MATH 122: CALCULUS I **NAME:**

FINAL EXAM: VERSION 1

You are allowed 2.5 hrs to complete this exam. You must ***show all work*** to get credit. You may use a scientific or graphing calculator, but may **not** use your cell phone for any reason at any time during this exam. If you are caught cheating, you will receive a score of zero on this exam and may be dropped from the class. Please raise your hand if you have any questions.

1. Let $f(x)=\dfrac{x^2-7x}{x^2-6x-7}$. $=\dfrac{x(x-7)}{(x+1)(x-7)}$

A. State all asymptotes of this function	**B.** List the locations and types of discontinuities	**C.** Find $\lim_{x\to 7} f(x)$ in two ways:	
VA: $x=-1$ $\lim_{x\to -1^+} f(x)=-\infty$ HA: $y=1$ $\lim_{x\to\infty} f(x)=1$	INFINITE: $x=-1$ REMOVABLE $x=7$	$\lim_{x\to 7}\dfrac{x}{x+1}=\dfrac{7}{8}$	$\lim_{x\to 7} f(x)$ L'H $=\lim_{x\to 7}\dfrac{2x-7}{2x-6}$ $=\dfrac{7}{8}$

3. Find the derivative of $f(x)=\dfrac{1}{x^2}$ in three different ways (state the rule you are using):

A. Definition of derivative	**B.** Rule#1: POWER RULE	**C.** Rule#2: QUOTIENT
$\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ $=\lim_{h\to 0}\dfrac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$ $=\lim_{h\to 0}\dfrac{x^2-x^2-2xh-h^2}{h(x^2)(x+h)^2}$ $=-2x^{-3}$	$f(x)=x^{-2}$ $f'(x)=-2x^{-3}$	$f'(x)=\dfrac{0\cdot x^2-2x\cdot 1}{x^4}$ $=-2x^{-3}$

4. Find the equation of the tangent line to $f(x)$ at $x=0$.

a. $f(x)=(4x-3)^3(x^2+1)^2$.

b. $f(x)=\left[\cos\left(x-\frac{\pi}{3}\right)\right]^{x^2+1}$

a) $f'(x) = 12(4x-3)^2(x^2+1)^2 + 4x(4x-3)^3(x^2+1)$

$y - y_1 = m(x - x_1) \rightarrow y + 27 = 108(x-0)$

b) $\ln f(x) = (x^2+1)\ln\left[\cos\left(x - \pi/3\right)\right]$

$\frac{f'(x)}{f(x)} = 2x\ln\left[\cos\left(x-\pi/3\right)\right] - (x^2+1)\sin\left(x-\frac{\pi}{3}\right)$

$m = f'(0) = \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$

5. A. Find $\lim_{x\to 0^+} x^2 \ln x$

$= \lim_{x\to 0^+} \frac{\ln x}{x^{-2}} = \lim_{x\to 0^+} \frac{1/x}{-2x^{-3}} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0$

B. Find $\lim_{x\to\infty}(1+x)^{3/x}$.

$\ln(1+x)^{3/x} = \frac{3}{x}\ln(1+x)$

$\lim_{x\to\infty} \frac{3\ln(1+x)}{x} \quad \frac{1}{1+x} \quad \lim_{x\to\infty} \frac{3/(1+x)}{1} = 0$

so $\lim_{x\to\infty}(1+x)^{3/x} = e^0 = 1$

6. Liquid is being poured into a cylinder with radius 2cm at a rate of 3.7cm^3/s. At what rate is the height of the liquid increasing when the volume is 16πcm^3?

$V = \pi r^2 h = 4\pi h$

$\frac{dV}{dt} = 4\pi\frac{dh}{dt}$

$3.7 = 4\pi\frac{dh}{dt}$

so

$\frac{dh}{dt} = \frac{3.7}{4\pi}\frac{cm}{s}$

7. State the behavior of the exponential and natural logarithmic functions (no need to show work):

$f(x)=\ln(x)$	$f(x)=e^x$
HA: NONE	HA: $\lim_{x\to-\infty} f(x) = 0$ / $y=0$
VA: $\lim_{x\to 0^+} f(x) = -\infty$ / $x=0$	VA: NONE
x-int: $\ln x = 0 \rightarrow x = 1$ $(1,0)$	x-int: $e^x = 0$ DNE
y-int: $f(0)$ DNE	y-int: $f(0) = e^0 = 1$ $(0,1)$

8. Let $f(x) = e^{-1/x}$.

A. Find the Domain and Asymptotes

Dom: $x \neq 0$

HA: $\lim_{x \to \infty} e^{-1/x} = 1 \to y = 1$

VA: $\lim_{x \to 0^-} e^{-1/x} = \infty \to x = 0$

B. Find x- and y- intercepts

y-int: $f(0)$ DNE

x-int: $e^{-1/x} = 0$ DNE

C. Determine Symmetry

$f(-x) = e^{-1/-x} = e^{1/x}$

no symmetry

D. Determine intervals of increase and decrease, and local extrema. Put the information in the table below as shown in class. You may not need the entire table.

$f'(x) = e^{-1/x} \cdot \frac{1}{x^2} = \frac{e^{-1/x}}{x^2}$

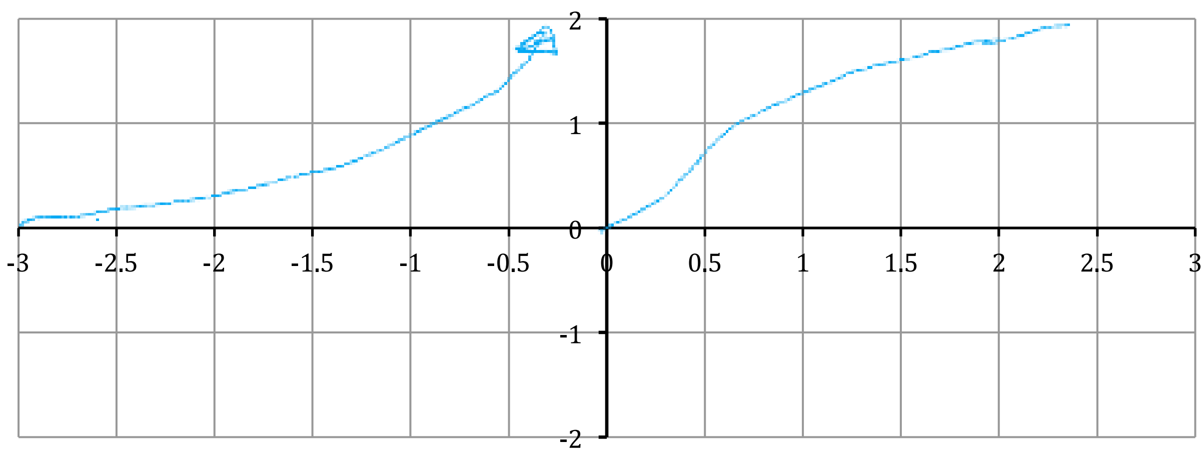
x	$(-\infty, 0)$	$(0, \infty)$	
$f'(x)$	>0	>0	
$f(x)$	inc	inc	

E. Determine intervals of concavity. Put the information in the table below as shown in class. You may not need the entire table.

$$f''(x) = \frac{\frac{e^{-1/x}}{x^2} \cdot x^2 - 2xe^{-1/x}}{x^4} = \frac{e^{-1/x}(1-2x)}{x^4}$$

x	$(-\infty, 0)$	$(0, 1/2)$	$(1/2, \infty)$
$f''(x)$	>0	>0	<0
$f(x)$	CU	CU	CD

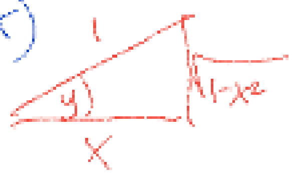
F. Sketch the graph, based on the above. Plot intercepts, local extrema and inflection points



9. A. Derive (prove) the formula for the derivative of $cos^{-1}x$.

$y = \cos^{-1} x \rightarrow \cos y = x$

$\frac{d}{dx}(\cos y) = \frac{d}{dx}(x)$



$-\sin y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}$

B. Let $H(x) = \cos^{-1}(1-3x)$. Find the derivative of $H(x)$.

$H'(x) = -\frac{1}{\sqrt{1-(1-3x)^2}} \cdot -3 = \frac{3}{\sqrt{1-(1-3x)^2}}$

C. Find the domain of $H(x)$ and of its derivative.

$-1 \le (1-3x) \le 1$

$-2 \le -3x \le 0$

$0 \le x \le \frac{2}{3}$

$1-(1-3x)^2 > 0$

$(1-3x) < 1$

$-1 < 1-3x < 1 \rightarrow 0 < x < \frac{2}{3}$

D. Find the limit as $x \to 0$ for $H(x)$ and its derivative.

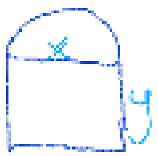
$\lim_{x \to 0} \cos^{-1}(1-3x)$

$= \cos^{-1}(1) = 0$

$\lim_{x \to 0} \frac{3}{\sqrt{1-(1-3x)^2}} = \text{DNE}$

So $\lim_{x \to 0} f'(x) = \text{DNE}$

10. A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle.) If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.



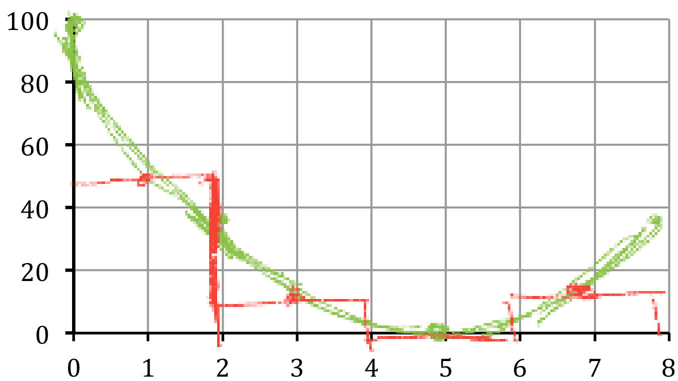
① $\frac{\pi x}{2} + x + 2y = 30 \rightarrow$ Solve for y

② $Q(x) = \pi\left(\frac{x}{2}\right)^2 + xy$ plug in and find max

11. Consider the function $f(x) = (2x - 10)^2$ on [0,8].

a. Estimate the area under the curve using M_4. Draw picture and show work.

$\Delta x = \frac{8-0}{4} = 2$



$M_4 = 2[f(1) + f(3) + f(5) + f(7)]$

$= 2[64 + 16 + 0 + 16] = 192$

b. Find the exact area by integrating in two different ways.

$A = \int_0^8 4x^2 - 40x + 100 \, dx$

$= \frac{4x^3}{3} - 20x^2 + 100x \Big|_0^8$

$= 202\frac{2}{3}$

let $u = 2x - 10$

$du = 2dx \rightarrow dx = \frac{1}{2}du$

$A = \frac{1}{2}\int_{-10}^{6} u^2 \, du$

$= \frac{1}{6}u^3 \Big|_{-10}^{6} = 202\frac{2}{3}$

12. If $g(x)=\int_{9x^3}^{17}\cos t^5\,dt$, then find $g'(x)$.

FTC: $g(x) = -\int_{17}^{9x^3} \cos t^5\,dt$

so $g'(x) = -\cos[9x^3]^5 \cdot 27x^2$

$= -27x^2\cos[9x^3]^5$

13. Evaluate

a. $\int 17\cos^3\theta\sin\theta\,d\theta$ $\int 17\cos^3 x\sin x\,dx$

let $u=\cos x$

$du = -\sin x\,dx \to -du = \sin x\,dx$

$= -17\int u^3\,du = -\frac{17}{4}u^4 + C$

$= -\frac{17}{4}\cos^4 x + C$

b. $\int \frac{x\,dx}{5-3x} = \int \frac{x}{5-3x}\,dx$

let $u = 5-3x \to x = \frac{5}{3}-\frac{1}{3}u$

$du = -3dx \to dx = -\frac{1}{3}du$

$= -\frac{5}{9}\int \frac{1}{u}du + \frac{1}{9}\int\frac{u}{u}du$

$= -\frac{5}{9}\ln|u| + \frac{1}{9}u + C$

$= -\frac{5}{9}\ln|5-3x| + \frac{1}{9}(5-3x) + C$

c. $\int_0^2 x^2\sqrt{x^3+1}\,dx$

let $u = x^3+1$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2dx$

$\frac{1}{3}\int u^{1/2}du = \frac{2}{9}u^{3/2}$

$= \frac{2}{9}(x^3+1)^{3/2}\Big|_0^2 = \frac{2}{9}\left[9^{3/2}-1^{3/2}\right]$

$= \frac{2}{9}[26] = \frac{52}{9}$