Math Day at the Beach 2010

Multiple Choice – *Write your name and school and mark your answers on the answer sheet. You have 45 minutes to work on these problems. No calculator is allowed.*

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1. Find the area of the region in the plane described by the inequality $2|x| + 3|y| \leq 1$.

   (A) $\frac{1}{12}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) 3 (E) 12

2. In a given arithmetic sequence, the first term is 2, the last term is 29, and the sum of all the terms is 155. The common difference is

   (A) 3 (B) 2 (C) $\frac{27}{19}$ (D) $\frac{27}{10}$ (E) $\frac{23}{38}$

3. If $\binom{n}{5} = 20\binom{n}{4}$, what does n equal?

   (A) 20 (B) 24 (C) 96 (D) 100 (E) 104

4. A bag contains three red dice, two green dice, and one blue die. The red dice are normal dice with the numbers $1, 2, 3, 4, 5, 6$ on the six sides. The green dice have numbers $1, 1, 2, 2, 3, 3$ and the blue die has numbers $1, 1, 1, 2, 2, 2$. One die is randomly chosen from the bag and rolled. If you know that the number that it shows is 1, what is the probability that it was a red die?

   (A) $\frac{1}{6}$ (B) $\frac{1}{4}$ (C) $\frac{3}{10}$ (D) $\frac{1}{3}$ (E) $\frac{1}{2}$

5. Consider the the infinite triangular array of numbers that starts as follows:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & & 3 & & \\ & 4 & & 5 & & 6 & \\ 7 & & 8 & & 9 & & 10 \end{array}$$

   Find the sum of the numbers on the 21st row.

   (A) 4631 (B) 4641 (C) 4851 (D) 4977 (E) 5082

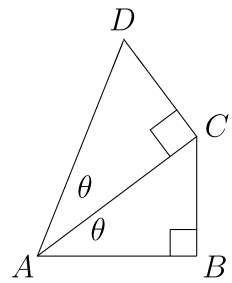
6. How many values of b are there such that the equation $x^2 - bx + 80$ has two different positive even integer solutions?

   (A) 0 (B) 3 (C) 5 (D) 6 (E) 10

7. Mr. Moya gave the same test to both of his algebra classes. The first class has 38 students and the second class 32 students. The first class had an average score of 74.6, while the two classes combined had an average score of 79.4. (All averages in this problem are rounded to the nearest tenth.) What was the average score of the second class?

   (A) 76.8 (B) 83.4 (C) 84.2 (D) 84.7 (E) 85.1

8. Suppose $\angle ABC$ and $\angle ACD$ are right angles, $\angle BAC = \angle CAD = \theta$, which is an acute angle. If $\frac{AD}{AB} = \frac{3}{2}$, compute $\sin\theta$.



(A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\sqrt{\frac{2}{3}}$ (D) $\frac{1}{\sqrt{2}}$ (E) $\frac{\sqrt{5}}{3}$

9. How many prime numbers p are there such that $\left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{p}{3} \right\rfloor + \left\lfloor \frac{p}{6} \right\rfloor$ is also prime? (Here $\lfloor x \rfloor$ stands for the greatest integer function of x, which is to say the largest integer that is less than or equal to x.)

(A) 1 (B) 2 (C) 3 (D) 4 (E) more than 4

10. How many 12-element subsets of $\{1, 2, \ldots, 2010\}$ contain no consecutive integers?

(A) $\binom{1998}{11}$ (B) $\binom{1999}{11}$ (C) $\binom{1999}{12}$ (D) $\binom{2000}{11}$ (E) $\binom{2000}{12}$

11. T_1 is an equilateral triangle inscribed in a circle of radius 1. The midpoints of the sides of T_1 are joined to form triangle T_2, and then the midpoints of the sides of T_2 are joined to form triangle T_3. This process is continued indefinitely, producing an infinite sequence of triangles. What is the sum of the areas of all of these triangles?

(A) $\frac{\sqrt{3}}{3}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{3\sqrt{3}}{2}$ (D) $\sqrt{3}$ (E) infinite

12. The equation $x^{12} - x^9 + x^8 - x^7 + 1 = 0$ has how many distinct real solutions?

(A) 0 (B) 2 (C) 4 (D) 5 (E) 12

13. In a right triangle, the three sides are all integers and form an arithmetic sequence. Which of the following could be one of the lengths?

(A) 61 (B) 71 (C) 81 (D) 91 (E) 101

14. How many three-digit numbers are there that are written with three different digits and have the property that the number written in the the opposite order is larger than the original number? 257 is an example of such a number since $752 > 257$. But 224 and 302 are not examples of such numbers, and 047 is also not an example.

(A) 36 (B) 45 (C) 288 (D) 360 (E) 450

15. Let S be the set $\{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, \ldots\}$ of integers whose only prime factors are $2, 3$, and 5. Find the sum of the reciprocals of the members of this set.

(A) π (B) $\frac{15}{4}$ (C) $\frac{19}{8}$ (D) 30 (E) infinity (*i.e.* the series diverges)

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INDIVIDUAL FREE RESPONSE – *Write your name and school and mark your answers on your answer sheet. You have 25 minutes to work on these problems. No calculator is allowed.*

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16. Find the positive number x such that $\arctan\left(\frac{x^2}{123}\right) + \arctan\left(\frac{1}{x^2+1}\right) = \frac{\pi}{4}$.

17. Suppose that $a_1 = \frac{1}{4}$ and for $n \geq 1$, $a_{n+1} = \dfrac{1}{4(1-a_n)}$. Find a_{2010}.

18. Points A, B, C, D lie in the plane. C lies between A and B in segment $\overline{AB}$, while D does not lie on line $\overleftrightarrow{AB}$. Suppose $AC = 2,\ BC = 5,\ CD = 7$, and $BD = 8$. Compute length AD.

19. Suppose a, b, c are positive integers such that the numbers $a+b$, $a+c$, and $b+c$ are consecutive perfect squares. Find the smallest possible value for $a^2 + b^2 + c^2$.

20. Choose a point (x, y, z) uniformly at random inside the right circular cone $\{(x,y,z) \mid \sqrt{x^2+y^2} < z < 1\}$. Let W be the nonnegative remainder obtained when dividing $\left\lfloor \log_2\left(\frac{z^2}{x^2+y^2}\right)\right\rfloor$ by 3. Compute the expected value of W.

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Team Round – *Write your name and school and mark your answers on the answer sheet. No calculator is allowed. You have 30 minutes to work on these problems.*

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1. If x and y are the integers written in base 3 as 21012 and 121, respectively, then find the base 3 representation of the product xy.

2. Find all solutions (real or complex) of the equation $2x^4 - 7x^3 + 7x^2 - 7x + 2 = 0$.

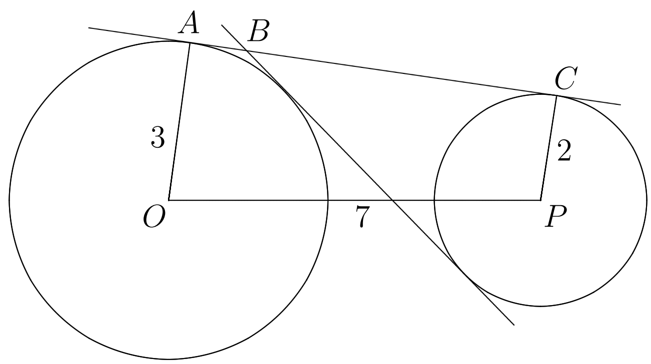
3. A tetrahedron (triangular pyramid, not regular) in the first octant ($x \geq 0, y \geq 0, z \geq 0$) is bounded by the three coordinate planes and by a fourth plane which cuts across the octant. If the point $(1, 1, 4)$ is on that fourth plane, what is the smallest possible volume of the tetrahedron?

4. How many subsets of $\{1, 2, 3, \ldots, 12\}$ are there such that the sum of the largest element and smallest element of that subset is 13?

5. Suppose $P(x)$ is a fourth degree polynomial such that $P(k) = \frac{k}{k+1}$ for $k = 0, 1, 2, 3, 4$. Compute $P(5)$.

6. Let $z_1, z_2, \ldots, z_{10}$ be complex numbers that form a regular decagon (10-sided polygon) in the complex plane, with that decagon inscribed in a circle of radius $\sqrt[5]{7}$ centered at 2. At least one of the z_k is real. Compute the product $z_1 z_2 \cdots z_{10}$.

7. The circle centered at O has radius 3, the circle centered at P has radius 2, and the distance between the two centers is 7. Two lines tangent to both circles intersect at B as shown. Find the product $AB \cdot BC$.



8. Xavier plays the following game. He rolls a fair six-sided die one, two, or three times. He wins \$999 times the number he rolls on his last throw. He may choose to stop after either the first or second throw, but if he continues to the third throw, he must accept the result of the third throw. If Xavier plays with his best possible strategy, what is the expected value of the amount he wins?