Math Day at the Beach 2013

MULTIPLE CHOICE – *Write your name and school and mark your answers on the answer sheet. You have 45 minutes to work on these problems. No calculator is allowed.*

---

1. Define $f(x) = \frac{1}{x}$ for $x > 0$. Find $f(f(3) + f(5))$.

   (A) 8 (B) $\frac{1}{8}$ (C) 15 (D) $\frac{8}{15}$ (E) $\frac{15}{8}$

2. A used car dealer bought two cars and then sold them, receiving \$560 for each car. One of these transactions amounted to a 40% profit for the dealer, whereas the other amounted to a 20% loss. What is the dealer's net profit on the two transactions?

   (A) \$20 (B) \$36 (C) \$56 (D) \$84 (E) \$112

3. The Aardvarks and the Wasps play a best of three series (*i.e.*, the series is over as soon as one team wins two games.) If the series is tied or the Wasps are ahead, the Aardvarks have a 60% chance of winning the next game, but if the Aardvarks are ahead, the Wasps have a 50% chance of winning the next game. What is the probability that the Aardvarks win the series in exactly three games?

   (A) .3 (B) .324 (C) .54 (D) .6 (E) .624

4. An equilateral triangle and a regular hexagon have equal perimeters. What is the ratio of the area of the triangle to the area of the hexagon?

   (A) 1 (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{1}{2}$ (D) $\frac{1}{3}$ (E) $\frac{2}{3}$

5. Let $X = \frac{1}{\log_2(100!)} + \frac{1}{\log_3(100!)} + \frac{1}{\log_4(100!)} + \cdots + \frac{1}{\log_{100}(100!)}$. Which of the following is true?

   (A) $X < .01$ (B) $.01 \leq X < 1$ (C) $X = 1$ (D) $1 < X \leq 10$ (E) $X > 10$

6. Suppose $f(x)$ is a polynomial with integer coefficients for which 3 and 13 are both roots. Which of the following could possibly be the value of $f(10)$?

   (A) -3 (B) 10 (C) -14 (D) 39 (E) -42

7. Three circles having centers A, B, and C, are all externally tangent to one another. The circle centered at A has radius 3, the circle centered at B has radius 5, and the radian measure of $\angle BAC$ is $\frac{\pi}{3}$. Find the radian measure of $\angle ABC$.

   (A) $\frac{\pi}{4}$ (B) $\arccos\left(\frac{11}{14}\right)$ (C) $\frac{\pi}{6}$ (D) $\arccos\left(\frac{5}{8}\right)$ (E) $\arccos\left(\frac{8}{15}\right)$

8. What is the remainder when $x^{200} - 2x^{199} + x^{50} - 2x^{49} + x^2 + x + 1$ is divided by $(x-1)(x-2)$?

   (A) $2x - 1$ (B) 7 (C) $2x + 3$ (D) 1 (E) $6x - 5$

9. Seven points are arranged in space so that no four of them are on the same plane. Use these points as vertices and draw triangle in such a way that no two triangles share a common edge. (Two triangles may share a common vertex.) At most how many triangles can be drawn?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

10. For how many primes p is the value of $p^2 + 21p - 1$ also prime?

(A) 0 (B) 1 (C) 3 (D) 5 (E) infinitely many

11. How many integers x in $\{1, 2, 3, \ldots, 99, 100\}$ are there such that $x^2 + x^3$ is the square of an integer?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

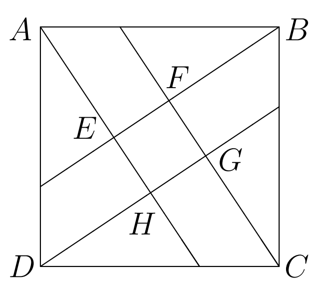
12. If $\sin x + \cos x = \frac{1}{2}$, find the value of $\sin^3 x + \cos^3 x$.

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$ (C) $\frac{9}{16}$ (D) $\frac{5}{8}$ (E) $\frac{11}{16}$

13. Let I_n be the number of functions from a set of n elements into itself that satisfy the condition that $f(f(x)) = f(x)$ for all x in that set. For instance, $I_1 = 1$ and $I_2 = 3$. What is the remainder obtained upon dividing I_6 by 5?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

14. In the diagram below, the square $ABCD$ has area 1, and each of the points on the sides of the square divides its side in the ratio of $1:2$. Find the area of the square $EFGH$.



(A) $\frac{1}{10}$ (B) $\frac{1}{11}$ (C) $\frac{1}{12}$ (D) $\frac{1}{13}$ (E) $\frac{1}{14}$

15. Let $N = 2013!$. (Here, 2013 is written in base 10.) Express N in base 5 notation. What is the rightmost nonzero digit of N written in base 5?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Math Day at the Beach 2013

INDIVIDUAL FREE RESPONSE – *Write your name and school and mark your answers on your answer sheet. You have 25 minutes to work on these problems. No calculator is allowed.*

---

16. Solve the equation

$$1 + \frac{1}{1 + \frac{1}{1+\frac{1}{x}}} = 2013.$$

17. Let $x_1, x_2, \ldots$ be the sequence of numbers that can be written as the sum of as one or more *different* powers of 3, with $x_1 < x_2 < \cdots$. (For example, $x_1 = 3^0 = 1$, $x_2 = 3^1 = 3$, $x_3 = 3^0 + 3^1 = 4$, *etc.* Find x_{100}.

18. Four balls of radius 1 are mutually tangent (in a tetrahedral arrangement). Each of these four balls is inside but tangent to a larger ball. What is the radius of the larger ball?

19. Place the numbers $1, 2, \ldots, 9$ at random so that they fill a 3×3 grid. What is the probability that each of the row sums and each of the column sums is odd?

20. Triangle ABC has a right angle at A with $AB = AC = 4$. Triangle PMN is inscribed in this triangle with P the midpoint of AB, M on AC, and N on BC. Find the minimum possible perimeter of triangle PMN.

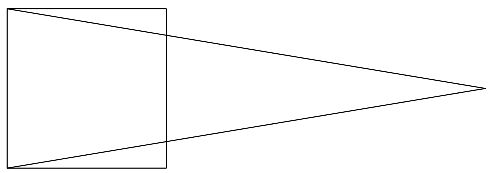
Math Day at the Beach 2013

TEAM ROUND – *Write your school name and mark your answers on the answer sheet. No calculator is allowed. You have 30 minutes to work on these problems.*

---

1. Springfield and Shelbyville are connected by a road. Xavier, Yan, and Zeb all start walking at the same time. Xavier starts in Springfield and walks toward Shelbyville at 2 miles per hour. Yan and Zeb start in Shelbyville and walk toward Springfield, Yan at 3 miles per hour and Zeb at 2 miles per hour. Xavier and Zeb meet on the road one hour after Xavier and Yan meet on the road. How far apart are Springfield and Shelbyville?

2. An isosceles triangle and a square share the same base. The area of the triangle is twice the area of the square. The square splits the larger triangle into a smaller triangle and a trapezoid. What is the ratio of the area of that smaller triangle to the area of the trapezoid?



3. Find a pair of positive integers (a, b) such that $2013^2 + 2001^2 = 2(a^2 + b^2)$. Express your answer as an ordered pair (a, b) with $a < b$.

4. Suppose that f is a function such that for every natural number
   (i) $f(n)$ is a natural number
   (ii) $f(n) < f(n+1)$
   (iii) $f(f(n)) = 3n$.

   What is $f(10)$?

5. A garden is fenced in the shape of a regular hexagon with each side 10 meters in length. A rabbit is tied on the outside of the fence by a 12 meter string attached to one of the corners of the garden fence. What is the area (in square meters) of lawn that the rabbit can reach?

6. Let point A in the plane be $(0, -a)$, for some positive number a. Let B and C be points on the curve $y = x^4$ such that $\overline{AB}$ and $\overline{AC}$ are tangent to that curve. Find the area of the region bounded by $\overline{AB}$, $\overline{AC}$ and the curve $y = x^4$. Express your answer in terms of a as a single term, with no addition or subtraction.

7. $\{z_1, z_2, \ldots, z_{12}\}$ is a set of 12 distinct complex numbers, and $1, i, 2+i$, and $1+2i$ are all members of this set. These numbers, when arranged on the complex plane, form the vertices of a regular dodecagon. Compute the product of these 12 numbers.

8. Three numbers are chosen at random, without replacement, from the set $\{1, 2, 3, \ldots, 101\}$. What is the probability that these three numbers form an arithmetic progression? Express your answer as a fraction in lowest terms.