x} \ln(x) - 2 \int x^{-1/2} dx$$

$$= \boxed{2\sqrt{x} \ln(x) - 4\sqrt{x} + C}$$

2. Evaluate $\int \frac{x^4}{(x+1)^2(x^2+1)} dx$

Step 1 $\deg(x^4) = 4 = 4 = \deg((x+1)^2(x^2+1))$

So, must divide

$$\begin{aligned} (x+1)^2(x^2+1) &= (x^2+2x+1)(x^2+1) = \cancel{x^3+2x^2+x^2+2x+1} \\ &= \cancel{x^3+3x^2+3x+1} \\ &= \cancel{x^4+2x^3+x^2+x^2+2x+1} \\ &= \cancel{x^4+\cancel{4}x^2+2x+1} \\ &= x^4+2x^3+x^2+x^2+2x+1 \\ &= x^4+2x^3+2x^2+2x+1 \end{aligned}$$

$$\begin{array}{r} x^4+2x^3+2x^2+2x+1 \overline{) x^4+0x^3+0x^2+0x+0} \\ \underline{-(x^4+2x^3+2x^2+2x+1)} \\ -2x^3-2x^2-2x-1 \end{array}$$

So, $\int \frac{x^4}{(x+1)^2(x^2+1)} dx = \int 1 + \frac{-2x^3-2x^2-2x-1}{(x+1)^2(x^2+1)} dx$

Step 2: $\frac{-2x^3-2x^2-2x-1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C(x+D)}{x^2+1}$

$$-2x^3-2x^2-2x-1 = A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2$$

if $x = -1$, then $2-2+2-1 = A \cdot 0 + B(2) + (Cx+D) \cdot 0$

$$\begin{aligned} 1 &= 2B \\ \boxed{B} &= \frac{1}{2} \end{aligned}$$

$$\text{So, } -2x^3 - 2x^2 - 2x - 1 = A(x^3 + x^2 + x^2 + 1) + \frac{1}{2}x^2 + \frac{1}{2} + (Cx + D)(x^2 + 2x + 1)$$

$$-2x^3 - 2x^2 - 2x - 1 = Ax^3 + \cancel{2A}x^2 + A + \frac{1}{2}x^2 + \frac{1}{2} + Cx^3 + 2Cx^2 + Cx + Dx^2 + 2Dx + D$$

$$-2x^3 - 2x^2 - 2x - 1 = (A + C)x^3 + (2A + \frac{1}{2} + 2C + D)x^2 + (C + 2D)x + (A + \frac{1}{2} + D)$$

$$\text{So, } \left. \begin{array}{l} A + C = -2 \\ C + 2D = -2 \\ A + \frac{1}{2} + D = -1 \end{array} \right\} \Rightarrow \left. \begin{array}{l} A - 2D = 0 \\ A + \frac{1}{2} + D = -1 \end{array} \right\} \begin{array}{l} 2D + \frac{1}{2} + D = -1 \\ 3D = -\frac{3}{2} \\ \boxed{D = -\frac{1}{2}} \end{array}$$

Since $D = -\frac{1}{2}$ and $A - 2D = 0$

then $A + 1 = 0$
 $\boxed{A = -1}$

Since $A = -1$ and $A + C = -2$

then $-1 + C = -2$ and $\boxed{C = -1}$

Step 3: Integrate

$$\int \frac{x^4}{(x+1)^2(x^2+1)} dx = \int 1 + \frac{-1}{x+1} + \frac{\frac{1}{2}}{(x+1)^2} + \frac{-x - \frac{1}{2}}{x^2+1} dx$$

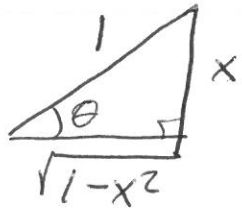
$$= x - \ln|x+1| + \left(-\frac{1}{2}\right)(x+1)^{-1} + \int \frac{-x}{x^2+1} dx + \int \frac{-\frac{1}{2}}{x^2+1} dx$$

$u = x^2 + 1$
 $du = 2x dx$

$$= \boxed{x - \ln|x+1| + \left(-\frac{1}{2}\right)(x+1)^{-1} - \frac{1}{2} \ln|x^2+1| - \frac{1}{2} \tan^{-1}(x) + C}$$

3. Evaluate $\int \frac{1}{(1-x^2)^{3/2}} dx$

Use trig sub.



$$\frac{x}{1} = \sin \theta$$
$$dx = \cos \theta d\theta$$

$$\frac{\sqrt{1-x^2}}{1} = \cos \theta$$

$$\int \frac{1}{(1-x^2)^{3/2}} dx = \int \frac{1}{(\sqrt{1-x^2})^3} dx = \int \frac{1}{(\cos \theta)^3} \cdot \cos \theta d\theta$$

$$= \int \sec^2 \theta d\theta$$

$$= \tan \theta + C$$

$$= \boxed{\frac{x}{\sqrt{1-x^2}} + C}$$

4. Evaluate $\int_{-\infty}^{\infty} \frac{(\tan^{-1}(x))^2}{x^2+1} dx = \lim_{t \rightarrow -\infty} \int_t^0 \frac{(\tan^{-1}(x))^2}{x^2+1} dx + \lim_{s \rightarrow \infty} \int_0^s \frac{(\tan^{-1}(x))^2}{x^2+1} dx$

use u-sub

let $u = \tan^{-1}(x)$

$du = \frac{1}{x^2+1} dx$

$= \lim_{t \rightarrow -\infty} \int_{x=t}^{x=0} u^2 du + \lim_{s \rightarrow \infty} \int_{x=0}^{x=s} u^2 du$

$= \lim_{t \rightarrow -\infty} \left. \frac{u^3}{3} \right|_{x=t}^{x=0} + \lim_{s \rightarrow \infty} \left. \frac{u^3}{3} \right|_{x=0}^{x=s}$

$= \lim_{t \rightarrow -\infty} \left. \frac{(\tan^{-1}(x))^3}{3} \right|_t^0 + \lim_{s \rightarrow \infty} \left. \frac{(\tan^{-1}(x))^3}{3} \right|_0^s$

$= 0 - \frac{\left(\frac{-\pi}{2}\right)^3}{3} + \frac{\left(\frac{\pi}{2}\right)^3}{3} - 0$

$= \frac{\pi^3}{24} + \frac{\pi^3}{24} = \boxed{\frac{\pi^3}{12}}$

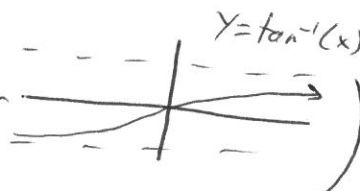
5. Show that $\int_0^1 \frac{\tan^{-1}(x)}{x^{1/3}} dx$ converges or diverges

Step 1: $\tan^{-1}(x) \geq 0$ for $x \in (0, 1)$
 $x^{1/3} \geq 0$ for $x \in (0, 1)$

Hence $\frac{\tan^{-1}(x)}{x^{1/3}} \geq 0$ for $x \in (0, 1)$

Step 2: Since $\int_0^1 \frac{\tan^{-1}(x)}{x^{1/3}} dx \sim \int_0^1 \frac{1}{x^{1/3}} dx$

we think it converges and we want to compare it to something like $\int_0^1 \frac{1}{x^{1/3}} dx$.

Step 3: $\tan^{-1}(x) \leq \frac{\pi}{2}$ (Remember )
 So, $\frac{\tan^{-1}(x)}{x^{1/3}} \leq \frac{\pi}{2 x^{1/3}}$

Now we must evaluate

$$\int_0^1 \frac{\pi}{2 x^{1/3}} dx = \lim_{t \rightarrow 0^+} \int_t^1 \frac{\pi}{2 x^{1/3}} = \lim_{t \rightarrow 0^+} \frac{\pi}{2} \cdot \frac{3}{2} x^{2/3} \Big|_t^1 = \frac{3\pi}{4}$$

Since $0 \leq \frac{\tan^{-1}(x)}{x^{1/3}} \leq \frac{\pi}{2 x^{1/3}}$ and $\int_0^1 \frac{\pi}{2 x^{1/3}} dx$ converges, then $\int_0^1 \frac{\tan^{-1}(x)}{x^{1/3}} dx$ converges.

6. Evaluate $\int_{-1}^2 \frac{1}{x^{2/3}} dx$

Note $\frac{1}{x^{2/3}}$ has an infinite discontinuity at $x=0$, so, this is an improper integral.

$$\int_{-1}^2 \frac{1}{x^{2/3}} dx = \lim_{t \rightarrow 0^-} \int_{-1}^t \frac{1}{x^{2/3}} dx + \lim_{s \rightarrow 0^+} \int_s^2 \frac{1}{x^{2/3}} dx$$

$$= \lim_{t \rightarrow 0^-} \left[3x^{1/3} \right]_{-1}^t + \lim_{s \rightarrow 0^+} \left[3x^{1/3} \right]_s^2$$

$$= \lim_{t \rightarrow 0^-} 3t^{1/3} - 3(-1)^{1/3} + \lim_{s \rightarrow 0^+} 3(2)^{1/3} - 3(s)^{1/3}$$

$$= \boxed{3 + 3\sqrt[3]{2}}$$

7. Evaluate $\int x \tan^{-1}(x) dx$

use by-parts

$$u = \tan^{-1}(x) \quad v' = x$$
$$u' = \frac{1}{x^2+1} \quad v = \frac{x^2}{2}$$

$$\int x \tan^{-1}(x) dx = \frac{x^2}{2} \tan^{-1}(x) - \int \frac{\frac{1}{2}x^2}{x^2+1} dx$$

↑
use partial fractions

Step I $\deg(\frac{1}{2}x^2) = 2 = \deg(x^2+1)$

So, must divide

$$\begin{array}{r} x^2+0x+1 \overline{) x^2+0x+0} \\ \underline{-(x^2+0x+1)} \\ -1 \end{array}$$

$$\text{So, } \frac{1}{2} \int \frac{x^2}{x^2+1} dx = \frac{1}{2} \int 1 + \frac{-1}{x^2+1} dx$$

$$= \frac{1}{2}x - \frac{1}{2} \tan^{-1}(x) + C$$

$$\text{So, } \int x \tan^{-1}(x) dx = \boxed{\frac{x^2}{2} \tan^{-1}(x) - \frac{1}{2}x + \frac{1}{2} \tan^{-1}(x) + C}$$