

## Math Day at the Beach 2014

MULTIPLE CHOICE – Write your name and school and mark your answers on the answer sheet. You have 45 minutes to work on these problems. No calculator is allowed.

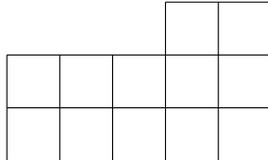
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- The price of purple widgets has dropped 60% since 2009. By how much does the price of purple widgets have to rise from its current value in order to return to its 2009 level?  
(A) 40% (B) 60% (C) 80% (D) 120% (E) 150%
- How many values of  $x$  are there in the interval  $[0, \pi]$  such that  $f(x) = g(x)$ , where  $f(x) = \cos(\sin x)$  and  $g(x) = \sin x$ ?  
(A) 1 (B) 2 (C) 3 (D) 4 (E) infinitely many
- Which of the following is equal to  $\log\left(3^{(5^7)}\right)$ ?  
(A)  $7\log(3^5)$  (B)  $35\log 3$  (C)  $7(\log 3)(\log 5)$  (D)  $7(\log 3 + \log 5)$  (E)  $5^7\log 3$
- Tina's car gets 3 more miles per gallon during highway driving than it does during city driving. On a recent trip, Tina drove 136 miles on the highway and 155 miles in the city, using a total of 9 gallons of gasoline. How many miles per gallon does Tina's car get during city driving?  
(A) 29 (B) 30 (C) 31 (D) 32 (E) 33
- Five real numbers lie in the interval  $[0, 360]$ . Their mean is 120. Let  $M$  be the largest possible values of the median of these numbers and let  $m$  be the smallest possible value of the median. Find  $M - m$ .  
(A) 60 (B) 180 (C) 200 (D) 240 (E) 360
- Choose three distinct numbers from  $1, 2, \dots, 20$  so that if  $a$  is chosen then  $a + 1$  and  $a + 2$  are not chosen. Count the number of ways you can do this.  
(A) 56 (B) 560 (C) 816 (D) 1140 (E) 4760
- A rational number is called "lucky" if it equals both  $a + \frac{b}{c}$  and  $a \cdot \left(\frac{b}{c}\right)$  for some positive integers  $a, b, c$ . How many lucky numbers are there between 5 and 10?  
(A) 4 (B) 5 (C) 6 (D) 8 (E) 10
- The three planes  $x = y$ ,  $y = z$ ,  $x = z$  cut the unit cube  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ ,  $0 \leq z \leq 1$  into  $n$  pieces. Find  $n$ .  
(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

9.  $f$  is a function defined on the whole real line which has the property that  $f(1+x) = f(2-x)$  for all  $x$ . Assume that the equation  $f(x) = 0$  has 8 distinct real roots. Find the sum of these roots.

- (A) 4    (B) 6    (C) 8    (D) 10    (E) 12

10. In how many ways can we tile the floor of a room (see figure) using some mixture of  $2 \times 1$  and  $1 \times 2$  tiles?



- (A) 8    (B) 10    (C) 11    (D) 12    (E) 66

11. Let  $a$  and  $b$  be any two distinct roots of the equation  $x^3 + 3x^2 - 1 = 0$ . Which one of the following equations must have  $ab$  as a root?

- (A)  $x^3 - 3x - 1 = 0$     (B)  $x^3 + x^2 - 3x + 1 = 0$     (C)  $x^3 + 3x^2 + 1 = 0$     (D)  $x^3 + x^2 + 3x - 1 = 0$   
 (E)  $x^3 - 3x^2 + 1 = 0$

12. On Monday, four trucks drove in a line. None of the trucks passed any of the others, so the order of the trucks never changed. How many ways are there to rearrange the order of the trucks so that on Tuesday, no truck is directly behind a truck that it was directly behind the day before?

- (A) 4    (B) 8    (C) 11    (D) 12    (E) 16

13. We have points  $A(0, 0)$ ,  $B(1, 0)$ ,  $C(0, 3)$ ,  $D(0, 4)$ , and  $E(3, 4)$ . Find the point  $(x, y)$  such that by rotating around  $(x, y)$  by a suitable angle, triangle  $ABC$  is rotated to triangle  $DCE$ .

- (A)  $(0, 0)$     (B)  $(0, 3)$     (C)  $(4, 1)$     (D)  $(0, 2)$     (E)  $(2, 2)$

14. Two circles of radius 1 intersect each other in such a way that the portion of the line passing through the centers of the circles that lies inside both circles has length  $\frac{1}{3}$ . Find the radius of a circle that is internally tangent to each of the circles and to the line passing through the centers of the circles.

- (A)  $\frac{\sqrt{3}}{6}$     (B)  $\frac{\sqrt{3}}{12}$     (C)  $\frac{1}{8}$     (D)  $\frac{3}{8}$     (E)  $\frac{11}{72}$

15. Place eight spheres of radius 1 in 3-dimensional coordinate space such that the centers are at  $(\pm 1, \pm 1, \pm 1)$  and each large sphere is externally tangent to three others. Place sphere  $A$  with center at the origin externally tangent to all 8 of the larger spheres. Then place sphere  $B$  with center on one of the coordinate axes such that sphere  $B$  is externally tangent to sphere  $A$  and to four of the larger spheres. Find the radius of sphere  $B$ .

- (A)  $\frac{3-\sqrt{3}}{3}$     (B)  $\frac{3+\sqrt{3}}{6}$     (C)  $\sqrt{3} - 1$     (D)  $\frac{3\sqrt{2}+2}{4}$     (E)  $\frac{2\sqrt{3}}{3}$

## Math Day at the Beach 2014

INDIVIDUAL FREE RESPONSE – Write your name and school and mark your answers on your answer sheet. You have 25 minutes to work on these problems. No calculator is allowed.

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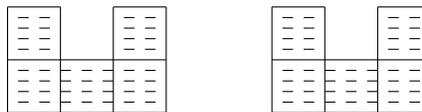
16. Suppose that  $a$  and  $x$  are two positive real numbers for which  $\log_a x + \log_x a = 15$ . Compute the value of  $(\log_a x)^2 + (\log_x a)^2$ .
17. Suppose  $p(x)$  is a nonzero polynomial such that  $xp(x-1) = (x-26)p(x)$  for all real  $x$ . Find the degree of the polynomial  $p$ .
18. Compute  $\sum_{n=1}^{99} [0.67n]$ , where the notation  $[x]$  means the greatest integer that is less than or equal to  $x$ .
19. Let  $z$  be a complex number such that  $z^3 = 1$  but  $z \neq 1$ . Compute the real part of  $\left(\frac{z+2}{z-1}\right)^{2014}$ .
20. A spherical ball of radius 1 foot is bouncing around randomly in a closed rectangular room of dimensions  $20 \times 12 \times 8$  (in feet). Find the volume (in cubic feet) of the space within the room that the ball cannot possibly touch no matter how it bounces. (Assume the ball does not deform at all when it bounces.)

## Math Day at the Beach 2014

TEAM ROUND – Write your school name and mark your answers on the answer sheet. No calculator is allowed. You have 30 minutes to work on these problems.

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- Find the set of all real numbers  $x$  such that  $(\log_3 x)^2 - \frac{30}{\log_3 x} = 19$ .
- $D, E, F$  lie on sides  $\overline{AB}, \overline{BC}, \overline{CA}$  of triangle  $ABC$  respectively such that  $\frac{AD}{DB} = \frac{BE}{EC} = \frac{CF}{FA} = \frac{2}{5}$ . Find the ratio  $\frac{\text{area}(\triangle DEF)}{\text{area}(\triangle ABC)}$ .
- A three dimensional structure is constructed by stacking cubical blocks on a  $3 \times 3$  grid of squares, with the grid aligned N-S and E-W. (A block on the second level must rest on a block on the first level.) The sketches below show the appearance of the structure when viewed from a great distance away to the west and to the south. Find the number of distinct ways such a structure could be built with 7 or fewer blocks.



- Draw the curve  $y = x^{2014}$ . Let  $A$  be a point in the first quadrant that is on this curve. The perpendicular from  $A$  to the  $x$ -axis intersects the  $x$ -axis at  $B$ . The tangent line to the curve at  $A$  intersects the  $x$ -axis at  $C$ . By “region  $OCA$ ” we mean the region bounded above by the curve  $y = x^{2014}$  and below by segments  $\overline{OC}$  and  $\overline{CA}$ . Compute the ratio  $\frac{\text{area}(\text{region } OCA)}{\text{area}(\triangle CBA)}$ .
- Let  $n$  be a positive integer. The positive divisors of  $n$  are  $d_1 < d_2 < d_3 < \dots$ , where  $d_1 = 1$ . Suppose that  $n = d_1^4 + d_2^4 + d_3^4 + d_4^4$ . Compute  $\lfloor \sqrt[5]{n} \rfloor$  (the integer part of the fifth root of  $n$ ).
- Let  $x = \{1, 2, 3, 4, 5\}$ . How many different functions  $f : X \rightarrow X$  are there such that  $f$  is invertible (that is, one-to-one and onto) but there is no  $x$  such that  $f(x) = x$ ?
- Suppose  $a_1, a_2, a_3, \dots$  is a sequence of positive integers such that each term is the sum of the two preceding terms. (That is, for  $n \geq 3$ ,  $a_n = a_{n-1} + a_{n-2}$ .) Suppose that  $a_1, a_2, \dots, 99, \dots$  is such a sequence and has the largest possible number of terms preceding the 99. Find the ordered pair  $(a_1, a_2)$ .
- From the set of polynomials  $x^2 + bx + c = 0$  with  $b$  and  $c$  real numbers uniformly chosen such that  $0 \leq b \leq 9$  and  $0 \leq c \leq 9$ , find the probability that the polynomial chosen has real roots.