1. (10 pts) Evaluate the following limit. Carefully justify your answer.

$$lim_{n\to\infty}\frac{sin(n)ln(n)}{n}$$

Use the squeeze thm.

$$-1 \le \sin(n) \le 1$$

$$\frac{-\ln(n)}{n} \le \frac{\sin(n)\ln(n)}{n} \le \frac{\ln(n)}{n}$$

Examine $\lim\limits_{n\to\infty} \frac{\ln(n)}{n}$ ← this is an indeterminant form of $\frac{\infty}{\infty}$.

Use L'Hospitals

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\lim_{n\to\infty} \frac{-\ln(n)}{n} = -\lim_{n\to\infty} \frac{\ln(n)}{n} = -0 = 0.$$

Since $\lim\limits_{n\to\infty} \frac{\ln(n)}{n} = \lim\limits_{n\to\infty} \frac{-\ln(n)}{n} = 0$ and

$\frac{-\ln(n)}{n} \le \frac{\sin(n)\ln(n)}{n} \le \frac{\ln(n)}{n}$, then

$$\lim_{n\to\infty} \frac{\sin(n)\ln(n)}{n} = 0.$$

2. (10 pts) Evaluate the following series

$$\Sigma_{n=0}^{\infty} \frac{3^{n-1}+4^{n+1}+1}{5^n}$$

Recall: $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ if $|r|<1$.

$$\sum_{n=0}^{\infty} \frac{3^{n-1}+4^{n+1}+1}{5^n} = \sum_{n=0}^{\infty} \frac{3^{n-1}}{5^n} + \frac{4^{n+1}}{5^n} + \frac{1}{5^n}$$

$$= \sum_{n=0}^{\infty} \frac{3^{n-1}}{5^n} + \sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n} + \sum_{n=0}^{\infty} \frac{1}{5^n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{3}\cdot\left(\frac{3}{5}\right)^n + \sum_{n=0}^{\infty} 4\left(\frac{4}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n$$

$$= \frac{1}{3}\left(\frac{1}{1-\frac{3}{5}}\right) + 4\left(\frac{1}{1-\frac{4}{5}}\right) + \frac{1}{1-\frac{1}{5}}$$

$$= \frac{1}{3}\left(\frac{5}{2}\right) + 4(5) + \frac{5}{4}$$

$$= \frac{10+240+15}{12}$$

$$= \boxed{\frac{265}{12}}$$

3. (10 pts) Find all values of k for which the following series converges. Carefully justify your answer.

$$\sum_{n=1}^{\infty} \frac{n+1}{kn^3+n^2+n+1}$$

Suppose $k=0$. Use limit Comparison

We know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test.

Examine $\lim_{n\to\infty} \frac{\frac{n+1}{n^2+n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+n}{n^2+n+1}$

Since degree of top equals degree of bottom

$= \frac{1}{1} = 1$

Hence, by the limit comparison test the series diverges when $k=0$.

Suppose $k \neq 0$. Use limit comparison test

we know $\sum_{n=1}^{\infty} \frac{1}{kn^2}$ converges by p-series test.

Evaluate $\lim_{n\to\infty} \frac{\frac{n+1}{kn^3+n^2+n+1}}{\frac{1}{kn^2}} = \lim_{n\to\infty} \frac{kn^3+kn^2}{kn^3+n^2+n+1}$

Since deg. of top = deg. of bottom

$= \frac{k}{k} = 1.$

Hence, by the limit comparison test, the series converges when $k \neq 0$.

4. (10 pts) Determine if the following series converges or diverges. Carefully justify your answer.

$$\Sigma_{n=1}^{\infty}(1+\frac{1}{n})^{\frac{1}{100}n^2}$$

use the root test

Examine $\lim_{n\to\infty} \sqrt[n]{\left|\left(1+\frac{1}{n}\right)^{\frac{n^2}{100}}\right|} = \lim_{n\to\infty} \sqrt[n]{\left(1+\frac{1}{n}\right)^{\frac{n^2}{100}}}$

$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{\frac{n}{100}}$

$= \left(\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n}\right)^{\frac{1}{100}}$

$= e^{\frac{1}{100}} > 1$

Hence, by the ~~lim~~ root test,

$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{\frac{n^2}{100}}$ diverges.

5. Show that the following series converges conditionally. Carefully justify your answer.

$$\Sigma_{n=1}^{\infty}\frac{(-1)^n ln(n)}{n}$$

Step 1: Show $\sum_{n=1}^{\infty}\left|\frac{(-1)^n \ln(n)}{n}\right| = \sum_{n=1}^{\infty}\frac{\ln(n)}{n}$ diverges

use integral test: Examine $\int_1^{\infty}\frac{\ln(x)}{x}dx$

let $u=\ln(x)$ $\quad du=\frac{1}{x}dx$

$$\int_?^? u\,du = \frac{u^2}{2}\Big|_?^?$$

$$= \lim_{a\to\infty}\frac{(\ln(x))^2}{2}\Big|_1^a$$

$$= \lim_{a\to\infty}\frac{(\ln(a))^2}{2} - 0 = \infty$$

Hence $\sum_{n=1}^{\infty}\frac{\ln(n)}{n}$ diverges by the integral test.

Step 2: Show $\sum_{n=1}^{\infty}\frac{(-1)^n\ln(n)}{n}$ converges

use Alternating Series test. with $b_n=\frac{\ln(n)}{n}$

① $\lim_{n\to\infty}\frac{\ln(n)}{n} \overset{L'H.}{=} \lim_{n\to\infty}\frac{\frac{1}{n}}{1} = \lim_{n\to\infty}\frac{1}{n} = 0$

② Show b_n is decreasing: Let $f(x)=\frac{\ln(x)}{x}$

$$f'(x) = \frac{x\cdot\frac{1}{x} - \ln(x)\cdot 1}{x^2} = \frac{1-\ln(x)}{x^2}$$

So, $f'(x)\le 0$ for $x \ge e$. Thus, b_n is decreasing

Hence, $\sum_{n=1}^{\infty}\frac{(-1)^n\ln(n)}{n}$ converges by the alt. Series test.

By steps 1 and 2 $\sum_{n=1}^{\infty}\frac{(-1)^n\ln(n)}{n}$ converges conditionally.

6. (10 pts) Find the interval of convergence for the following power series.

$$\Sigma_{n=1}^{\infty} \frac{(n!)^2 x^n}{(2n)!}$$

Step 1: Find the radius of convergence.

$$R = \lim_{n\to\infty} \left| \frac{\frac{(n!)^2}{(2n)!}}{\frac{((n+1)!)^2}{(2n+2)!}} \right| = \lim_{n\to\infty} \frac{(n!)^2\,(2n+2)!}{(2n)!\,((n+1)!)^2}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)\,(n+1)}$$

Since deg. of top = deg. of bottom

$$= \frac{4}{1} = \boxed{4}$$

Step 2: Determine convergence when $x=4, -4$.

$x=4$: Examine $\sum_{n=1}^{\infty} \frac{(n!)^2 4^n}{(2n)!}$ use the test for divergence.

Look at $\lim_{n\to\infty} \frac{(n!)^2 4^n}{2n\,!} = \lim_{n\to\infty} \frac{n\cdot n\,(n-1)(n-1)(n-2)(n-2)\cdots 3\cdot 3\cdot 2\cdot 2\cdot 1\cdot 1\cdot 4^n}{2n\,(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)\cdots 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$

$$= \lim_{n\to\infty} \frac{(2n)(2n)(2n-2)(2n-2)(2n-4)(2n-4)\cdots 6\cdot 6\cdot 4\cdot 4\cdot 2\cdot 2}{2n\,(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)\cdots 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= \lim_{n\to\infty} \frac{2n}{2n-1}\cdot\frac{2n-2}{2n-3}\cdot\frac{2n-4}{2n-5}\cdot\cdots\cdot\frac{6}{5}\cdot\frac{4}{3}\cdot\frac{2}{1}$$

each of these is <u>bigger</u> than 1, so ≥ 1

Hence $\lim_{n\to\infty} \frac{(n!)4^n}{(2n)!} \geq 1 \neq 0$. Thus, $\sum_{n=1}^{\infty} \frac{(n!)4^n}{(2n)!}$ diverges.

The same argument shows $\sum_{n=1}^{\infty} \frac{(n!)(-4)^n}{(2n)!}$ <u>diverges.</u>

Thus, the interval of convergence is $\boxed{(-4, 4)}$

7. (10 pts) Let $f(x) = cos(x)sin(x)$.

A) Find the cubic polynomial representing the first four terms of the Macluarin series for $f(x)$. Recall $sin(x) = (x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots)$

$$cos(x) = (1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots)$$

$$cos(x) \cdot sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

$$+ \cancel{x^3} - \frac{x^3}{2} + \frac{x^5}{12} - \frac{x^7}{240} + \cdots$$

$$+ \frac{x^5}{24} - \frac{x^7}{144} + \cdots$$

$$= x + (-\frac{1}{6} - \frac{1}{2})x^3 + \cdots$$

$$= \boxed{x - \frac{2}{3}x^3} + \cdots$$

B) Use Taylor's formula to estimate the error in approximating $f(x)$ by the polynomial found in part A) on the interval on $[\frac{-\pi}{4}, \frac{\pi}{4}]$.

Step 1: Find M s.t. $|f^{(4)}(t)| \leq M$ for $t \in [\frac{-\pi}{4}, \frac{\pi}{4}]$

$f(x) = cos(x)\,sin(x) = \frac{1}{2}sin(2x)$ $\quad f'(x) = cos(2x)$

$f''(x) = -2\,sin(2x)$ $\quad f'''(x) = -4\,cos(2x)$ $\quad f^{(4)}(x) = 8\,sin(2x)$

$|f^{(4)}(t)| \leq |8\,sin(2t)| \leq 8|1| = \boxed{8}$ on $[\frac{-\pi}{4}, \frac{\pi}{4}]$

Step 2: Use Taylor's Formula

$$|R_3(x)| < M\frac{|x|^4}{4!} \leq 8\frac{(\frac{\pi}{4})^4}{24} = \boxed{\frac{\pi^4}{768}}$$