Math Day at the Beach 2012

MULTIPLE CHOICE – *Write your name and school and mark your answers on the answer sheet. You have 45 minutes to work on these problems. No calculator is allowed.*

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1. For all real numbers a and b, $\sqrt{(a-b)^2} - \sqrt{(b-a)^2} =$

   (A) $2a - 2b$ (B) $2a$ (C) $2b$ (D) $2a + 2b$ (E) 0

2. How many ways are there to arrange 10 students in 5 pairs to play tennis?

   (A) 10 (B) 45 (C) 120 (D) 720 (E) 945

3. An isosceles triangle has sides of length x, x, and $2y$. Find the area of the triangle.

   (A) $y\sqrt{x^2 - y^2}$ (B) $\frac{1}{2}x^2$ (C) $\frac{1}{2}y\sqrt{x^2 - 4y^2}$ (D) $xy - y^2$ (E) $x\sqrt{y^2 - x^2}$

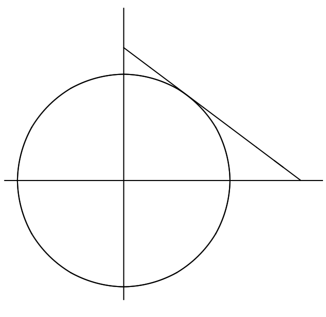
4. A bag contains red, orange, yellow, and blue marbles. There are twice as many orange marbles as red, half as many red marbles as yellow marbles, and 15% of the marbles are red. What is the smallest possible number of blue marbles?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

5. How many real solutions are there to the equation $2x + 2 = |x + 3| - |x - 1|$?

   (A) 1 (B) 2 (C) 3 (D) 0 (E) infinitely many

6. The circle in the diagram has equation $x^2 + y^2 = 25$, and the line L is tangent to the circle at the point $(3, 4)$. Find the area of the triangle formed by L and the coordinate axes, and round that area to the nearest whole number.



   (A) 12 (B) 24 (C) 25 (D) 26 (E) 29

7. In the four-digit number $\underline{abcd}$ the digits a, b, c, and d are all distinct. Find the maximum of $|a - b| + |b - c| + |c - d| + |d - a|$.

   (A) 9 (B) 24 (C) 28 (D) 32 (E) 36

8. What is the largest integer n such that $\dfrac{100!}{10^n}$ is an integer? (Note: $m!$ is defined as the product $m! = 1 \cdot 2 \cdot 3 \cdots m$ of the first m integers multiplied together.)

   (A) 23 (B) 24 (C) 25 (D) 26 (E) 27

9. Two positive numbers have the property that if a is the difference of the two numbers, b is the sum of the two numbers, and c is the product of the two numbers, then $a : b : c = 13 : 41 : 63$. Compute c (the product of the two numbers).

   (A) $\frac{21}{8}$ (B) $\frac{21}{2}$ (C) $\frac{63}{2}$ (D) 63 (E) 378

10. A boy and a girl agree to meet in the park. Each will show up at a random time between 4:00 p.m. and 5:00 p.m. and wait for the other person for 20 minutes, but not past 5:00. What is the probability that they will actually meet?

    (A) $\frac{2}{9}$ (B) $\frac{1}{3}$ (C) $\frac{4}{9}$ (D) $\frac{5}{9}$ (E) $\frac{2}{3}$

11. Let $x = 2^{1000}, y = 3^{600}$, and $z = 10^{300}$. Arrrange x, y, and z in increasing order.

    (A) $x < y < z$ (B) $x < z < y$ (C) $y < x < z$ (D) $y < z < x$ (E) $z < y < x$

12. A drawer contains n socks, of which r are red. If two socks are chosen at random (without replacement) from the drawer, the probability that both of them are red is $\frac{5}{14}$. What is the smallest possible value of $n - r$? (That is, the number of non-red socks.)

    (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

13. Let f be a function such that $f(x+y) = f(x)f(y)$ for all real numbers x and y. If $f(1) = \frac{1}{16}$, then the value of $f(-1)$ is

    (A) 16 (B) $\frac{1}{16}$ (C) $-\frac{1}{16}$ (D) -16 (E) 1

14. What is the smallest positive angle θ such that $8\cos^3\theta - 6\cos\theta = \sqrt{3}$?

    (A) $5°$ (B) $10°$ (C) $15°$ (D) $20°$ (E) $25°$

15. Assume f is an integer coefficient polynomial. Suppose that $f(0) = 36$ and that $f(x_1) = f(x_2) = \cdots = f(x_n) = 2012$ for distinct positive integers $x_1, x_2, \ldots, x_n$. The largest possible value of n is ?

    (A) 4 (B) 5 (C) 6 (D) 8 (E) 16

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INDIVIDUAL FREE RESPONSE – *Write your name and school and mark your answers on your answer sheet. You have 25 minutes to work on these problems. No calculator is allowed.*

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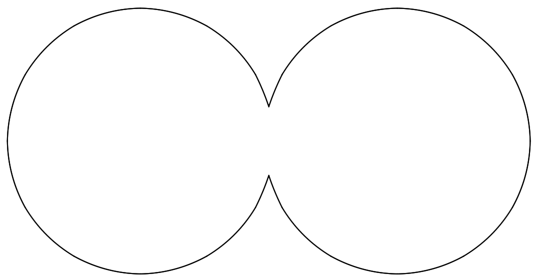
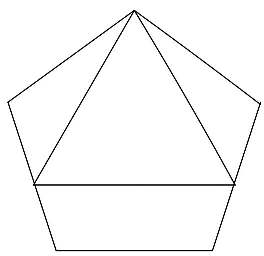
16. In racing over a given distance at uniform speed, Al can beat Bob by 20 meters, Bob can beat Chris by 10 meters, and Al can beat Chris by 28 meters. How many meters is the distance over which they are racing?

17. The region depicted below is the union of two overlapping circular discs of radius 1. The boundary of the region has arclength $\dfrac{11\pi}{3}$. Find the area of the region.



18. A perfectly spherical boulder of volume $288\pi\,\mathrm{m}^3$ sits on perfectly flat ground. We build a small tower on top of the bounder to hold a point-source light exactly 2 meters above the exact top of the boulder. Find the area, in square meters, of the shadow of the boulder cast by this light onto the flat ground.

19. The figure below contains a regular pentagon and an equilateral triangle. Let $a < b < c < d < e$ be all the different measures of all of the angles in the picture. Compute
$$\frac{b}{a} + \frac{e}{d} + \frac{d}{b}.$$



20. A convex polyhedron has faces of which some are pentagons and some are hexagons. Exactly three faces meet at each vertex. Let x be the smallest possible number of pentagons and let y be the largest possible number of pentagons. Compute xy.

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TEAM ROUND – *Write your school name and mark your answers on the answer sheet. No calculator is allowed. You have 30 minutes to work on these problems.*

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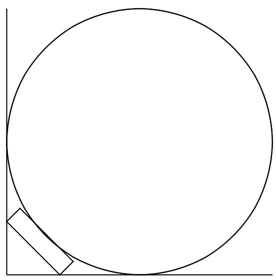
1. Each of two congruent equilateral triangles with side s has center that is a vertex of the other triangle. What is the area of the overlap, in terms of s?

2. Let S be a set of integers. The set of all possible sums of two different elements of S is $\{7, 8, 10, 11, 13, 14, 16, 19, 20, 22\}$. Each of these sums happens in only one way. If X is the mean of the set S and Y is the median of the set S, find $X + Y$.

3. Consider the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{14}$, where x and y must be positive integers. Find the sum of all the distinct x values that occur as solutions.

4. When the furniture in a room all got pushed into one right-angled corner of the room, a circular table with radius 2 feet wound up touching both walls and a rectangular file cabinet fit into the space between the table and the corner. Find the area, in square feet, of the base of largest file cabinet that would fit. You may assume that the sides of the file cabinet meet the walls at $45°$ angles.



5. Find the minimum value of $f(x) = |\cos x| + |\cos 2x|$, where x can be any real number.

6. Suppose $f(x+1) = \frac{1+f(x)}{1-f(x)}$ for all $x \in \mathbb{R}$. If $f(1) = 2$, find
$$f(1) + f(2) + f(3) + \cdots + f(2012).$$

7. Solve for x : $\left(\sqrt{4} - \sqrt{3}\right)^4 = \sqrt{x} - \sqrt{x-1}$.

8. Find the sum of all of the roots for $0 \leq \theta < 2\pi$ of the equation $\cos(2012\theta) = \cos(2001\theta)$. Express angles as radians in this problem.