Math Day at the Beach 2011

MULTIPLE CHOICE – *Write your name and school and mark your answers on the answer sheet. You have 45 minutes to work on these problems. No calculator is allowed.*

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1. A ladder is leaning against a wall, with its legs on a level floor. Now move the ladder so that it is still leaning against the wall but its top end is twice as high off the floor as before. Then the *slope* of the ladder is now

   (A) Less than the previous slope (B) The same as the previous slope (C) Greater than before but less than twice the previous slope (D) Twice the previous slope (E) Greater than twice the previous slope

2. It takes a steamboat 5 days to go downstream on the Mississippi River from St. Louis to New Orleans, but 7 days for the same boat to go upstream from New Orleans to St. Louis. How many days would it take a raft to drift with the current from St. Louis to New Orleans?

   (A) 12 (B) 17.5 (C) 35 (D) 36 (E) 70

3. A bowl of fruit has apples, bananas, and oranges. There are twice as many apples as oranges, and 20% of the fruit are bananas. What is the smallest possible number of apples?

   (A) 4 (B) 5 (C) 6 (D) 7 (E) 8

4. 200 ml of water containing 1 gram per liter of salt is mixed with 300 ml of water containing 3 grams per liter of salt. What is the salt concentration, in grams per liter, of the resulting solution?

   (A) 2 (B) 2.1 (C) 2.2 (D) 2.4 (E) 4

5. A hemispherical dome of radius r sits on flat ground, and a square-based pyramid fits tightly into to that dome; all of the vertices of the pyramid touch the dome. What is the area of a single triangular face of the pyramid?

   (A) $\frac{r^2}{2}$ (B) $\frac{\sqrt{2}r^2}{2}$ (C) $\frac{\sqrt{3}r^2}{2}$ (D) r^2 (E) $\sqrt{2}r^2$

6. Out of 100 customers leaving Macy's, 70 had purchased lotions, 83 had purchased sweets, 58 had purchased shoes, and 98 had purchased clothing. What is the least possible number of customers that bought all four items?

   (A) 2 (B) 9 (C) 58 (D) 98 (E) Cannot be determined

7. Assume $x, y, z \in \left(0, \frac{\pi}{2}\right)$, and assume that $\cos x = x, \sin(\cos y) = y$, and $\cos(\sin z) = z$. Which one of the following is true?

   (A) $y < z < x$ (B) $z < x < y$ (C) $y < x < z$
   (D) $x < y < z$ (E) At least two of x, y, z are equal

8. The largestinteger n such that $n^{200} < 3^{300}$ is

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

9. When $(2009^{2009} + 2011^{2011})$ is divided by $(2009 + 2011)$, the remainder is

(A) 0 (B) 1 (C) 2006 (D) 2008 (E) 2010

10. Let A be the interval $(1, \infty)$, B the interval $(0, 1)$, and C the interval $(-\infty, 0)$. Suppose that x_1 belongs to the set A. For $n \geq 1$, define $x_{n+1} = \frac{1}{1-x_n}$. Which of the following statements about the 2011th term of the sequence is true?

(A) $x_{2011} \in A$ (B) $x_{2011} \in B$ (C) $x_{2011} \in C$ (D) $x_{2011} = 1$ (E) x_{2011} is undefined.

11. If n is a positive integer, let $r(n)$ denote the number obtained by reversing the order of the digits of n. For example $r(16) = 61$. Let w be the number of 2-digit positive integers for which $n + r(n)$ a square of a positive integer. What is the remainder obtained upon dividing w by 5?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

12. Find the *sum* of all θ in the interval $0 \leq \theta < 2\pi$ such that

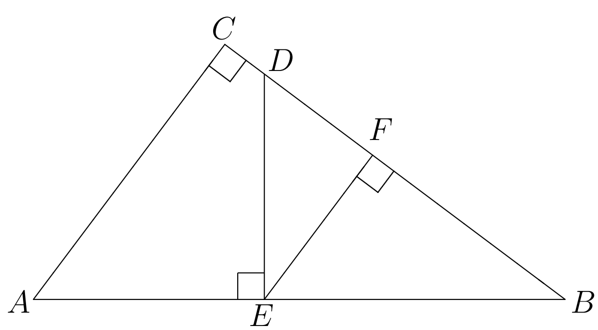
$$2\sin^3\theta - \cos^2\theta - 5\sin\theta + 3 = 0.$$

(A) There are no such θ (B) $\frac{\pi}{2}$ (C) π (D) $\frac{3\pi}{2}$ (E) $\frac{2\pi}{3}$

13. For $10 \leq x \leq 10000$, define the function $f(x) = x^{4-\log_{10} x}$. Let M be the maximum value that $f(x)$ takes on and m be the minimum value that $f(x)$ takes on. Find the ratio $\frac{M}{m}$.

(A) 2 (B) 10 (C) 100 (D) 1000 (E) 10000

14. In the figure below, $\triangle ABC$ is a right triangle with right angle at C, $AC = 12, AB = 20$, and $\overline{DE}$ bisects the area of $\triangle ABC$ with $\overline{DE} \perp \overline{AB}$ and $\overline{EF} \perp \overline{BC}$. Compute length EF.



(A) 6 (B) $\frac{9\sqrt{2}}{2}$ (C) $\frac{18\sqrt{2}}{5}$ (D) $\frac{24}{5}$ (E) $\frac{24\sqrt{2}}{5}$

15. Three people A, B, C play ping-pong. Two of them at a time play a game, while the third sits and watches. At the end of each game, the loser sits down and the winner and the watcher play the next game. We know that A played 8 games, B sat and watched 2 games, and C played 5 games. Which two were the players in game #9?

(A) A and B (B) A and C (C) B and C (D) Insufficient information (E) (Not the answer)

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Individual Free Response – *Write your name and school and mark your answers on your answer sheet. You have 25 minutes to work on these problems. No calculator is allowed.*

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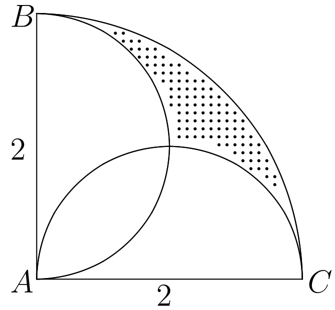
16. In the figure below, $AB = AC = 2$ and all of the arcs are portions of circles – in particular, the semicircles with diameters $\overline{AB}$ and $\overline{AC}$ and a quadrant of the circle centered at A. Compute the area of the shaded region.



17. Out of all possible sequences of length five made up of the symbols $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$, how many have an even number of **A**'s?

18. Bird Birdy is 2 miles away from home. Confused with the directions, she chooses a random direction and files 2 miles in that direction. What is the probability that she is within $2\sqrt{2-\sqrt{3}}$ from home?

19. $P(x) = ax^4+bx^3+cx^2+dx$ for some constants a, b, c, d. If $P(-3) = P(-1) = P(1) = P(2) = 1$, find $P(3)$.

20. Two 9th graders and n 10th graders play a chess tournament. Every student plays every other student once. A student scores one point for winning a match, one half of a point for drawing a match, and zero points for losing a match. The total number of points scored by the two 9th graders was 8. Each 10th grader scored the same number of points as each other. The two 9th grade students each had scores lower than any 10th grader. How many 10th grade students were there?

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TEAM ROUND – *Write your name and school and mark your answers on the answer sheet. No calculator is allowed. You have 30 minutes to work on these problems.*

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1. Chose three digits x, y, z uniformly and independently at random from $\{0, 1, 2, \ldots, 9\}$. What is the probability that $x + y + z \geq 20$?
2. Ant Z starts a random walk on the edges of a cube. She starts at point A and each time chooses one of the three vertices connected to where she is, each being chosen equally likely. Continuing this way, what is the probability that she will be at the starting point A after exactly 8 steps?
3. Suppose $z^3 = 3 - 4i, w^3 = -3 + 4i$, and $z + w \neq 0$. Find $|z + w|^3$.
4. Let $\lfloor x \rfloor$ be the greatest integer that is less than or equal to x. Compute $\displaystyle\sum_{n=1}^{250} \lfloor \log_2 n \rfloor$.
5. A cylinder has radius 2 and height 22. Find the maximum possible number of non-overlapping balls of radius 1 that will fit inside this cylinder.
6. Find all nonzero integers a such that $x^2 - (a - 10)x + a = 0$ has two integer roots.
7. Let S be the set $\{1, 2, 3, \ldots, 12, 13\}$. How many subsets of S have sum greater than 45?
8. A line tangent to the parabola $y = \frac{1}{4}x^2$ that passes through the point $(0, -\frac{1}{4})$ is also tangent to the parabola $y = -(x - a)^2 + 2$, where a is some positive number. Find a.