An Elliptic Proof of Heron's Area Formula

Abstract

The standard proof of Heron's formula makes use of Law of Cosines. In this article we give a proof based on elementary properties of the ellipse. This approach may be suitable as an exploration/enrichment project in a Precalculus or Geometry class.

Heron's (side-side-side) area formula expresses the area of $\triangle ABC$ in terms of the lengths of its sides $a = BC$, $b = AC$, $c = AB$:

$$\mathcal{A}(\triangle ABC) = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where} \quad s = \frac{1}{2}(a+b+c).$$

Let $f = \frac{c}{2}$. Then we may assume that A, B, C have coordinates $A(f,0), B(-f,0), C(g,h)$ with $g, h > 0$ (Figure 1). By Pythagorean Theorem, $a^2 - b^2 = ((f+g)^2 + h^2) - (|f-g|^2 + h^2) = 4fg$, or

$$(\frac{a-b}{2})(\frac{a+b}{2}) = fg. \tag{1}$$

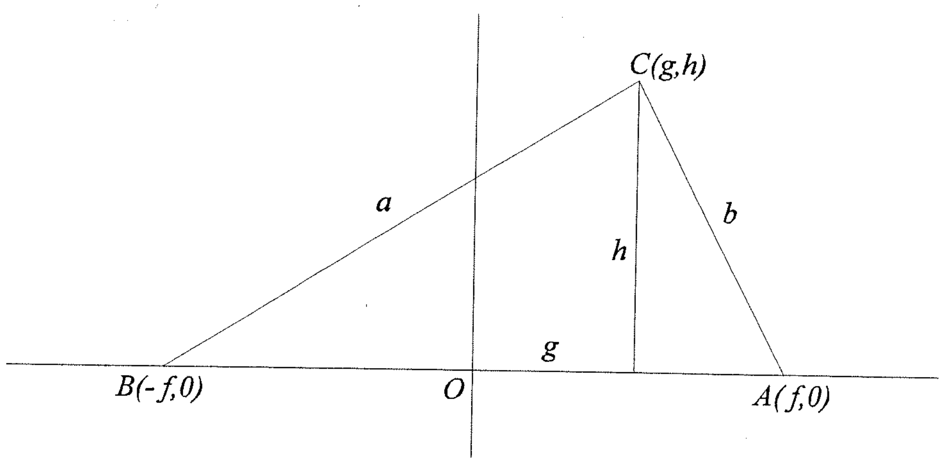

Figure 1.

The Elliptic shell $\mathcal{E}$ of $\triangle ABC$. Consider the ellipse $\mathcal{E}$ which contains the vertex C and has foci at the vertices A, B (see Figure 2). Let ℓ (resp. m) be the length of the semimajor axis OL (resp. semiminor axis OM) of $\mathcal{E}$. Since the ellipse $\mathcal{E}$ is the locus of points with constant distance sum 2ℓ from its foci A, B ([1], [4]) we have $CA + CB = b + a = 2\ell$ and $MA + MB = 2\ell$. Thus

$$\ell = \frac{a+b}{2} \quad \text{and} \quad MA = MB = \ell \tag{2}$$

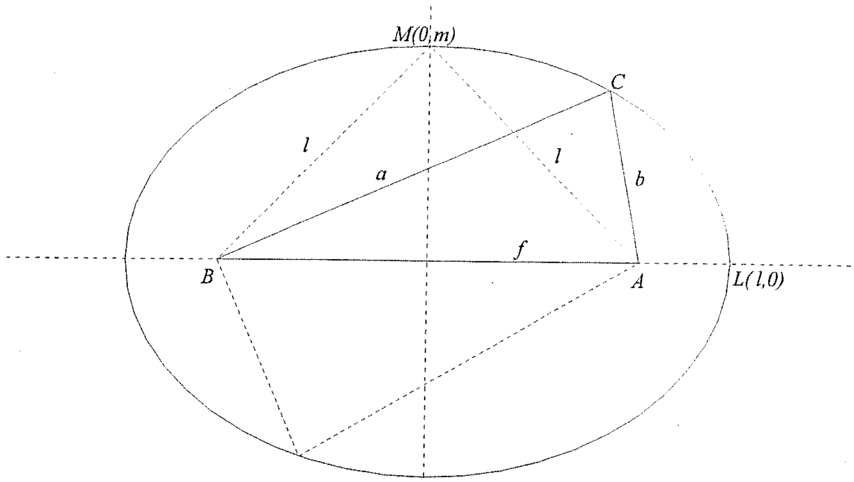

Figure 2.

Lemma 1 $\sqrt{s(s-c)} = m$, *the length of the semiminor axis of the elliptic shell* $\mathcal{E}$ *of* $\triangle ABC$.

Proof. $s(s-c) = (\frac{a+b+c}{2})(\frac{a+b-c}{2}) = (\ell+f)(\ell-f) = \ell^2 - f^2 = MA^2 - OA^2 = OM^2 = m^2$.

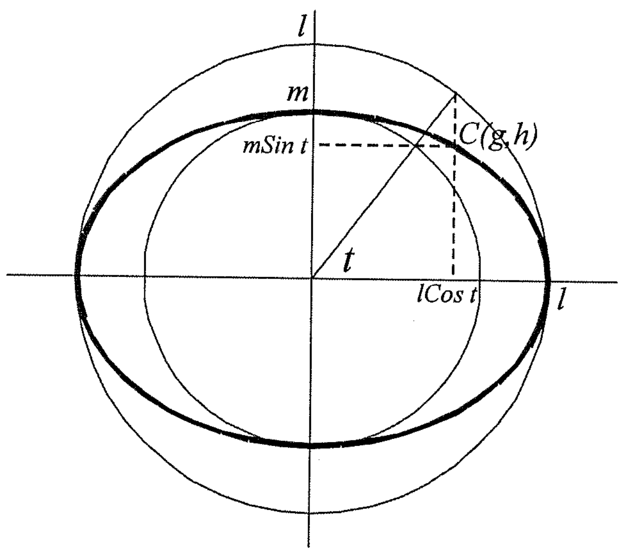

Figure 3.

The eccentric angle of C on the ellipse $\mathcal{E}$. Since $\mathcal{E}$ has equation $x^2/\ell^2 + y^2/m^2 = 1$ the point C has coordinates: $(g,h) = (\ell\cos t,\, m\sin t)$, for some $0 < t < \pi/2$ (Figure 3). By (1) and (2),

$$\frac{a-b}{2} = \frac{fg}{\ell} = \frac{f\ell\cos t}{\ell} = f\cos t. \tag{3}$$

Moreover, since $\mathcal{A}(\triangle ABC) = ch/2 = fh = fm\sin t$, Heron's formula follows immediately from Lemma 1 and the next Lemma.

Lemma 2 $\sqrt{(s-a)(s-b)} = \frac{c}{2}\sin t = f\sin t$, *where* t *is the eccentric angle of* C *on* $\mathcal{E}$.

Proof. Since $(s-a)+(s-b) = 2s-a-b = c$ there is a (unique) point T on $\overline{AB}$ with $TA = s-a$ and $TB = s-b$. (In fact, T is a point of tangency of the incircle, see Figure 4.)

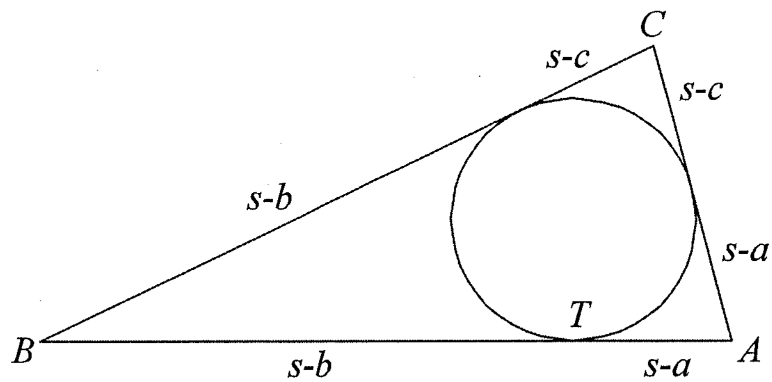

Figure 4.

Let D be the point on the semicircle with diameter $AB = c = 2f$ such that $\overline{DT} \perp \overline{AB}$ (Figure 5). Then DT is the geometric mean of AT and BT: $DT = \sqrt{(AT)(BT)} = \sqrt{(s-a)(s-b)}$. (Since $\triangle ABD$ is a right triangle with height $\overline{DT}$, $\triangle DTB$ is similar to $\triangle ATD$ and $DT/AT = BT/DT$.)

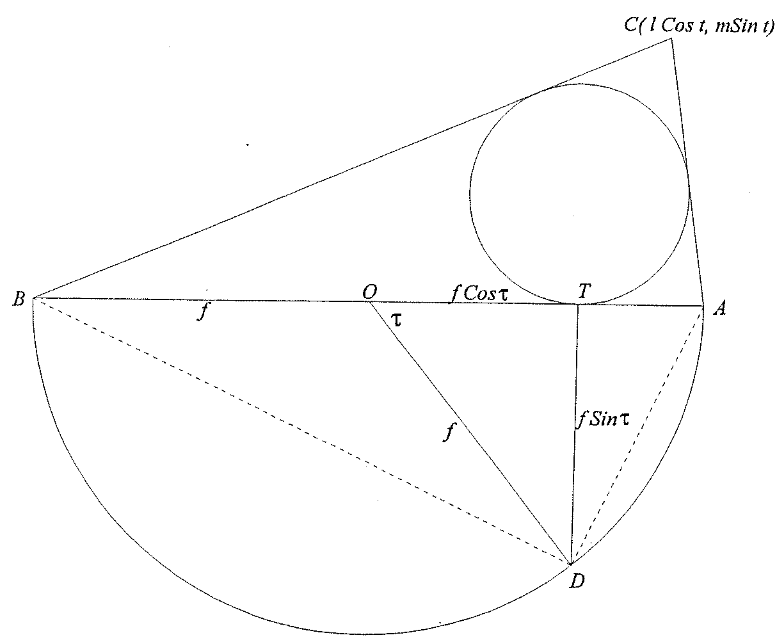

Figure 5.

On the other hand, letting $\tau = \angle DOT$, $DT = OD \sin\tau = f \sin\tau$. Hence $\sqrt{(s-a)(s-b)} = f \sin\tau$ and, to complete the proof of Lemma 2 (and Heron's formula), we need only show "$\tau = t$".

Proof of "$\tau = t$". Clearly, $f \cos\tau = OD \cos\tau = OT$. On the other hand, in view of Figure 4 and equation (3),

$$OT = OA - TA = f - (s-a) = \frac{c}{2} - \frac{a+b+c}{2} + \frac{2a}{2} = \frac{a-b}{2} \underset{(3)}{=} f \cos t$$

Thus $f \cos\tau = f \cos t$ or $\tau = t$ (as both t and τ are between 0 and $\pi/2$).

Remark For a "proof without words" of Heron's formula and other proofs with geometric flavor, see [3] and the references given there. The proof in [5] makes use of more sophiscated properties of ellipse and hyperbola. In [2] it is shown that $(s-b)(s-c)$ equals rr_a, where r is the radius of the incircle of $\triangle ABC$ and r_a is the radius of the tritangent circle outside $\triangle ABC$ and opposite vertex A.

References

[1] C. G. Gibson, *Elementary Euclidean Geometry*, Cambridge University Press, 2003.

[2] Sidney H. Kung, Another elementary proof of Heron's formula, *Mathematics Magazine* **65**(1992), 337-338.

[3] Roger B. Nelson, Heron's formula via proofs without words, *College Mathematics Journal* **32**(2001), 290-292.

[4] J. Stewart, R. Redlin and S. Watson: *Precalculus*, 4-th ed. Thomson, 2002.

[5] Victor Thébault, The Area of a triangle as a function of the sides, *The American Mathematical Monthly* **52**(1945), 508-509.