Stiff Well-Posedness and Asymptotic Convergence for a Class of Linear Relaxation Systems in a Quarter Plane

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In this paper we study the asymptotic equivalence of a general linear system of 1-dimensional conservation laws and the corresponding relaxation model proposed by S. Jin and Z. Xin (1995, Comm. Pure Appl. Math. 48, 235–277) in the limit of small relaxation rate. The main interest is this asymptotic equivalence in the presence of physical boundaries. We identify and rigorously justify a necessary and sufficient condition (which we call the Stiff Kreiss Condition, or SKC in short) on the boundary condition to guarantee the uniform well-posedness of the initial boundary value problem for the relaxation system independent of the rate of relaxation. The SKC is derived and simplified by using a normal mode analysis and a conformal mapping theorem. The asymptotic convergence and boundary layer behavior are studied by the Laplace transform and a matched asymptotic analysis. An optimal rate of convergence is obtained.

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1. INTRODUCTION

We study the linear version of the following hyperbolic relaxation model proposed by Jin and Xin in [3]

\[
\begin{align*}
\partial_t u' + \partial_x v' &= 0, \\
\partial_t v' + a \partial_x u' &= -\frac{1}{\varepsilon} (v' - f(u')),
\end{align*}
\]

where \( u' \), \( v' \in \mathbb{R}^n \), \( a > 0 \), \( \varepsilon > 0 \). (1)

Our main purpose in this paper is to understand the boundary layer behavior of the solution \((u', v')\) and its asymptotic convergence to the solution of the corresponding equilibrium system

\[
\begin{align*}
\partial_t u + \partial_x f(u) &= 0, \\
v &= f(u),
\end{align*}
\]

as the rate of relaxation \( \varepsilon \) goes to zero, and most of all, the precise stability requirements implied on the boundary conditions for the corresponding initial-boundary value problem of (1).

We assume \( f(u) \) is linear, i.e.,

\[ f(u) = f'u \]

for some constant \( n \times n \) real matrix \( f' \). Furthermore we assume \( f' \) has \( n \) real eigenvalues and a complete set of eigenvectors. Therefore

\[ Lf' R = A = \text{diag}\{\lambda_1, \ldots, \lambda_n\}, \quad LR = I_n. \] (4)

We are mostly interested in the initial-boundary value problem (IBVP) in the quarter plane \( x > 0, \ t > 0 \). Therefore, we complete (1) with the necessary initial data

\[ u'(x, 0) = u_0(x), \quad v'(x, 0) = v_0(x) \]

and linear boundary condition

\[ B_u u(0, t) + B_v v(0, t) = b(t), \]

where \( B_u \) and \( B_v \) are constant \( n \times n \) real matrices. For simplicity, we also assume the initial data \( U_0(x) = ((u_0(x), v_0(x)) \) and the boundary data \( b(t) \) are sufficiently compatible at the space-time corner \( x = 0, \ t = 0 \), say,

\[ U_0(0) = U'_0(0) = 0, \quad b(0) = b'(0) = 0. \] (7)
It is easy to see that system (1) is diagonalizable with Riemann invariants $\sqrt{a} u^\pm$ and eigenvalues $\pm \sqrt{a}$. Therefore the boundary condition (6) has to satisfy the Uniform Kreiss Condition (UKC)

$$\det(B_u + \sqrt{a} B_v) \neq 0$$

so that on the boundary $x = 0$ the incoming flow $\sqrt{a} u^\pm + v^\pm$ can be expressed in terms of the outgoing flow $\sqrt{a} u^\pm - v^\pm$ and the data $b(t)$, and therefore the IBVP (1), (5)-(6) is well-posed for each fixed $\varepsilon$.

Due to the stiff source term, the relaxation approximation (1) is a highly singular process and its dissipative mechanism is rather weak. In order for asymptotic convergence to hold, i.e., solution of (1) tending to that of (2) in the limit $\varepsilon \downarrow 0$, certain stability conditions have to be satisfied. The most well-known is the following sub-characteristic condition [8, 14]

$$a - f'(u)^2 > 0 \quad \text{for all } u \text{ under consideration.}$$

The Cauchy problem is relatively well understood and various asymptotic convergence results have been obtained under the sub-characteristic assumptions, see, for example, [1, 3, 8-10, 12, 15, 16]. However, the corresponding initial-boundary value problem is much more difficult and much less is known [7, 13, 17].

In this paper, we concentrate on the one-dimensional linear model (1) with the general linear boundary condition (6). First of all, we shall examine the issue of stiff well-posedness [9] (see the definitions below) of the IBVP (1), (5)-(6). This is a uniform version of the usual well-posedness [6, 11] for all $0 < \varepsilon \leq \varepsilon_0$ and is closely related to the asymptotic convergence part of our problem. For Cauchy problem, the stability condition required is a slightly weaker version of the sub-characteristic condition (9). For the IBVP, however, the sub-characteristic condition and UKC are not enough and a more stringent restriction has to be imposed on the structure of the boundary condition (6). The bulk of this paper is devoted to the study of this extra condition (Stiff Kreiss Condition, SKC).

The necessity of the SKC can be seen from a simple normal mode analysis. The main difficulty is with the sufficiency proof, particularly, when the initial data $U_0(x)$ is nonzero. To isolate the effects of the possible boundary layer and avoid the complicated interactions of boundary and initial layers, we will consider the simpler homogeneous initial data case first. The IBVP (1), (5)-(6) will be solved explicitly by the method of Laplace transform. On the other hand, the boundary layer structure and the formal asymptotic limit can be identified by a matched asymptotic analysis. Based on a detailed study of the explicit solution representation, all the required estimates, including the rigorous justification of the boundary layer structure and optimal convergence results, can be obtained rather directly.
The nonzero initial data case, on the other hand, is much more difficult. This is due to the complicated interactions among the initial data, the boundary condition and the stiff relaxation term. The initial data alone, besides being responsible for the initial layer, can excite both types of boundary layers (characteristic and non-characteristic), and may also produce a nontrivial equilibrium limit. The proof requires a combination of several techniques. See Section 5 for details.

One of the main motivations of the present study is the numerical treatment of boundary for the relaxation schemes for systems of nonlinear hyperbolic conservation laws. One of the major issues in the theory of the relaxation approximations to equilibrium system of conservation laws is the almost surely appearance of stiff boundary layers in the presence of physical or numerical boundaries due to the additional characteristic speeds introduced in the relaxation systems. Thus, how to formulate boundary conditions for the relaxation systems to guarantee the uniform stability, and to minimize and/or localize the artificial boundary layers are crucial to the success of the relaxation schemes.

**Definition 1.1.** The Cauchy problem (1), (5) is said to be stiffly well-posed if the solution $U^\varepsilon=(u^\varepsilon, v^\varepsilon)$ satisfies

$$\int_{-\infty}^{\infty} |U^\varepsilon(x, t)|^2 \, dx \leq K_T \int_{-\infty}^{\infty} |U_0(x)|^2 \, dx \quad \forall t \in [0, T] \tag{10}$$

for some positive constant $K_T$ independent of $\varepsilon$ and for all $U_0=(u_0, v_0) \in L^2(\mathbb{R})$, and $\varepsilon \in (0, \varepsilon_0]$.

**Definition 1.2.** The IBVP problem (1), (5)–(6) is stiffly well-posed if

$$\int_0^{T} \int_{-\infty}^{\infty} |U^\varepsilon(x, t)|^2 \, dx \, dt + \int_0^{T} |U^\varepsilon(0, t)|^2 \, dt \leq K_T \int_0^{T} |b(t)|^2 \, dt + K_T \int_{-\infty}^{\infty} |U_0(x)|^2 \, dx \tag{11}$$

for some positive constant $K_T$ independent of $\varepsilon$ and for all $U_0 \in L^2(\mathbb{R}^+)$, $b \in L^2(\mathbb{R}^+)$ and $\varepsilon \in (0, \varepsilon_0]$.

We now state our main theorems as follows:

**Theorem 1.1 (IBVP: $n=1$).** Let $f(u)=\lambda u$, $\lambda \in \mathbb{R}$ and assume $\lambda > 0$ satisfies the sub-characteristic condition

$$\lambda \geq \lambda^2. \tag{12}$$
1. The IBVP (1), (5)-(6) is stiffly well-posed if and only if the boundary condition (6) satisfies the following SKC:

$$B_v = 0 \quad \text{or} \quad \frac{B_u}{B_v} \geq \left[-\sqrt{a}, -\frac{\dot{\lambda} + |\lambda|}{2}\right].$$  \hfill (13)

2. Assume (12)-(13) and $b(t) \in L^2(\mathbb{R}^+)$, $U_0(x) \in H^1(\mathbb{R}^+)$, $U_0(0) = 0$.
Then there exists a unique solution $U = (u, v)$ of (2) such that

$$\int_{0}^{\infty} \int_{0}^{\infty} |U^e - U|^2 (x, t) \, e^{-2\alpha t} \, dx \, dt \to 0 \quad \text{as} \quad \epsilon \downarrow 0$$ \hfill (14)

for any $\alpha > 0$.

3. If we further assume $b(t) \in H^2(\mathbb{R}^+)$, $U_0(x) \in H^2(\mathbb{R}^+)$ with the compatibility condition $b(0) = b'(0) = 0$, $U(0) = U'_0(0) = 0$, then

$$\int_{0}^{\infty} \int_{0}^{\infty} |U^e - U|^2 (x, t) \, e^{-2\alpha t} \, dx \, dt$$

$$\leq O(1) \, \epsilon \, \|v_0 - f(u_0)\|_{L^2}^2 + O(1) \, \epsilon^2 \|U_0\|_{H^2}^2$$

$$+ \begin{cases} O(1) \, \epsilon^2 \|b\|_{L^2}^2 & \text{if} \quad \dot{\lambda} > 0, \\ O(1) \, \epsilon \|b\|_{L^2}^2 & \text{if} \quad \dot{\lambda} < 0, \\ O(1) \, \epsilon^{1/2} \|b\|_{L^2}^2 & \text{if} \quad \dot{\lambda} = 0. \end{cases}$$ \hfill (15)

4. There exist an initial layer

$$U^{il} = U^{il}(x, t/e)$$ \hfill (16)

and a boundary layer

$$U^{bl} = \begin{cases} 0 & \text{if} \quad \dot{\lambda} > 0, \\ U^{bl}(x/e, t) & \text{if} \quad \dot{\lambda} < 0, \\ U^{bl}(x/\sqrt{\epsilon}, t) & \text{if} \quad \dot{\lambda} = 0. \end{cases}$$ \hfill (17)

with $u^{il} = 0$ and $v^{bl} = 0$ such that

$$\int_{0}^{\infty} \int_{0}^{\infty} |U^e - U^{il} - U^{bl}|^2 (x, t) \, e^{-2\alpha t} \, dx \, dt$$

$$\leq \begin{cases} O(1) \, \epsilon^2 \|b\|_{L^2}^2 + O(1) \, \epsilon^2 \|U_0\|_{H^2}^2 & \text{if} \quad \dot{\lambda} \neq 0, \\ O(1) \, \epsilon^{3/2} \|b\|_{L^2}^2 + O(1) \, \epsilon^2 \|U_0\|_{H^2}^2 & \text{if} \quad \dot{\lambda} = 0. \end{cases}$$ \hfill (18)
In the above (and henceforth), $O(1)$ represent some absolute constants which might depend on $\alpha$, but otherwise independent of $c$, $t$, $b(t)$ or $U_0(x)$.

**Theorem 1.2** (IBVP; $n > 1$). Assume $a > 0$ satisfies the following sub-characteristic condition

$$a \geq \max_{1 \leq i \leq n} \lambda_i^2.$$  \hspace{1cm} (19)

1. The IBVP (1), (5)--(6) is stiffly well-posed if and only if the boundary condition (6) satisfies the following Stiff Kreiss Condition

$$|\det(B_uR + B_vRG(\xi))| \geq C$$  \hspace{1cm} (20)

for some $C > 0$ and for all $\xi$ with $\Re \xi \geq 0$. Here

$$G(\xi) = \text{diag}\{g_1(\xi), g_2(\xi), \ldots, g_n(\xi)\},$$

$$g_j(\xi) = \dot{\lambda}_j + \sqrt{\dot{\lambda}_j^2 + 4a(1 + \xi)}.$$  \hspace{1cm} (21)

2. Assume (19)--(20) and $b(t) \in L^2(\mathbb{R}^+)$, $U_0(x) \in H^1(\mathbb{R}^+)$ with $U_0(0) = 0$. Then there exists a unique solution $U = (u, v)$ of (2) such that

$$\int_0^\infty \int_0^\infty |U^e - U|^2(x, t) e^{-2at} \, dx \, dt \to 0 \quad \text{as} \quad \varepsilon \downarrow 0.$$  \hspace{1cm} (22)

3. If we further assume $b(t) \in H^2(\mathbb{R}^+)$, $U_0(x) \in H^2(\mathbb{R}^+)$ with the compatibility condition $b(0) = b'(0) = 0$, $U(0) = U_0(0) = 0$, then

$$\int_0^\infty \int_0^\infty |U^e - U|^2(x, t) e^{-2at} \, dx \, dt$$

$$\leq O(1) \varepsilon^{1/2} \|b\|_{H^2}^2 + O(1) \varepsilon^2 \|U_0\|_{H^2}^2 + O(1) \varepsilon \|v_0 - f(u_0)\|_{L^2}^2.$$  \hspace{1cm} (23)

4. There exist an initial layer $U^{il} = U^{il}(x, t/\varepsilon)$ and a boundary layer $U^{bl}$ (of mixed type) with $u^{il} = 0$ and $v^{il} = 0$ such that

$$\int_0^\infty \int_0^\infty |U^e - U^{il} - U^{bl}|^2(x, t) e^{-2at} \, dx \, dt$$

$$\leq O(1) \varepsilon \|b\|_{H^2}^2 + O(1) \varepsilon^2 \|U_0\|_{H^2}^2.$$  \hspace{1cm} (24)

For completeness, we will also prove the following theorem for the simpler Cauchy problem.
Theorem 1.3 (Cauchy Problem). 1. The Cauchy problem (1), (5) is stiffly well-posed if and only if $a > 0$ satisfies the following weak sub-characteristic condition
\begin{equation}
    a \geq \max_{1 \leq i < n} \lambda_i^2. \tag{25}
\end{equation}

2. Let (25) be satisfied and $U_0 = (u_0, v_0) \in L^2(\mathbb{R})$. Then there exists a unique solution $U = (u, v)$ of the equilibrium system (2) such that
\begin{equation}
    \int_{-\infty}^{\infty} |U^e - U|^2 (x, t) \, dx \to 0 \quad \text{as} \quad \varepsilon \downarrow 0 \tag{26}
\end{equation}
for any $t > 0$.

3. Assume further $U_0 \in H^2(\mathbb{R})$. Then
\begin{equation}
    \int_{-\infty}^{\infty} |U^e - U|^2 (x, t) \, dx \leq O(1) \varepsilon^2 (1 + t^2) \| U_0 \|_{H^2}^2 + O(1) \varepsilon e^{-t/\varepsilon} \| v_0 - f(u_0) \|_{L^2}^2. \tag{27}
\end{equation}

4. There exists an initial layer $U^{it} = U^{it}(x, t/\varepsilon)$ with $u^{it} = 0$ such that
\begin{equation}
    \int_{-\infty}^{\infty} |U^e - U^{it}|^2 (x, t) \, dx \leq O(1) \varepsilon^2 (1 + t^2) \| U_0 \|_{H^2}^2. \tag{28}
\end{equation}
In particular, this implies
\begin{equation}
    \int_{-\infty}^{\infty} |u^e - u|^2 (x, t) \, dx \leq O(1) \varepsilon^2 (1 + t^2) \| U_0 \|_{H^2}^2. \tag{29}
\end{equation}

Remarks. 1. Under the strict sub-characteristic condition and the same SKC as in Theorem 1.1 and 1.2, one can show that the IBVP (1), (5)-(6) is also stiffly well-posed in the following sense:
\begin{equation}
    \sup_{0 \leq t < T} \int_{0}^{\infty} |U^i(x, t)|^2 \, dx + \int_{0}^{T} |U^i(0, t)|^2 \, dt \leq K_T \int_{0}^{T} |b(t)|^2 \, dt + K_T \int_{0}^{\infty} |U_0(x)|^2 \, dx. \tag{30}
\end{equation}
This can be proved by combining the energy method (see Section 5.3) with the boundary estimate in (11).

2. By a change of variable
\begin{equation}
    u^e \to Ru^e, \quad v^e \to Rv^e, \tag{31}
\end{equation}
(1) can be decomposed into \( n \times 2 \times 2 \) relaxation systems of type \((1)\). Therefore the Cauchy problem is completely decoupled and it suffices to prove Theorem 1.3 for \( n = 1 \). For IBVP, the \( n \times 2 \times 2 \) relaxation systems are coupled through boundary conditions. Without loss of generality, we assume \( f' = A \) and \( R = I_n \).

3. Lorenz and Schroll [9] studied the Cauchy problem for a general multi-dimensional linear constant-coefficient relaxation system. They have obtained necessary and sufficient conditions for stiff well-posedness. Asymptotic convergence was also proved in [9]. However their stability conditions are in a very abstract form and therefore hard to verify. The first result in Theorem 1.3 may be viewed as a reduction of their stability conditions for the Cauchy problem of \((1)\).

4. The stability condition for Cauchy problem in \((25)\) is weaker than the usual sub-characteristic condition \((9)\).

5. The term \( e^{-\lambda t} \|x_0 - f(u_0)\|_{L^2} \) in \((27)\) (and also the term \( O(1) \|x_0 - f(u_0)\|_{L^2} \) in \((15)\) and \((23)\)) is due to an initial layer in the \( v \) components

\[
e^{-\lambda t}(x, t) = e^{-\lambda t}(v_0(x) - f(u_0(x)))
\]

which decays exponentially fast in time. The initial layer does not occur to the \( u \) variables, see \((29)\). For local equilibrium initial data, i.e.,

\[
v_0(x) = f(u_0(x)),
\]

no initial layer occurs. The rates of convergence in \((27)-(29)\) are optimal.

6. Yong [17] considered the (non-characteristic) initial-boundary value problem for a general multi-dimensional linear constant coefficient relaxation system and derived the Generalized Kreiss Condition (GKC) in the same spirit of deriving Uniform Kreiss Condition for multi-dimensional linear hyperbolic IBVP. But his GKC is extremely complicated and no sufficiency or asymptotic convergence result was proved. For our model in the case \( n = 1 \), by using conformal mapping theorem, we are able to simplify the GKC to an explicit form, see \((13)\). More importantly, we are able to prove the sufficiency of SKC for the IBVP to be stiffly well-posed even in the case of characteristic boundary conditions. Various asymptotic convergence results with optimal convergence rates are also obtained.

7. The different (optimal) convergence rates in \((15)\) are due to the different types of boundary layer behaviors. No boundary layer develops when \( \lambda > 0 \). For \( \lambda < 0 \), the boundary layer lives on a scale of order \( \varepsilon \) near \( x = 0 \) and decays exponentially fast. In the case \( \lambda = 0 \), the boundary \( x = 0 \) becomes uniformly characteristic for \((2)\); the corresponding boundary layer
is of diffusion type and lives on a larger scale of order $\varepsilon^{1/2}$. The estimates in (18) establish the validity of such boundary layers.

8. The UKC (8) can be recovered from the SKC (20) by taking the limit $\xi \to \infty$.

9. The formal matched asymptotic expansions may be performed under a much weaker assumption than (1.4), that is, assuming (1.4) for $\xi = 0$ only.

10. The simplification of SKC in the general system case ($n > 1$) is still possible when $B_x$ and $B_u$ are both tridiagonal or can be made so by left multiplying a common nonsingular matrix. However, when the boundary conditions are strongly coupled, finding the range of the complex analytic function $\det(B_x + B_u G(\xi)) (\Re \xi \geq 0)$ analytically would be much more difficult than in the simplest case $n = 1$ (see Section 3.2). Instead, one may choose to plot the boundary curve (in the complex plane) $\{ \det(B_x + B_u G(\xi)) : \xi \in i\mathbb{R} \}$ numerically and check if the origin lies in the exterior of the curve. On the other hand, due to the coupling of boundary conditions, interesting new phenomena may also occur in the system case. For example, the reflection of outgoing waves may produce an extra equilibrium limit in the incoming waves and both types of boundary layers will generally be present simultaneously; in particular, the existence of characteristic boundary layers brings down the overall convergence rates in (23) and (24).

2. CAUCHY PROBLEM

We start with the easy case, the Cauchy problem (1), (5), and prove Theorem 1.3 in this section. We shall only prove the case $n = 1$ with $f(u) = \lambda u, \lambda \in \mathbb{R}$. The results easily extend to the more general case of $n > 1$, see Remark 1.

Denote

$$U^e = \begin{pmatrix} u^e \\ v^e \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ \lambda & -1 \end{pmatrix},$$

then we can rewrite (1) and (5) more concisely as

$$\partial_t U^e + A \partial_x U^e = \frac{1}{\varepsilon} SU^e, \quad (35)$$

$$U^e(x, 0) = U_0(x). \quad (36)$$
2.1. Normal Mode Analysis and Sub-characteristic Condition

In this subsection, we apply the elementary normal mode (Fourier) analysis to (35) and derive a necessary condition for stiff well-posedness. This necessary condition turns out to be exactly the weak sub-characteristic condition (25) and will later be shown to be also sufficient.

Consider the plane wave ansatz

\[ U(x, t) = e^{(\omega t + i k x) / \epsilon} U, \quad (37) \]

where \( k \in \mathbb{R} \) and \( U \in \mathbb{R}^2 \) is some constant vector. Plug (37) into (35), we obtain

\[ (S - i k A) U = \omega U, \quad (38) \]

which means \( \omega = \omega(k) \) is an eigenvalue of the matrix \( S - i k A \) and \( U \) is a corresponding eigenvector. Therefore

\[ \omega^2 + \omega + (a k^2 + i k \lambda) = 0, \quad (39) \]

and

\[ \omega = \omega_\pm(k) = -\frac{1 \pm \sqrt{1 - 4 a k^2 - 4 i k \lambda}}{2}. \quad (40) \]

We observe that, if for some \( k \in \mathbb{R} \), \( \text{Re} \, \omega = \text{Re} \, \omega_\pm(k) > 0 \), then the Fourier mode (37) grows exponentially in time and the uniform \( L^2 \) estimate (10) is violated as \( \epsilon \downarrow 0 \). The difficulty of \( U'(\cdot, t) \notin L^2(\mathbb{R}) \) can be overcome by a suitable cutoff in the \( x \) variable, see [2]. Therefore, for the sake of stiff well-posedness, it is necessary that

\[ \text{Re} \, \omega_\pm(k) < 0, \quad (41) \]

i.e.,

\[ \text{Re} \sqrt{1 - 4 a k^2 - 4 i k \lambda} = \frac{1 - 4 a k^2 + \sqrt{(1 - 4 a k^2)^2 + (4 k \lambda)^2}}{2} \leq 1 \quad (42) \]

for all \( k \in \mathbb{R} \).

A straightforward simplification of (42) yields

\[ a - \lambda^2 \geq 0. \quad (43) \]

This proves the necessary part for stiff well-posedness. We remark that (43) is the non-strict version of the usual sub-characteristic condition (9) in the case \( n = 1 \). The only difference is with the critical case \( a = \lambda^2 \) for which
\[ \omega_+(k) = -ik \lambda \] and \( \text{Re} \, \omega_+(k) \equiv 0 \). With (9), we have \( \text{Re} \, \omega_+(k) < 0 \) except \( \omega_+(0) = 0 \).

2.2. Sufficiency of Sub-characteristic Condition

We now show that the weak sub-characteristic condition (43) is also sufficient for the Cauchy problem (35)–(36) to be stiffly well-posed. For this purpose, we solve (35)–(36) explicitly by Fourier transform and estimate \( U^* \) using Parseval’s equality.

Denote \( \hat{U}^*(k, t) \) the Fourier transform (in \( x \)) of \( U^*(x, t) \):

\[
\hat{U}^*(k, t) = \mathcal{F} U^* = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} U^*(x, t) \, dx;
\]

then for each \( k \in \mathbb{R} \), \( \hat{U}^* \) satisfies the following linear ODE:

\[
\partial_t \hat{U}^* = \left( \frac{1}{\xi} S - ikA \right) \hat{U}^*;
\]

hence

\[
\hat{U}^*(k, t) = e^{(S - ikA) \cdot \xi} \hat{U}_0(k).
\]

Note that for \( \lambda \neq 0 \), we have \( \omega_+(k) \neq \omega_-(k) \) for all \( k \in \mathbb{R} \). Therefore

\[
S - ikA = P(k) \begin{pmatrix} \omega_+(k) \\ \omega_-(k) \end{pmatrix} P^{-1}(k),
\]

where

\[
P(k) = \begin{pmatrix} 1 & \frac{k}{\text{i} \omega_-(k)} \\ \frac{\text{i} \omega_+(k)}{k} & 1 \end{pmatrix},
\]

and

\[
P^{-1}(k) = \frac{1}{2} \left( 1 + \frac{1}{\omega_+(k) - \omega_-(k)} \begin{pmatrix} 1 & \frac{ik}{\omega_-(k)} \\ \omega_+(k) & 1 \end{pmatrix} \right).
\]
For $\lambda \neq 0$, both $P(k)$ and $P^{-1}(k)$ are uniformly bounded for all $k \in \mathbb{R}$. On the other hand, the weak sub-characteristic condition (43) implies $\Re \omega_\pm(k) \leq 0$, and hence $|e^{\omega_\pm(k)t}| \leq 1$. Therefore the matrix
\[ e^{(S - i\lambda A)t} = P(k) \begin{pmatrix} e^{\omega_+(k)t} & e^{\omega_-(k)t} \\ e^{\omega_-(k)t} & e^{\omega_+(k)t} \end{pmatrix} P^{-1}(k) \] (50)
is uniformly bounded for all $k \in \mathbb{R}$ and $t \geq 0$.

The same conclusion holds in the case $\lambda = 0$. Note that in this case, we have $\omega_+ = \omega_- = -\frac{1}{2}$ when $k = \pm 1/\sqrt{\alpha}$ and therefore the matrix $P(k)$ becomes singular. However the above argument remains valid for $k$ away from $\pm 1/\sqrt{\alpha}$. For $k$ near $\pm 1/\sqrt{\alpha}$, we can rewrite (50) as
\[ e^{(S - i\lambda A)t} = \begin{pmatrix} e^{\omega_+(k)t} & e^{\omega_-(k)t} \\ e^{\omega_-(k)t} & e^{\omega_+(k)t} \end{pmatrix} - \frac{1}{\omega_+(k) - \omega_-(k)} \begin{pmatrix} \omega_+ & ki \\ iak - \lambda & -\omega_+ \end{pmatrix}, \]
(51)
where $\omega_\pm = \omega_\pm(k)$. This, together with
\[ \left| \frac{e^{\omega_+(k)t} - e^{\omega_-(k)t}}{\omega_+(k) - \omega_-(k)} \right| = \frac{1}{\omega_+(k) - \omega_-(k)} \left| \frac{e^{\omega_+(k)t} - e^{\omega_-(k)t}}{\omega_+(k) - \omega_-(k)} \right| \leq O(1) \]
(52)
establishes the uniform boundedness of $e^{(S - i\lambda A)t}$ for $k$ near $\pm 1/\sqrt{\alpha}$.

By a scaling in $k$ and $t$, we conclude that $e^{(S - i\lambda A)t}$ is uniformly bounded for all $\varepsilon > 0$, $k \in \mathbb{R}$ and $t \geq 0$. The desired uniform $L^2$ estimate
\[ \int_{-\infty}^{\infty} |U(x, t)|^2 \, dx \leq O(1) \int_{-\infty}^{\infty} |U_\varepsilon(x)|^2 \, dx \]
(53)
now follows easily from Parseval’s equality.

2.3. Asymptotic Convergence

Next, we show that in the limit $\varepsilon \downarrow 0$, the above solution $(u^\varepsilon, v^\varepsilon)$ of the Cauchy problem (35)-(36) converges to some solution $(u, v)$ of the corresponding equilibrium system
\[ \begin{align*}
\partial_x u + \lambda \cdot \partial_x u &= 0, \\
v &= i \lambda \cdot u.
\end{align*} \]
(54)

We only need to specify the initial data $u(x, 0)$ in order to find the limiting solution from (54). The obvious choice is
\[ u(x, 0) = u_\varepsilon(x). \]
(55)
This is indeed the right one. This is because the initial layer only occurs in the $v$ component. Thus the $u$ component of the Hilbert solution must take
the full initial data $u_0(x)$, and the initial difference $v_0(x) - \lambda u_0(x)$ in the $v$ component is taken up by the initial layer $v^l(x, t) = e^{-l/t}(v_0(x) - \lambda u_0(x))$.

Therefore, we obtain

\[ u(x, t) = u_0(x - \lambda t) \]
\[ v(x, t) = \lambda u_0(x - \lambda t) \]  

(56)

with Fourier transform

\[ \hat{u}(k, t) = \mathcal{F} u = e^{-\lambda k^2 t} \hat{u}_0(k) \]
\[ \hat{v}(k, t) = \mathcal{F} v = \lambda e^{-\lambda k^2 t} \hat{u}_0(k) \]  

(57)

i.e.,

\[ \hat{U}(k, t) = \begin{pmatrix} \hat{u}(k, t) \\ \hat{v}(k, t) \end{pmatrix} = \begin{pmatrix} e^{-\lambda k^2 t} & 0 \\ \lambda e^{-\lambda k^2 t} & 0 \end{pmatrix} \hat{U}_0(k). \]  

(58)

The formal convergence of $U^l(x, t) \to U(x, t)$, or more precisely, the pointwise convergence of $\hat{U}^l(k, t) \to \hat{U}(k, t)$ for $t > 0, k \in \mathbb{R}$ as $\varepsilon \downarrow 0$ can be easily verified by studying the limiting behavior of the solution operator (in Fourier space) $e^{(S - \omega_+(k)A^l)} \hat{v}$, see (62) below.

First, we note that

\[ \lim_{k \to 0} \omega_-(k) = -1, \quad \lim_{k \to 0} \omega_+(k) = 0, \quad \lim_{k \to 0} \frac{\omega_+(k)}{k} = -i\lambda. \]  

(59)

Therefore, for $t > 0$ and $k \in \mathbb{R}$, we obtain, as $\varepsilon \downarrow 0$,

\[ e^{\omega_+(ik) l \varepsilon} \to e^{-\lambda k^2 t}, \quad e^{\omega_-(ik) l \varepsilon} \to 0, \]  

(60)

and

\[ P(\varepsilon k) \to \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \quad P^{-1}(\varepsilon k) \to \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}, \]  

(61)

and hence

\[ e^{(S - \omega l A^l) l \varepsilon} \to \begin{pmatrix} e^{-\lambda k^2 t} & 0 \\ \lambda e^{-\lambda k^2 t} & 0 \end{pmatrix}. \]  

(62)
On the other hand, the solution operator $e^{(S - i k A) t / \varepsilon}$ ($t \geq 0$, $\varepsilon > 0$) is uniformly bounded and $\hat{U}_0 \in L^2(\mathbb{R})$, therefore, by Lebesgue's dominated convergence theorem, we conclude

$$\int_{-\infty}^{\infty} |U^\varepsilon - U|^2 (x, t) \, dx = \int_{-\infty}^{\infty} |\hat{U}^\varepsilon - \hat{U}|^2 (k, t) \, dk \to 0, \quad \text{as} \quad \varepsilon \downarrow 0 \quad (63)$$

for any $t > 0$.

### 2.4. Convergence Rate and Initial Layer Effect

Due to the initial layer effect, the convergence in (62) is non-uniform in $t$ (and in $k$). This makes it impossible to get a uniform rate of convergence in $t$. However, as the matched asymptotic expansion indicates, the initial layer only occurs in the $v$ component and becomes negligible after an initial transient time of order $\varepsilon$. In addition, for local equilibrium initial data, no initial layer occurs. Here we give some more detailed analysis on the structure of the solution operator $e^{(S - i k A) t / \varepsilon}$ and improve the convergence result in the previous subsection.

First, we rearrange (62) as

$$e^{(S - i k A) t / \varepsilon} = e^{-i k \lambda t / \varepsilon} \left( \begin{matrix} 1 & 0 \\ \lambda & 0 \end{matrix} \right)$$

$$= -\frac{e^{\omega_+(ik) t / \varepsilon} - e^{\omega_-(ik) t / \varepsilon}}{\omega_+(ik) - \omega_-(ik)} \left( \begin{array}{cc} \omega_+(ik) & i k \\ i k & -\omega_+(ik) \end{array} \right)$$

$$+ \frac{1}{\omega_+(ik) - \omega_-(ik)} \left(e^{\omega_+(ik) t / \varepsilon} - e^{\omega_-(ik) t / \varepsilon}\right) \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right)$$

$$+ \left(e^{\omega_+(ik) t / \varepsilon} - e^{-i k \lambda t / \varepsilon}\right) \left( \begin{array}{cc} 1 & 0 \\ 0 & -\lambda \end{array} \right) \left( e^{\omega_+(ik) t / \varepsilon} - e^{\omega_-(ik) t / \varepsilon}\right). \quad (64)$$

Each term on the right hand side of (64) tends to zero as $\varepsilon \downarrow 0$. Besides, we have the following estimates

$$\frac{e^{\omega_+(ik) t / \varepsilon} - e^{\omega_-(ik) t / \varepsilon}}{\omega_+(ik) - \omega_-(ik)} \leq O(1), \quad (65)$$

$$|\omega_+(ik)| \leq O(1) \, \varepsilon \, |k|, \quad \text{Re} \, \omega_-(ik) \leq -1/2, \quad (66)$$

$$\left| \frac{1}{\omega_+(ik) - \omega_-(ik)} - 1 \right| \left(e^{\omega_+(ik) t / \varepsilon} - e^{\omega_-(ik) t / \varepsilon}\right) \leq O(1) \, \varepsilon \, |k|, \quad (67)$$

$$|\omega_+(ik) t / \varepsilon + i k \lambda t| \leq O(1) \, \varepsilon k^2 t, \quad |e^{\varepsilon - 1}| \leq O(1) \, |z| \quad \text{for} \quad \text{Re} \, z < 0, \quad (68)$$
Therefore, we obtain
\[ |\hat{U}^a(k, t) - \hat{U}^a(k, 0)|^2 \leq O(1) e^{c(k^2 + t^2k^4)} |\hat{U}_0(k)|^2 + O(1) e^{-t/c} |\hat{a}_0(k) - \hat{\lambda}_a(k)|^2, \tag{70} \]
and hence
\[
\int_{-\infty}^{\infty} |U'(x, t) - U(x, t)|^2 \, dx \leq O(1) e^c \int_{-\infty}^{\infty} (|U'_0(x)|^2 + |U_0(x)|^2) \, dx + O(1) e^{-t/c} \int_{-\infty}^{\infty} |v_0(x) - \lambda u_0(x)|^2 \, dx. \tag{71} 
\]
This proves (27). Finally, we notice that the initial layer effect is only reflected in the last term \( e^{c(k^2 + t^2k^4)} \) in (64). With \( \hat{v}^L = 0, \hat{v}^L = e^{-t/c} \hat{v}_0 - \hat{\lambda}_a \) and the inequality,
\[ |e^{o_+ (k) t/c} - e^{-t/c}| = |e^{o_+ (k) t/c} (1 - e^{o_+ (k) t/c})| \leq O(1) e^{-t/c} |o_+ (k) t/c| \leq O(1) e |k|, \tag{72} \]
(28) can be proved by the same type of analysis as above.
This completes the proof of Theorem 1.3.

3. IBVP WITH HOMOGENEOUS INITIAL DATA: \( N = 1 \) CASE

We now turn to our main task of studying the IBVP (1), (5)-(6). We start with the case \( n = 1 \) and prove Theorem 1.1 under the additional assumption \( U_0(x) = 0 \) in this section. We remark that the homogeneous initial condition \( U_0(x) = 0 \) allows us to focus on the boundary layer effects and to avoid the complicated interactions between the boundary and initial layers. The full IBVP will be studied in Section 5.

With the same notation as in (34) and \( B = (B_x, B_y) \), we rewrite (1), (5) and (6) as
\[
\partial_t U^* + A \partial_x U^* = \frac{1}{\xi} SU^*, \tag{73} 
\]
\[ U^*(x, 0) = 0, \tag{74} \]
\[ BU^*(0, t) = h(t). \tag{75} \]
Following Kreiss [6], see also [17], we shall first apply the normal mode analysis to derive a necessary condition for stiff well-posedness. We call the necessary condition *Stiff Kreiss Condition*. The SKC is then simplified and its explicit equivalent form (13) is obtained by a conformal mapping theorem. Under the assumption of SKC, the solution $U^*$ can then be constructed by the method of Laplace transform. Stiff well-posedness can be proved rather directly. Finally, with the help of matched asymptotic expansions, the limiting equilibrium solution and various boundary layer behaviors are identified.

3.1. Derivation of the Stiff Boundary Condition

We look for (nontrivial) solutions of (73) satisfying the homogeneous boundary condition

$$BU^* = 0 \quad \text{at} \quad x = 0 \quad (76)$$

and of the form

$$U^*(x, t) = e^{\zeta t} \phi(x/c) \quad (77)$$

with $\text{Re} \, \zeta > 0$, $\phi \in L^2(\mathbb{R}^+)$. Such solutions, if they exist, clearly violate the $c$-uniform $L^2$ estimates in (11).

Plugging (77) into (73), one obtains the following “eigenvalue problem”

$$\phi' = M \phi, \quad (78)$$

where

$$M = M(\zeta) = A^{-1}(S - \zeta I) = \frac{1}{a} \begin{pmatrix} \lambda & -1 + \zeta \\ -a\zeta & 0 \end{pmatrix}. \quad (79)$$

The eigenvalues of $M$ can be easily found to be

$$\mu_{\pm} = \mu_{\pm}(\zeta) = \frac{\lambda \pm \sqrt{\lambda^2 + 4a\zeta(1 + \zeta)}}{2a} \quad (80)$$

with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ a\mu_{\pm} \\ 1 + \zeta \end{pmatrix}. \quad (81)$$

Under the sub-characteristic condition (12), the complex function

$$h(\zeta) = \sqrt{\lambda^2 + 4a\zeta(1 + \zeta)} \quad (82)$$
is analytic in \( \text{Re} \, \xi \geq \frac{1}{2}(-1 + \sqrt{1 - \frac{\lambda^2}{a}}) \), hence in the half plane \( \text{Re} \, \xi \geq 0 \).

(As usual, we take \( \sqrt{z} \) to be the principal branch with the branch cut along the negative real axis.)

Let \( \xi = \pi + i \beta, \beta \geq 0 \), and

\[
p = \lambda^2 + 4ax(1 + x) - 4a\beta^2, \quad q = 4a(1 + 2x) \beta.
\]  

(83)

Then,

\[
\text{Re} \, h(\xi) = \text{Re} \, \sqrt{p + iq} = \frac{p + \sqrt{p^2 + q^2}}{2}.
\]  

(84)

Now we observe that

\[
\sqrt{p^2 + q^2} = \sqrt{(\lambda^2 + 4ax(1 + x) - 4a\beta^2)^2 + (4a(1 + 2x) \beta)^2} \\
\geq \lambda^2 + 4ax(1 + x) + 4a\beta^2.
\]  

(85)

where the key estimate is an application of the sub-characteristic condition (12). Therefore,

\[
\text{Re} \, h(\xi) \geq \sqrt{\lambda^2 + 4ax(1 + x)} \geq |\lambda| (1 + 2x).
\]  

(86)

We further note that by using the basic inequality \( \sqrt{1 + x} \leq 1 + x/2 \) \((x \geq -1)\), we can also obtain from (85) a close upper bound for \( \text{Re} \, h(\xi) \)

\[
\text{Re} \, h(\xi) \leq \sqrt{a(1 + 2x)}.
\]  

(87)

In particular, we have from the above

\[
\text{Re} \, \mu_+(\xi) > 0, \quad \text{Re} \, \mu_-(\xi) < 0, \quad \text{for} \quad \text{Re} \, \xi > 0.
\]  

(88)

Thus the general solution of (78) satisfying \( \phi \in L^2(\mathbb{R}^+) \) can be represented as

\[
\phi(y) = ce^{\mu^- y} \begin{pmatrix} 1 \\ a\mu_+ \\ 1 + \xi \end{pmatrix}
\]  

(89)

for any constant \( c \) and thus

\[
U^\xi(x, t) = ce^{\xi t}e^{\mu^- x} \begin{pmatrix} 1 \\ a\mu_+ \\ 1 + \xi \end{pmatrix}.
\]  

(90)
The boundary condition (76) now reduces to

\[(B_u, B_v) \left( \begin{array}{c} 1 \\ \frac{\alpha \mu_+}{1 + \zeta} \end{array} \right) c = 0. \tag{91} \]

Clearly if

\[(B_u, B_v) \left( \begin{array}{c} 1 \\ \frac{\alpha \mu_+}{1 + \zeta} \end{array} \right) = B_u + \frac{\alpha \mu_+}{1 + \zeta} B_v = 0, \tag{92} \]

then there exists a nontrivial solution of (73) of the form (77) which violates the uniform $L^2$ estimates. Therefore, it is necessary that

\[B_u + \frac{\alpha \mu_+ (\zeta)}{1 + \zeta} B_v \neq 0 \quad \text{for all } \text{Re} \, \zeta > 0. \tag{93} \]

Actually, a uniform version of (93) is needed. This leads to the following SKC which is slightly more restrictive than (93),

\[\left| B_u + \frac{\alpha \mu_+ (\zeta)}{1 + \zeta} B_v \right| \geq C, \tag{94} \]

for some $C > 0$ and for all $\text{Re} \, \zeta \geq 0$.

We comment that, in general, one also needs a normalization of the eigenvector $(1, a \mu_+ (\zeta)/(1 + \zeta))$ and therefore requires, instead of (94),

\[\left| B_u + \frac{\alpha \mu_+ (\zeta)}{1 + \zeta} B_v \right| \geq C \sqrt{1 + \left| \frac{\alpha \mu_+}{1 + \zeta} \right|^2} \tag{95} \]

for some $C > 0$ and for all $\text{Re} \, \zeta \geq 0$.

In our case, however, (95) and (94) are actually equivalent and the normalization is therefore not necessary. This is because, by our choice, the eigenvector $(1, a \mu_+ (\zeta)/(1 + \zeta))$ in (81) is uniformly bounded and away from 0 for all $\text{Re} \, \zeta \geq 0$.

3.2. Simplification of SKC

The SKC in the form of (94) is still very complicated and hard to verify. Here we want to derive a simple equivalent condition of (94).
For this purpose, it is necessary to study the range of
\[
g(\zeta) = \frac{a \mu(\zeta)}{1 + \zeta} = \frac{\lambda + \sqrt{\lambda^2 + 4a\zeta(1 + \zeta)}}{2(1 + \zeta)}
\] (96)
for \(\text{Re} \, \zeta \geq 0\).

The complex function \(g(\zeta)\) is analytic and bounded in \(\text{Re} \, \zeta \geq 0\). It is also one-to-one in \(\text{Re} \, \zeta \geq 0\) when the strict sub-characteristic condition \(a > \lambda^2\) is satisfied.

Therefore, by the conformal mapping theorem, \(g(\zeta)\) maps the half plane \(\text{Re} \, \zeta \geq 0\) to a simply connected closed bounded domain \(\Omega \subset \mathbb{C}\) whose boundary corresponds to the image of the imaginary axis \(\text{Re} \, \zeta = 0\) under \(g\).

The boundary curve
\[
g(i\beta) = \frac{\lambda + \sqrt{\lambda^2 - 4a\beta^2 + 4a \beta i}}{2(1 + i\beta)}, \quad -\infty < \beta < \infty
\] (97)
is a closed curve which intersects the real axis only at \(\beta = 0\) and at \(\beta = \pm \infty\) with \(g(0) = \frac{\lambda + |\lambda|}{2}\), \(g(\pm \infty) = \sqrt{a}\). Besides, the curve is transversal to the real axis.

We observe that (94) holds trivially when \(B_v = 0\). For \(B_v \neq 0\), (94) can be reformulated as
\[
-\frac{B_u}{B_v} \# \text{closure}\{g(\zeta) : \text{Re} \, \zeta \geq 0\} \quad \text{(which is } \Omega). \] (98)

As \(B_u, B_v\) are assumed to be real, (98) can be further simplified to
\[
-\frac{B_u}{B_v} \# \Omega \cap \mathbb{R}. \] (99)

Therefore, we obtain
\[
\frac{B_u}{B_v} \# \left[ -\sqrt{a}, -\frac{\lambda + |\lambda|}{2} \right]. \] (100)

Finally, we remark that the SKC (100) is still true in the case \(a = \lambda^2\). The above analysis applies equally well for \(\lambda = -\sqrt{a}\) as \(g(\zeta) = \sqrt{a} \zeta/(1 + \zeta)\) is again one-to-one. The other case \(\lambda = \sqrt{a}\) becomes extremely simple as \(g(\zeta) = \sqrt{a}\).
3.3. Solution by Laplace Transform

Now we solve (73)–(75) by the method of Laplace transform. Let

\[ \widehat{U}^\epsilon(x, \xi) = \mathcal{L} U^\epsilon = \int_0^\infty e^{-\xi t} U^\epsilon(x, t) \, dt, \quad \text{Re} \xi > 0. \]  

(101)

With \( U_0(x) \equiv 0 \), we have

\[ \mathcal{L}(\partial_t U^\epsilon) = \xi \widehat{U}^\epsilon(x, \xi) - U^\epsilon(x, 0) = \xi \widehat{U}^\epsilon(x, \xi), \]  

(102)

and therefore (73)–(75) become

\[ \partial_x \widehat{U}^\epsilon = A^{-1} \left( \frac{1}{\epsilon} S - \xi I \right) \widehat{U}^\epsilon = \frac{1}{\epsilon} M(\epsilon \xi) \widehat{U}^\epsilon, \]  

(103)

\[ B \widehat{U}^\epsilon = \overline{b}(\xi) \quad \text{at} \quad x = 0, \]  

(104)

where

\[ \overline{b}(\xi) = \mathcal{L} b = \int_0^\infty e^{-\xi t} b(t) \, dt, \]  

(105)

and the matrix \( M \) is the same as in (79).

Note that the eigenvalues \( \mu_{\pm}(\xi) \) of the matrix \( M(\xi) \) satisfy

\[ \text{Re} \mu_-(\xi) < 0, \quad \text{Re} \mu_+(\xi) > 0 \quad \text{for} \quad \text{Re} \xi > 0. \]  

(106)

Therefore the solution to (103)–(104) is given by

\[ \widehat{U}^\epsilon(x, \xi) = c(\xi) e^{\mu_-(\epsilon \xi)x} \left( \frac{1}{\epsilon \mu_-(\epsilon \xi)} \right) = c(\xi) e^{\mu_-(\epsilon \xi)x} \left( \frac{1}{g(\epsilon \xi)} \right) \]  

(107)

for some constant \( c = c(\xi) \).

The constant \( c(\xi) \) can be determined easily from the boundary condition (104)

\[ c(\xi) = \frac{\overline{b}(\xi)}{B_u + g(\epsilon \xi) B_u}. \]  

(108)
Therefore,

$$\hat{U}(x, \zeta) = \frac{\tilde{b}(\zeta)}{B_u + g(e^\zeta) B_v} e^{\rho - (x + i\beta) \zeta} \left( \frac{1}{g(e^\zeta)} \right). \quad (109)$$

With \( \hat{U} \) found, the solution \( U \) to (73)-(75) can then be obtained by inverting the Laplace transform

$$U(x, t) = \mathcal{L}^{-1} \hat{U} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{(x + i\beta) t} \hat{U}(x, \alpha + i\beta) d\beta, \quad \alpha > 0. \quad (110)$$

### 3.4. Stiff Well-Posedness for IBVP (73)-(75)

Now we prove the uniform \( L^2 \) estimate in (11). By an application of the following Parseval’s identity [6],

$$\int_0^\infty e^{-2\pi t} |U(x, t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{U}(x, \alpha + i\beta)|^2 d\beta, \quad \alpha > 0, \quad (111)$$

we have

$$\int_0^\infty e^{-\pi t} |U(x, t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{U}(0, \alpha + i\beta)|^2 d\beta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{b}(\zeta)}{B_u + g(e^\zeta) B_v} \left( 1 + |g(e^\zeta)|^2 \right) d\beta. \quad (112)$$

where \( \zeta = x + i\beta \). We fix \( \eta > 0 \) from now on.

Our analysis of the SKC shows that \( B_u + g(\zeta) B_v \) is uniformly bounded away from 0 for \( \text{Re} \zeta \geq 0 \), see (94). On the other hand, since \( g(\zeta) \) is uniformly bounded in \( \text{Re} \zeta \geq 0 \), we obtain

$$\int_0^\infty e^{-2\pi t} |U(0, t)|^2 dt \leq O(1) \int_{-\infty}^{\infty} |\tilde{b}(\alpha + i\beta)|^2 d\beta$$

$$\leq O(1) \int_0^\infty e^{-2\pi t} |b(t)|^2 dt. \quad (113)$$

This, together with the hyperbolicity of (1), implies the desired boundary estimate

$$\int_0^T |U(x, t)|^2 dt \leq K_T \int_0^T |b(t)|^2 dt. \quad (114)$$
Similarly, we have
\[
\int_0^\infty \int_0^\infty e^{-2\alpha t} |U'(x, t)|^2 \, dx \, dt = \frac{1}{2\pi} \int_0^\infty \int_0^\infty \left| \bar{U}'(x, x + i\beta) \right|^2 \, dx \, d\beta
\]
\[
= \frac{1}{2\pi} \int_0^\infty \int_0^\infty \left| \frac{\bar{b}(\zeta)}{B_\alpha + g(\zeta) B_v} \right|^2 e^{\alpha \epsilon \zeta^2} \left( 1 + |g(\zeta)|^2 \right) \, dx \, d\beta
\]
\[
= \frac{1}{2\pi} \int_0^\infty \int_0^\infty \left| \frac{\bar{b}(\zeta)}{B_\alpha + g(\zeta) B_v} \right|^2 \frac{e}{2 \Re \mu_+(\zeta)} (1 + |g(\zeta)|^2) \, d\beta
\]
\[
\leq O(1) \sup_{\beta} \frac{e}{-2 \Re \mu_+(\zeta)} \int_0^\infty e^{-2\alpha t} |b(t)|^2 \, dt \tag{115}
\]

In order to complete the proof of (11), it suffices to establish the uniform boundedness of \(-\frac{\alpha}{\Re \mu_+(\zeta)}\). For \(\lambda < 0\), we have
\[
-2\alpha \Re \mu_+(\zeta) \geq -\lambda > 0, \tag{116}
\]
therefore,
\[
\int_0^\infty \int_0^\infty e^{-2\alpha t} |U'(x, t)|^2 \, dx \, dt \leq O(1) e \int_0^\infty e^{-2\alpha t} |b(t)|^2 \, dt \quad (\lambda < 0) \tag{117}
\]

For \(\lambda = 0\), we have from (86)
\[
-2\alpha \Re \mu_+(\zeta) = \Re b(\zeta) \geq \sqrt{4\alpha \lambda (1 + \alpha \zeta)}, \tag{118}
\]
and thus
\[
\int_0^\infty \int_0^\infty e^{-2\alpha t} |U'(x, t)|^2 \, dx \, dt \leq O(1) e^{1/2} \int_0^\infty e^{-2\alpha t} |b(t)|^2 \, dt \quad (\lambda = 0) \tag{119}
\]
Similarly, for \(\lambda > 0\), we have
\[
-2\alpha \Re \mu_+(\zeta) = \Re b(\zeta) - \lambda \geq 2\alpha \lambda \zeta. \tag{120}
\]
Therefore,
\[
\int_0^\infty \int_0^\infty e^{-2\alpha t} |U'(x, t)|^2 \, dx \, dt \leq O(1) \int_0^\infty e^{-2\alpha t} |b(t)|^2 \, dt \quad (\lambda > 0). \tag{121}
\]
3.5. Asymptotic Convergence: Non-characteristic Case

We now turn to the question of asymptotic convergence. In order to determine the limiting solution \((u(x, t), v(x, t))\) of the IBVP (73)-(75), we have to complete (54) with suitable initial data, and further boundary condition may also be necessary depending on the sign of \(\lambda\). All these can be achieved by a matched asymptotic analysis. We refer to [5] for an interesting discussion on the formal asymptotic analysis of this model.

The initial data \(u_0(x)\) should again be chosen homogeneous since there is no initial layer for solutions of (73)-(75) due to the homogeneity of the initial data in (74). The boundary condition can be obtained by the matched asymptotic analysis of Hilbert solution and boundary layer solution. The key point is that the combined approximate solution should satisfy the boundary condition (75).

Hence we propose the following expansions for solutions of (73)-(75)

\[
\begin{align*}
   u(x, t) &= u(x, t) + u^{bl}(y, t) + O(1) \varepsilon \\
   v(x, t) &= v(x, t) + v^{bl}(y, t) + O(1) \varepsilon
\end{align*}
\]  

(122)

with the localized boundary layer \((u^{bl}, v^{bl})\) decaying exponentially fast in \(y = x/\varepsilon\).

Plugging (122) into (73) and noticing the equilibrium equation (54) for \((u, v)\), we obtain the following boundary layer equation

\[
\begin{align*}
   \partial_y v^{bl} &= 0, \\
   a \partial_y u^{bl} &= \lambda u^{bl} - v^{bl}.
\end{align*}
\]  

(123)

Therefore we have

\[
\begin{align*}
   u^{bl}(y, t) &= e^{(\lambda/y)} u^{bl}(0, t), \\
   v^{bl}(y, t) &= 0.
\end{align*}
\]  

(124)

It is clear that in the case \(\lambda > 0\), the only choice for the data \(u^{bl}(0, t)\) is \(u^{bl}(0, t) = 0\), as \(u^{bl}(y, t)\) would otherwise grow exponentially in \(y\).

Now we match the boundary condition by requiring

\[
B_a u(0, t) + u^{bl}(0, t) + B_v (v(0, t) + v^{bl}(0, t)) = b(t).
\]  

(125)

In the case \(\lambda < 0\), it gives

\[
u^{bl}(0, t) = \frac{b(t)}{B_a}.
\]  

(126)
Therefore, we get
\[ u(x, t) = 0, \quad u^b(y, t) = \frac{b(t)}{B_u} e^{(\lambda/\beta) y}. \] (127)

No boundary condition for \( u(x, t) \) is needed in this case.

The estimate in (117) already shows the asymptotic convergence as we expected and the \((L^2)\) convergence rate of order \( \epsilon^{1/2} \) is actually optimal due to the presence of boundary layer.

In the case \( \lambda > 0 \), no boundary layer occurs and (125) yields exactly the necessary boundary condition for (54)
\[ u(0, t) = b(t), \] (128)

and hence
\[ u(x, t) = \begin{cases} 0 & x \geq \lambda t, \\ \frac{1}{B_u + \lambda B_v} b(t - x/\lambda) & x \leq \lambda t, \end{cases} \] (129)
\[ \tilde{u}(x, \xi) = \mathcal{P} u = \frac{\tilde{b}(\xi)}{B_u + \lambda B_v} e^{-i(\xi/\lambda) x}. \] (130)

Therefore,
\[ \int_{\lambda}^{\infty} \int_{0}^{\infty} e^{-2\epsilon t} |u^\epsilon - u|^2(x, t) \, dx \, dt \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\tilde{u}^\epsilon - \tilde{u}|^2(x, \alpha + i\beta) \, dx \, d\beta \]
\[ \leq O(1) \int_{-\infty}^{\infty} |\tilde{b}(\xi)|^2 R(\epsilon, \xi) \, d\beta, \] (131)

where
\[ R(\epsilon, \xi) = \int_{0}^{\infty} \left| \frac{1}{B_u + g(\epsilon \xi) B_v} e^{\mu_{\xi - \alpha} x/\alpha} - \frac{1}{B_u + \lambda B_v} e^{-\xi x/\lambda} \right|^2 \, dx \]
\[ \leq I_1(\epsilon, \xi) + I_2(\epsilon, \xi) \] (132)
with

\[ I_1(\varepsilon, \zeta) = \int_0^\infty \left| \frac{1}{B_{\varepsilon} + g(\varepsilon \zeta) B_{\varepsilon}^{-1}} e^{\mu(-\varepsilon \zeta) x/\varepsilon} - \frac{1}{B_{\varepsilon} + \lambda B_{\varepsilon}^{-1}} e^{\mu(-\varepsilon \zeta) x/\varepsilon} \right|^2 dx \]

\[ = \frac{\varepsilon}{-2 \Re \mu(-\varepsilon \zeta)} \left| \frac{1}{B_{\varepsilon} + g(\varepsilon \zeta) B_{\varepsilon}^{-1}} - \frac{1}{B_{\varepsilon} + \lambda B_{\varepsilon}^{-1}} \right|^2 \]

\[ = O(1) \left| g(\varepsilon \zeta) - \lambda \right|^2, \quad (133) \]

and

\[ I_2(\varepsilon, \zeta) = \int_0^\infty \left| \frac{1}{B_{\varepsilon} + \lambda B_{\varepsilon}^{-1}} \left( e^{\mu(-\varepsilon \zeta) x/\varepsilon} - e^{\varepsilon \zeta x/\lambda} \right) \right|^2 dx \]

\[ = \left| \frac{1}{B_{\varepsilon} + \lambda B_{\varepsilon}^{-1}} \right|^2 \left( -\frac{1}{2 \Re \mu(-\varepsilon \zeta)/\varepsilon} + \frac{1}{2\lambda} \right) \left| -\mu(-\varepsilon \zeta)/\varepsilon - \frac{\zeta}{\lambda} \right|^2 \]

\[ = O(1) \left| \frac{\mu(-\varepsilon \zeta)}{\varepsilon} + \frac{\zeta}{\lambda} \right|^2. \quad (134) \]

The pointwise convergence of \( I(\varepsilon, \zeta) \to 0 \) as \( \varepsilon \downarrow 0 \) now follows easily from the following estimates

\[ g(\varepsilon \zeta) - \lambda = \frac{2(a - \lambda^2) \varepsilon \zeta}{\sqrt{\lambda^2 + 4a \varepsilon \zeta(1 + \varepsilon \zeta) + \lambda(1 + 2\varepsilon \zeta)}} = O(1) \varepsilon \zeta, \quad (135) \]

and

\[ \frac{\mu(-\varepsilon \zeta)}{\varepsilon} + \frac{\varepsilon \zeta}{\lambda} = \frac{2(a - \lambda^2) \varepsilon \zeta^2}{\lambda \sqrt{\lambda^2 + 4a \varepsilon \zeta(1 + \varepsilon \zeta) + \lambda^2 + 2a \varepsilon \zeta}} = O(1) \varepsilon \zeta^2. \quad (136) \]

On the other hand, \( I(\varepsilon, \zeta) \) is uniformly bounded and \( \tilde{h}(\alpha + i \cdot) \in L^2(\mathbb{R}) \), therefore, by Lebesgue’s dominated convergence theorem, we obtain the asymptotic convergence in the \( u \) component

\[ \int_{-\infty}^{\infty} \int_{\mathbb{R}} e^{-2\pi} |u^t - u|^2(x, t) \, dx \, dt \to 0 \quad \text{as} \quad \varepsilon \downarrow 0 \quad (\lambda > 0). \quad (137) \]
An optimal convergence rate of order $\varepsilon$ can be obtained if we assume $b(t) \in H^2(\mathbb{R}^+)$ and $b(t)$ satisfies the compatibility condition $b(0) = b'(0) = 0$ as

$$
\int_{-\infty}^{\infty} |\tilde{b}(\xi)|^2 I(\varepsilon, \xi) d\xi \leq O(1) \varepsilon^2 \int_{-\infty}^{\infty} (|\xi|^2 + |\xi|^4) |\tilde{b}(\xi)|^2 d\xi
$$

$$
= O(1) \varepsilon^2 \int_0^{\infty} e^{-2\varepsilon b'(t)^2 + b''(t)^2} dt
$$

$$
\leq O(1) \varepsilon^2 \|b\|^2_{H^2}.
$$

(138)

The analysis of the $v$ component can be done similarly.

3.6. Convergence Analysis: Uniformly Characteristic Boundary Case

We observe that the matched asymptotic expansion in the form (122) fails to capture the boundary layer behavior for the solution $U^\varepsilon$ of (73)–(75) in the case $\lambda = 0$. The boundary layer solution in (124) would vanish identically and the matching of boundary condition as in (125) would be impossible. However it is already clear from (119) that the relaxation solution $(u^r, v^r)$ converges to the trivial solution $u(x, t) \equiv 0$ of the equilibrium equation in $L^2$ with a slower rate of order $\varepsilon^{1/4}$. These facts clearly indicates, in the present case of uniformly characteristic boundary, the boundary layer should be of a completely different nature and lives on a larger scale near the boundary.

Therefore we propose the following type of boundary layer expansion

$$
u^l(x, t) = \nu(x, t) + \nu^l_1(y, t) + \sqrt{\varepsilon} \nu^l_2(y, t) + O(1) \varepsilon
$$

$$
u^l(x, t) = \nu(x, t) + \nu^l_1(y, t) + \sqrt{\varepsilon} \nu^l_2(y, t) + O(1) \varepsilon
$$

where $y = x/\sqrt{\varepsilon}$ and $(\nu^l, v^l), (u^l_1, v^l_1) \to 0$ as $y \to \infty$.

Plug (139) into (73), one obtains

$$
\partial_y v^l = 0,
$$

$$
\partial_y \partial_y v^l + \partial_y v^l_1 = 0,
$$

$$
\partial_y \partial_y v^l = -v^l_1.
$$

(140)

Therefore,

$$
\partial_y v^l = \partial_y^2 v^l_1,
$$

$$
v^l = 0.
$$

(141)
Note that (141) is a diffusion equation, both initial data and boundary data are necessary to determine the solution. The initial data should again be chosen homogeneous and the necessary boundary data can be determined in the same way as in (125). Therefore

\begin{align}
  u^b(j,0) &= 0, \\
  u^b(0,t) &= \frac{b(t)}{B_a}. 
\end{align}  \tag{142}

The solution to the IBVP (141)–(142) may be given in closed form [4]. For our purpose, however, it is more convenient to solve (141)–(142) by Laplace transform which gives

\begin{align}
  \tilde{u}^b(x,\xi) &= e^{-\sqrt{-\mu_x}} \tilde{b}(\xi) \frac{B_a}{B_a} = e^{-\sqrt{-\mu_x}} \tilde{b}(\xi).
\end{align}  \tag{143}

We end this section by verifying the above boundary layer structure

\begin{align}
  \int_0^\infty \int_0^\infty e^{-2at} |u^e - u^b(x,t)|^2 (x,t) \, dx \, dt \leq O(1) e^{3/2} \|b\|_{L^2}^2. \tag{144}
\end{align}

To prove (144), we rewrite \( \tilde{u}^e - \tilde{u}^b \) as

\begin{align}
  \tilde{u}^e(x,\xi) - \tilde{u}^b(x,\xi) &= \left( \frac{1}{B_a + g(\xi)} - 1 \right) \tilde{b}(\xi) e^{\mu_x (\xi) x} + \tilde{b}(\xi) \left( e^{\mu_x (\xi) x} - e^{-\sqrt{-\mu_x} x} \right). \tag{145}
\end{align}

Note that

\begin{align}
  \int_0^\infty \int_{-\infty}^\infty \left| \frac{1}{B_a + g(\xi)} - 1 \right|^2 |\tilde{b}(\xi)|^2 e^{2 \Re\mu_x (\xi) x} \, dx \, d\beta \\
  &\leq O(1) \sup_{\beta} \left| \frac{e}{-2 \Re\mu_x (\xi)} \right| \int_{-\infty}^\infty |g(\xi)|^2 |\tilde{b}(\xi)|^2 \, d\beta \\
  &\leq O(1) e^{3/2} \int_{-\infty}^\infty |\xi| |\tilde{b}(\xi)|^2 \, d\beta. \tag{146}
\end{align}
and
\[
\frac{\beta(\xi)}{\beta(x)} \left| e^{\mu_-(x) x e} - e^{-\sqrt{\mu_e x}} \right|^2 dx dB 
\leq O(1) \int_{-\infty}^{\infty} \left( \frac{1}{-2 \text{Re} \mu_-(\xi) x e} + \frac{1}{2 \text{Re} \sqrt{\mu_e x}} \right) 
\times \left| \frac{\mu_-(\xi)}{x} - \frac{\sqrt{\xi}}{ae} \right|^2 \left| \beta(\xi) \right|^2 dB 
\int_{-\infty}^{\infty} |\xi|^4 \left| \beta(\xi) \right|^2 dB. \tag{147}
\]

Therefore,
\[
\int_{-\infty}^{\infty} |\tilde{u} - \tilde{u}_b|^2 (x, \xi) dx dB \leq O(1) \varepsilon^{3/2} \int_{-\infty}^{\infty} (1 + |\xi|^4) \left| \tilde{\beta}(\xi) \right|^2 dB. \tag{148}
\]

The estimate (144) now follows provided that \( b(t) \in H^2(\mathbb{R}^+) \) and \( b(t) \) satisfies the compatibility condition \( b(0) = b'(0) = 0 \). This justifies the above uniformly characteristic boundary layer structure in the \( u \) component. On the other hand, as \( g(\xi) = O(1) \varepsilon^{1/2} |\xi|^{1/2} (\lambda \equiv 0) \), a similar estimate holds obviously for the \( v \) component with \( v^b \equiv 0 \).

The validity of the non-characteristic boundary layer given in (127) can be verified similarly.

4. IBVP WITH HOMOGENEOUS INITIAL DATA: \( n > 1 \) CASE

We now study the IBVP (1), (5)-(6) (again with \( U_0(x) = 0 \)) in the general system case \( n > 1 \) and prove Theorem 1.2. As we remarked earlier, we may assume
\[
f(u) = Au, \quad A = \text{diag} \{ \lambda_1, \lambda_2, \ldots, \lambda_n \},
\tag{149}
\lambda_1, \ldots, \lambda_p > 0, \quad \lambda_{p+1} = \cdots = \lambda_q = 0, \quad \lambda_{q+1}, \ldots, \lambda_n < 0. \tag{150}
\]
For convenience, we rewrite (1), (5)-(6) in the same form as (73)-(75) with

\[
A = \begin{pmatrix} 0 & I_n \\ aI_n & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ A & -I_n \end{pmatrix}.
\]

(151)

4.1. The Stiff Kreiss Condition

Again we look for solutions of the form

\[
U(t, x) = e^{\xi t} \phi(x/t)
\]

(152)

with \( \Re \xi > 0, \phi \in L^2(\mathbb{R}^+) \).

Then \( \phi \) solves the following eigenvalue problem

\[
\phi' = M \phi
\]

(153)

with

\[
M = \begin{pmatrix} A & -(1 + \xi) I_n \\ -a\xi I_n & 0 \end{pmatrix}^{-1} = \frac{1}{a} \left( \begin{array}{cc} A & -(1 + \xi) I_n \\ -a\xi I_n & 0 \end{array} \right).
\]

(154)

The \( 2n \times 2n \) matrix \( M \) is diagonalizable with eigenvalues

\[
\mu_j^\pm = \mu_j^\pm(\xi) = \frac{\lambda_j \pm \sqrt{\lambda_j^2 + 4a\xi(1 + \xi)}}{2a}
\]

(155)

and corresponding eigenvectors

\[
\begin{pmatrix} e_j \\ \frac{a}{1 + \xi} \mu_j^\pm e_j \end{pmatrix}
\]

(156)

where \( j = 1, ..., n \) and \( e_1 = (1, 0, ..., 0), ..., e_n = (0, ..., 0, 1) \) are the standard basis vectors of \( \mathbb{R}^n \).

From the discussion in the previous section, it is clear that

\[
\Re \mu_j^+ > 0, \quad \Re \mu_j^- < 0, \quad \text{for} \quad \Re \xi > 0, \quad 1 \leq j \leq n
\]

(157)

holds under the sub-characteristic condition (19). Thus the general solution of (153) satisfying \( \phi \in L^2(\mathbb{R}^+) \) can be represented as

\[
\phi(y) = \sum_j e_j e^{\mu_j^+ y} \left( \frac{a}{1 + \xi} \mu_j^+ e_j \right),
\]

(158)
and hence

$$U^*(x, t) = \sum_j c_j e^{i \omega_j \mu^+ x} e^{a_j (1 + \xi)/\mu^+ x}$$

$$= e^{i \omega x} \begin{pmatrix} I_n \\ \frac{a}{1 + \xi} \mu^+ \end{pmatrix} e^{\mu^+ x} \begin{pmatrix} e_j \\ \vdots \\ e_n \end{pmatrix},$$

(159)

where

$$\mu^\pm = \text{diag} \{ \mu_1^\pm, \ldots, \mu_n^\pm \},$$

(160)

and $c_j, 1 \leq j \leq n$ are constants.

However, such solutions, unless $U^* = 0$, violate the uniform $L^2$ estimate in (11) and thus have to be excluded by the boundary condition (75). Therefore it is necessary that

$$\det \left( B_u + B_v \frac{a \mu^+}{1 + \xi} \right) = \det(B_u + B_v G(\xi)) \neq 0$$

(161)

for all $\xi$ with $\text{Re} \, \xi > 0$ where

$$G(\xi) = \text{diag} \{ g_1(\xi), g_2(\xi), \ldots, g_n(\xi) \}$$

(162)

and

$$g_j(\xi) = \frac{a \mu_j^+ (\xi)}{1 + \xi}, \quad 1 \leq j \leq n.$$  

(163)

The SKC is a uniform version of (161) and requires

$$|\det(B_u + B_v G(\xi))| \geq C \sqrt{\det(I_n + G(\xi) G^*(\xi))}$$

(164)

for some $C > 0$ and for all $\text{Re} \, \xi \geq 0$.

Due to the uniform boundedness of $G(\xi)$, (164) is equivalent to

$$|\det(B_u + B_v G(\xi))| \geq C$$

(165)

for some $C > 0$ and for all $\xi$ with $\text{Re} \, \xi \geq 0$.

4.2. Solution by Laplace Transform

With the SKC, the IBVP (73)-(75) can be solved by Laplace transform in much the same way as in Section 3.3.
First, it is clear that the Laplace transform $\tilde{U}(x, \xi)$ of $U(x, t)$ satisfies the same equations in (103)--(104) with the new $M(\xi)$ given in (154).

From (155)--(157), it follows

$$\tilde{U}(x, \xi) = \sum_{j=1}^{n} c_{j} e^{\nu_{j}(\xi) x} \begin{pmatrix} e_{j} \\ g_{j}(\xi) e_{j} \end{pmatrix}.$$ (166)

Again the constant $c = c(\xi) \in \mathbb{R}^{n}$ should be determined from the boundary condition (104) which now becomes

$$(B_{u}, B_{v})(I_{n} G(\xi)) c(\xi) = (B_{u} + B_{v} G(\xi)) c(\xi) = \tilde{b}(\xi).$$ (167)

The SKC (165) guarantees the solvability of $c(\xi)$ from the above linear system since the coefficient matrix $B_{u} + B_{v} G(\xi)$ is uniformly invertible in $\text{Re} \, \xi > 0$. Therefore, we obtain

$$c(\xi) = (B_{u} + B_{v} G(\xi))^{-1} \tilde{b}(\xi),$$ (168)

and

$$\tilde{U}(x, \xi) = \left( I_{n} G(\xi) \right) e^{\nu(\xi) x / \nu} (B_{u} + B_{v} G(\xi))^{-1} \tilde{b}(\xi).$$ (169)

### 4.3. Stiff Well-Posedness

With the solution explicitly given in (169) by Laplace transform and the result in the previous section, the stiff well-posedness follows almost obviously.

The matrix $B_{u} + B_{v} G(\xi)$ is clearly uniformly bounded in $\text{Re} \, \xi \geq 0$. On the other hand, by the SKC (165), it is also uniformly invertible. Hence its inverse $(B_{u} + B_{v} G(\xi))^{-1}$ is uniformly bounded for all $\text{Re} \, \xi \geq 0$ and $\nu > 0$.

Therefore, we obtain

$$\int_{0}^{\infty} e^{-2\pi \tau} \left| U(0, t) \right|^{2} \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2\pi \beta} \left| \tilde{U}(0, \xi) \right|^{2} \, d\beta$$

$$\leq O(1) \int_{-\infty}^{\infty} \left| \tilde{b}(\xi) \right|^{2} \, d\beta$$

$$\leq O(1) \int_{0}^{\infty} e^{-2\pi \tau} \left| b(t) \right|^{2} \, dt,$$ (170)
and
\[ \int_0^\infty \int_0^\infty e^{-2s} |U^s(x, t)|^2 \, dx \, dt \]
\[ = \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty |\tilde{U}^s(x, \xi)|^2 \, dx \, d\beta \]
\[ \leq O(1) \int_0^\infty \int_{-\infty}^\infty \sum_{j=1}^n |e^{\mu_j^0(x\xi)}| |\tilde{\beta}(\xi)|^2 \, dx \, d\beta \]
\[ \leq O(1) \sum_{j=1}^n \sup_{\beta} \frac{\varepsilon}{2 \Re \mu_j^0(\xi)} \int_{-\infty}^\infty |\tilde{\beta}(\xi)|^2 \, d\beta \]
\[ \leq O(1) \int_0^\infty e^{-2s} |b(t)|^2 \, dt, \] (171)
where we have used the following estimates
\[ \sup_{\beta} \frac{\varepsilon}{2 \Re \mu_j^0(\xi)} \leq \begin{cases} O(1) & \text{if } \lambda_j > 0, \\ O(1) \varepsilon & \text{if } \lambda_j < 0, \\ O(1) \varepsilon^{1/2} & \text{if } \lambda_j = 0. \end{cases} \] (172)

With (170) and (171), the uniform estimate (11) follows easily.

4.4. Matched Asymptotic Expansions

To identify the relaxation limit of the solutions to (73)–(75), and understand their boundary layer behavior, we will carry out a formal matched asymptotic analysis to derive the formal leading asymptotic ansatz. To this end, we propose the following three-scale asymptotic expansion for solutions of (73)–(75):
\[ u^s(x, t) = u(x, t) + u^{BL}(y, t) + \sqrt{\varepsilon} u^{BL}(z, t) + O(1) \varepsilon \]
\[ v^s(x, t) = v(x, t) + v^{BL}(y, t) + \sqrt{\varepsilon} v^{BL}(z, t) + O(1) \varepsilon \] (173)
where \( y = x/\varepsilon, z = y/\sqrt{\varepsilon} \) and
\[ (u^{BL}, v^{BL}) \to 0 \quad \text{as} \quad y \to +\infty, \]
\[ (u^{BL}, v^{BL}, u^{BL}_1, v^{BL}_1) \to 0 \quad \text{as} \quad z \to +\infty. \] (174)
Substituting (173) into (73) and matching the orders of $\varepsilon$, we obtain
\[ \partial_t u + A \partial_x u = 0, \]
\[ v = Au, \]
\[ a \partial_x u^{BL} = Au^{BL} - v^{BL}, \]
\[ \partial_x v^{BL} = 0, \]
\[ Au^{BL} = 0, \]
\[ \partial_x v^{BL} = 0, \]
\[ a \partial_x u^{BL} = Au^{BL} - v^{BL}, \]
\[ \partial_x u^{BL} + \partial_x v^{BL} = 0. \]

There is no difficulty in solving the above equations. Indeed, the same analysis in the previous section can be applied to each component $(u_j, v_j)$ and therefore most of the results remain the same as in the case $n = 1$:

\[ u_j(x, t) = 0 \quad (\dot{\lambda}_j \leq 0), \]
\[ u_j(x, t) = 0 \quad (\dot{\lambda}_j > 0), \]
\[ v_j(x, t) = \lambda_j u_j(x, t), \]
\[ u_j^{BL}(y, t) = \begin{cases} 0 & (\dot{\lambda}_j > 0), \\ e^{\gamma_j y} u_j^{BL}(0, t) & (\dot{\lambda}_j < 0), \end{cases} \]
\[ v_j^{BL}(y, t) = 0, \]
and $u_j^{BL}$ (corresponding to $\dot{\lambda}_j = 0$) solves the parabolic IBVP
\[ \partial_t u_j^{BL} = a \partial_x^2 u_j^{BL}, \]
\[ u_j^{BL}(z, 0) = 0, \quad (\dot{\lambda}_j = 0) \]
\[ u_j^{BL}(0, t) \quad \text{to be given}, \]
with
\[ v_j^{BL}(z, t) = 0. \]

It is also clear that the two types of boundary layers are actually separated:
\[ u_j^{BL}(y, t) = 0 \quad \text{if} \quad \dot{\lambda}_j = 0, \]
and
\[ u^B_L(z, t) = 0 \quad \text{if} \quad \lambda_j \neq 0. \quad (187) \]

That is, the non-characteristic type boundary layers only occur to the \( u_j \) components corresponding to \( \lambda_j \neq 0 \) and the uniformly characteristic type boundary layers only occur to the \( u_j \) components corresponding to \( \lambda_j = 0 \).

We now turn to determine the necessary boundary data

\[ u_1(0, t), ..., u_p(0, t); \quad u^B_L(0, t), ..., u^B_L(p+1, t); \quad u^B^L_j(0, t), ..., u^B^L_n(0, t) \]

by requiring

\[ Bu(u(0, t) + u^B_L(0, t)) + B(u(0, t) + u^B_L(0, t)) = b(t). \quad (188) \]

The above matching of boundary conditions turns out to be very simple

\[ (B_u + B_u A_+) \begin{pmatrix} u_1(0, t) \\ \vdots \\ u_p(0, t) \\ u^B_L(0, t) \\ \vdots \\ u^B^L_n(0, t) \end{pmatrix} = b(t), \quad (189) \]

where

\[ A_+ = \text{diag}\{\lambda_1, ..., \lambda_p, 0, ..., 0\} \quad (190) \]

is the positive part of the matrix \( A \).

The invertibility of the matrix \( B_u + B_u A_+ \) can be easily seen from the SKC (165) by taking \( \xi = 0 \). Therefore we have from (189)

\[ u_j(0, t) = e_j(B_u + B_u A_+)^{-1} b(t) \quad 1 \leq j \leq p, \]
\[ u^B_L(0, t) = e_j(B_u + B_u A_+)^{-1} b(t) \quad p + 1 \leq j \leq q, \]
\[ u^B^L_j(0, t) = e_j(B_u + B_u A_+)^{-1} b(t) \quad q + 1 \leq j \leq n. \quad (191) \]
With the above boundary data, the equilibrium solution \((u, v)\), the boundary layers \((u_{bl}, v_{bl})\) and \((u_{BL}, v_{BL})\) can all be uniquely determined. In particular,

\[
u_j(x, t) = \begin{cases} 0 & x \geq \lambda_j t \\ e_j(B_u + B_v A_+)^{-1} b(t - x/\lambda_j) & x \leq \lambda_j t \end{cases} \quad 1 \leq j \leq p, \tag{192}
\]

\[u_j(x, t) = 0 \quad p + 1 \leq j \leq n \tag{193}\]

with Laplace transform

\[	ilde{u}_j(x, \xi) = \begin{cases} e_j(B_u + B_v A_+)^{-1} \tilde{b}(\xi) e^{-\xi x/\lambda_j} & 1 \leq j \leq p, \\ 0 & p + 1 \leq j \leq n. \tag{194}\]

4.5. Asymptotic Convergence

We now establish the asymptotic convergence results for the \(u\) components. First, we note that the same estimate in (171) yields

\[
\int_0^\infty \int_0^\infty e^{-2\pi rt} |u'_j(x, t)|^2 \,dx \,dt \\
\leq O(1) \sup_{\rho} \frac{\rho}{2} \text{Re} \mu_j(-\rho) \int_0^\infty e^{-2\pi s \rho} |b(t)|^2 \,dt \\
\leq \begin{cases} O(1) \rho^{1/2} \int_0^\infty e^{-2\pi s \rho} |b(t)|^2 \,dt & (\lambda_j = 0), \\ O(1) \rho \int_0^\infty e^{-2\pi s \rho} |b(t)|^2 \,dt & (\lambda_j < 0). \end{cases} \tag{195}\]

This establishes the convergence of \(u'_j \to u_j \equiv 0\) corresponding to \(\lambda_j = 0\) and \(\lambda_j < 0\).

Finally, for the components corresponding to \(\lambda_j > 0\) \(1 \leq j \leq p\), we note that

\[
\tilde{u}_j(x, \xi) - \tilde{u}_j(x, \xi) = e_j(B_u + B_v G(\xi))^{-1} \tilde{b}(\xi) (e^{\rho_j(\xi) x/\xi} - e^{-\xi x/\lambda_j}) \\
+ e_j((B_u + B_v G(\xi))^{-1} - (B_u + B_v A_+)^{-1}) \tilde{b}(\xi) e^{-\xi x/\lambda_j}, \tag{196}\]

\[
\int_0^\infty |e^{\rho_j(\xi) x/\xi} - e^{-\xi x/\lambda_j}|^2 \,dx = O(1) \rho^{2+\lambda_j} \quad (\lambda_j > 0), \tag{197}\]

\[
(B_u + B_v G(\xi))^{-1} - (B_u + B_v A_+)^{-1} \\
= (B_u + B_v G(\xi))^{-1} B_+(A_+ - G(\xi))(B_u + B_v A_+)^{-1}, \tag{198}\]

\[
g_j(\xi) = \begin{cases} O(1) \rho^j & q + 1 \leq j \leq n \quad (\lambda_j < 0), \\ O(1) \rho^{1/2} \xi^{1/2} & p + 1 \leq j \leq q \quad (\lambda_j = 0). \end{cases} \tag{199}\]
and
\[ g_j(\varepsilon \zeta) - \hat{\lambda}_j = O(1) \varepsilon^p \quad 1 \leq j \leq p, \quad (\hat{\lambda}_j > 0); \quad (200) \]
see (134)–(136). Therefore, by the same type of argument as before, we obtain the convergence in the \( u_j \) components corresponding to \( \hat{\lambda}_j > 0 \)
\[ \int_0^\infty \int_0^\infty e^{-2\lambda t} |u'_j - u_j|^2 (x, t) \, dx \, dt \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad (\hat{\lambda}_j > 0). \quad (201) \]
Under the additional assumption of \( b(t) \in H^2(\mathbb{R}^+) \) and \( b(0) = b'(0) = 0 \), we also have
\[ \int_0^\infty \int_0^\infty e^{-2\lambda t} |u'_j - u_j|^2 (x, t) \, dx \, dt \]
\[ = \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty |\bar{u}'_j - \bar{u}_j|^2 (x, \zeta) \, dx \, d\beta \]
\[ \leq O(1) \varepsilon \int_{-\infty}^\infty (|\zeta| + |\zeta|^4) |\tilde{b}(\zeta)|^2 d\beta \]
\[ \leq O(1) \varepsilon \int_0^\infty e^{-2\lambda t}(b(t)^2 + b'(t)^2 + b''(t)^2) \, dt. \quad (202) \]
The convergence in the \( v \) variables can be proved similarly. Furthermore, it is clear that (24) holds with \( U^{b,L} \) replaced by \( (u^{b,L} + u^{R,L}, 0) \). This verifies the two-scale boundary layer structure \( (u^{b,L}(x/c, t) + u^{R,L}(x/\sqrt{\varepsilon}, t)) \) in the general system case.

Summarizing the above results, we finish the proof of Theorem 1.2 in the case of homogeneous initial data \( (U_0(x) = 0) \).

5. NONZERO INITIAL DATA EFFECT

Now we look at the nonzero initial data effect and prove Theorem 1.1 and Theorem 1.2 for the full IBVP
\[ \partial_t U^v + A \partial_x U^v = \frac{1}{\varepsilon} S U^v, \]
\[ U^v(x, 0) = U_0(x), \quad (203) \]
\[ BU^v(0, t) = b(t). \]
First, we note that, by linearity, we can break up the above IBVP into two simpler problems, one with homogeneous initial condition
\[
\begin{align*}
\partial_t U^x + A \partial_x U^x &= \frac{1}{\varepsilon} SU^x, \\
U^x(x, 0) &= 0, \\
BU^x(0, t) &= b(t),
\end{align*}
\]
(204)
and the other with homogeneous boundary condition
\[
\begin{align*}
\partial_t U^x + A \partial_x U^x &= \frac{1}{\varepsilon} SU^x, \\
U^x(x, 0) &= U_0(x), \\
BU^x(0, t) &= 0.
\end{align*}
\]
(205)
The first one, (204), has been studied extensively in Section 3 (for \( n = 1 \)) and in Section 4 (for \( n > 1 \)). Our focus in this final section is on the second one, (205). The proof of Theorem 1.1 and 1.2 for the full IBVP (203) will be complete if we can prove the same theorems for (205).

5.1. Solution by Laplace Transform

We shall consider the case \( n = 1 \) first. Again, we solve the IBVP (205) explicitly by the method of Laplace transform. With
\[
\begin{align*}
L(U^x) &= \mathcal{L}(U^x) = \int_0^\infty e^{-\xi t} U^x(x, t) \, dt, \quad \text{Re} \, \xi > 0, \\
U^x(x, 0) &= U_0(x),
\end{align*}
\]
(206)
and
\[
U^x(x, 0) = U_0(x), \quad \text{we get}
\]
\[
\mathcal{L}(\partial_t U^x) = \xi \mathcal{L}(U^x) - U^x(x, 0) = \xi \mathcal{L}(U^x) - U_0(x).
\]
(207)
Therefore, (205) becomes
\[
\partial_x \mathcal{L}(U^x) = \frac{1}{\varepsilon} M(\xi) \mathcal{L}(U^x) + A^{-1} U_0,
\]
(208)
where \( M(\xi) \) is the same as in (79).

The general solution \( \mathcal{L}(U^x) \) can be represented as
\[
\mathcal{L}(U^x) = e^{(1/\varepsilon) M(\xi) x} \left( \mathcal{L}(U^x(0, \xi)) + \int_0^x e^{-((1/\varepsilon) M(\xi) y) A^{-1} U_0(y)} \, dy \right),
\]
(209)
where

\[ e^{M(t)} = e^{m(t)} \Phi_+ (t) + e^{-m(t)} \Phi_- (t), \]

\[ \Phi_+ (t) = \frac{1}{g(t) - k(t)} \left( 1 - \frac{k(t)}{g(t)} \right), \]

\[ \Phi_- (t) = \frac{1}{g(t) - k(t)} \left( 1 - \frac{-k(t)}{g(t)} \right), \]

\[ k(t) = \frac{a}{g(t)} \left( 1 + \frac{\lambda}{1 + \xi} \right), \]

and \( g(t) \) is the same as in (96). Therefore,

\[ U(x, t) = e^{m(t)} \Phi_+ (t) \left( U(t, 0, t) + \int_0^t e^{-m(t)} U(t, y, t) dy \right) \]

\[ + e^{-m(t)} \Phi_- (t) \left( U(t, 0, t) + \int_0^t e^{-m(t)} U(t, y, t) dy \right). \]

The boundary data \( \tilde{U}(0, t) \) remains to be determined. The boundary condition at \( x = 0 \) supplies one such condition

\[ B \tilde{U}(0, t) = B_u \tilde{u}(0, t) + B_v \tilde{v}(0, t) = 0, \]

and the other condition comes from

\[ \Phi_+ (0) \left( \tilde{U}(0, t) + \int_0^t e^{-m(t)} U(t, y, t) dy \right) = 0. \]

We note that (216) can be viewed as a natural boundary condition at \( x = +\infty \), i.e., \( \tilde{U}(+\infty, t) = 0 \) since we expect \( \tilde{U}(\cdot, t) \in L^2(\mathbb{R}^+) \). We further remark that if (208) is to be solved on the whole line \( x \in \mathbb{R} \), the boundary condition (215) at \( x = 0 \) should then be replaced by the boundary condition \( \tilde{U}(-\infty, t) = 0 \) at \( x = -\infty \). This is the case when one solves the Cauchy problem (223) by Laplace transform.

For convenience, we denote

\[ \tilde{u}(t) = \int_0^t e^{-m(t)} U(t, y, t) dy \]

\[ \tilde{v}(t) = \int_0^t e^{-m(t)} \left( u(t, y) - \frac{1}{a} g(t) v(y) \right) dy. \]

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Then, (216) can be rewritten as
\[ g(\varepsilon_\xi) \hat{u}(0, \xi) - \hat{v}(0, \xi) = \hat{w}(\xi). \] (218)
Again, the SKC guarantees the solvability of \( \hat{U}(0, \xi) \) from (215) and (218).
Therefore, we obtain
\[ \hat{U}(0, \xi) = \frac{\hat{w}(\xi)}{B_u + B_v g(\varepsilon_\xi)} \left( B_u - B_v \right). \] (219)
and
\[ \hat{U}(x, \xi) = \frac{1}{g(\varepsilon_\xi) - k(\varepsilon_\xi)} \times \left( \left( \frac{1}{k(\varepsilon_\xi)} \right) \int_0^x e^{\mu_+ (\varepsilon_\xi)(x - y)/\alpha} \left( u_0(y) - \frac{1}{\alpha} g(\varepsilon_\xi) \right) v_0(y) \, dy + \left( \frac{1}{g(\varepsilon_\xi)} \right) \int_0^x e^{\mu_- (\varepsilon_\xi)(x - y)/\alpha} \left( u_0(y) - \frac{1}{\alpha} k(\varepsilon_\xi) \right) v_0(y) \, dy \right) \right) \]
\[ - \frac{B_u + B_v k(\varepsilon_\xi)}{B_u + B_v g(\varepsilon_\xi)} \hat{U}(\xi) e^{\mu_+ (\varepsilon_\xi) x/\alpha} \left( \frac{1}{g(\varepsilon_\xi)} \right). \] (220)

### 5.2. Purely Initial Data Effect and Boundary Effect

The solution representation of \( \hat{U}^t \) for (205) as in the above is much more complicated than that for the homogeneous initial data problem (204) which we considered earlier. Indeed, a straightforward derivation of the \( \varepsilon \)-uniform estimates in (11) would be extremely difficult, if at all possible. See [11] for the treatment of general nonzero initial data for classical hyperbolic initial-boundary value problems.

The above solution \( \hat{U}^t \) consists of two parts:

\[ \hat{U}^t_1(x, \xi) = \frac{1}{g(\varepsilon_\xi) - k(\varepsilon_\xi)} \times \left( \left( \frac{1}{k(\varepsilon_\xi)} \right) \int_0^x e^{\mu_+ (\varepsilon_\xi)(x - y)/\alpha} \left( u_0(y) - \frac{1}{\alpha} g(\varepsilon_\xi) \right) v_0(y) \, dy + \left( \frac{1}{g(\varepsilon_\xi)} \right) \int_0^x e^{\mu_- (\varepsilon_\xi)(x - y)/\alpha} \left( u_0(y) - \frac{1}{\alpha} k(\varepsilon_\xi) \right) v_0(y) \, dy \right) \]
\[ - \frac{B_u + B_v k(\varepsilon_\xi)}{B_u + B_v g(\varepsilon_\xi)} \hat{U}(\xi) e^{\mu_+ (\varepsilon_\xi) x/\alpha} \left( \frac{1}{g(\varepsilon_\xi)} \right). \] (221)

and

\[ \hat{U}^t_2(x, \xi) = - \frac{\hat{w}(\xi)}{g(\varepsilon_\xi) - k(\varepsilon_\xi)} \frac{B_u + B_v k(\varepsilon_\xi)}{B_u + B_v g(\varepsilon_\xi)} e^{\mu_- (\varepsilon_\xi) x/\alpha} \left( \frac{1}{g(\varepsilon_\xi)} \right). \] (222)
It is a little surprising to note that the first part $\tilde{U}_1^r$ involves only the initial data and has nothing to do with the boundary condition while the second part $\tilde{U}_2^r$ incorporates both initial data and boundary condition. We point out that $\tilde{U}_1^r$ represents the purely initial data effect in the IBVP (205) and $\tilde{U}_2^r$ reflects the boundary effect of the initial data. Our approach depends on a separate treatment of these effects.

Indeed, it is easy to verify that, $\tilde{U}_1^r$ corresponds to the Laplace transform of the solution $U_1^r$ of the following extended Cauchy problem

$$\partial_t U_1^r + A \partial_x U_1^r = \frac{1}{\varepsilon} \mathcal{L}U_1^r,$$
$$U_1^r(x, 0) = U_0(x),$$

(223)

and $\tilde{U}_2^r$ corresponds to the Laplace transform of the solution $U_2^r$ of the following IBVP of type (204)

$$\partial_t U_2^r + A \partial_x U_2^r = \frac{1}{\varepsilon} U_2^r,$$
$$U_2^r(x, 0) = 0,$$
$$BU_2(0, t) = -BU_2^r(0, t),$$

(224)

where in (223), we extend the initial data $U_0(x)$ to the whole line by setting $U_0(x) = 0$ for $x < 0$.

The Cauchy problem (223) is more conveniently studied by Fourier transform (in $x$) than by Laplace transform (in $t$). Therefore, the difficulty arising from estimating $\tilde{U}_1^r$ directly can now be avoided since, from (53), we have

$$\int_{-\infty}^{\infty} |U_1^r(x, t)|^2 dx \leq O(1) \int_{-\infty}^{\infty} |U_0(x)|^2 dx \quad \text{for all} \quad t > 0.$$ (225)

Integrating with respect to $t$ and noticing $U_0(x) = 0$ for $x < 0$, we get

$$\int_{0}^{\infty} \int_{-\infty}^{\infty} e^{-2\lambda t} |U_1^r(x, t)|^2 dx dt \leq O(1) \lambda^{-1} \int_{0}^{\infty} |U_0(x)|^2 dx.$$ (226)

Thus, by Parseval's equality, we obtain

$$\int_{0}^{\infty} \int_{-\infty}^{\infty} |\tilde{U}_1^r(x, \chi + i\beta)|^2 dx d\beta \leq O(1) \chi^{-1} \int_{0}^{\infty} |U_0(x)|^2 dx.$$ (227)
This gives the desired uniform estimate on $\tilde{U}_1^\varepsilon$. We remark that the above approach of resorting to the Cauchy problem \((223)\) is not only a technical convenience in estimating $\tilde{U}_1^\varepsilon$. More importantly, in the present case of $n = 1$, the extended Cauchy problem \((223)\) admits exactly the same asymptotic limit $U(x, t)$ and leading initial layer $U^{\varepsilon}\varepsilon(x, t/\varepsilon)$ as those of the IBVP \((205)\) (restricted to $x > 0$). Neither the equilibrium limit nor the initial layer is affected by the zero initial data $U_0(x) = 0$ on $x < 0$ in \((223)\).

The convergence of $U_1^\varepsilon \to U$ as $\varepsilon \downarrow 0$, the optimal convergence rate and the corresponding initial layer behavior have been proved in Theorem 1.3 in a slightly different sense. However, we note that, the same proof of \((26)\) in Section 2 can be applied to show

\[
\int_0^\infty \int_0^\infty |U_1^\varepsilon - U|^2(x, t) e^{-2\sigma t} \, dx \, dt \to 0 \quad \text{as} \quad \varepsilon \downarrow 0. \tag{228}
\]

The other two estimates, namely,

\[
\int_0^\infty \int_0^\infty |U_1^\varepsilon - U|^2(x, t) e^{-2\sigma t} \, dx \, dt \leq O(1) \varepsilon^2 \|U_0\|_{L^2}^2 + O(1) \varepsilon \|v_0 - \dot{\lambda}u_0\|_{L^2}^2, \tag{229}
\]

and

\[
\int_0^\infty \int_0^\infty |U_1^\varepsilon - U^{\varepsilon}\varepsilon|^2(x, t) e^{-2\sigma t} \, dx \, dt \leq O(1) \varepsilon^2 \|U_0\|_{L^2}^2, \tag{230}
\]

can be obtained by integrating \((27)\) and \((28)\) with respect to $t$ directly.

The boundary effect of \((205)\), on the other hand, is reflected in $U_I^\varepsilon$. We note that the boundary layer vanishes in the cases $\lambda > 0$ and $\lambda = 0$, but is nontrivial when $\lambda < 0$:

\[
u_b^\lambda(x, t) = -\frac{B_u + \dot{\lambda}B_v}{B_a} e^{(\lambda \omega(s/c) - \dot{\lambda}t)} \quad \text{and} \quad v_b^\lambda = 0 \tag{231}
\]

with Laplace transform

\[
u_b^\lambda(x, \zeta) = \frac{B_u + \dot{\lambda}B_v}{\lambda} \int_0^\infty e^{(\lambda \omega(s/c) - \dot{\lambda}t)} \eta \, dy, \quad \tilde{v}_b^\lambda = 0. \tag{232}
\]

Both $U'(0, t)$ and $U_I^\varepsilon$ are closely related to $\tilde{\nu}^\varepsilon$, see \((217)\), \((219)\)–\((220)\). Indeed, the following estimate on $\tilde{\nu}^\varepsilon$

\[
\int_{-\infty}^{\infty} |\tilde{\nu}^\varepsilon(x + i\beta)|^2 \, d\beta \leq O(1) \int_0^\infty |U_\Gamma(x)|^2 \, dx \quad \text{for all} \quad x \geq 0 \tag{233}
\]
turns out to be essential to the rest of our proof. We remark that in the special case \( \lambda = \frac{\xi}{\sqrt{a}} \), the above estimate follows from Parseval’s identity since \( \mu_\pm(\xi) = \frac{\xi}{\sqrt{a}} \) is linear in \( \xi \). In the general case when \( \mu_\pm(\xi) \) is nonlinear, (233) can be viewed as a version of Parseval’s identity along nonstandard integral curves. This is best seen by taking \( \nu_i(x) \equiv 0 \):

\[
\int_{-\infty}^{\infty} \left( \int_0^{\infty} e^{-\mu_\pm(\xi) y} u_d(y) \, dy \right)^2 \, d\beta \leq O(1) \int_0^{\infty} |u_d(y)|^2 \, dy. \tag{234}
\]

With (233), the desired boundary estimate

\[
\int_0^{\infty} |\hat{U}(0, \xi)|^2 \, d\beta \leq O(1) \int_0^{\infty} |U_d(x)|^2 \, dx \tag{235}
\]

then follows immediately from the SKC.

The estimates (233) and (235) are actually equivalent. This is because

\[
0 < c_1 \leq |B_\pm + B_\pm g(\xi)| \leq c_2 < \infty \quad \text{for all } \xi \text{ with } \Re \xi \neq 0, \tag{236}
\]

and thus

\[
|\hat{U}(0, \xi)| \approx |\hat{w}(\xi)|. \tag{237}
\]

The proof of (233) is highly nontrivial. A naive estimate would yield

\[
|\hat{w}(\xi)|^2 = \left| \int_0^{\infty} e^{-\mu_\pm(\xi) y} \left( u_d(y) - \frac{1}{a} g(\xi^2) \nu_i(y) \right) \, dy \right|^2 \\
\leq \int_0^{\infty} |e^{-\mu_\pm(\xi) y}|^2 \, dy \int_0^{\infty} \left| u_d(y) - \frac{1}{a} g(\xi^2) \nu_i(y) \right|^2 \, dy \\
\leq O(1) \frac{\varepsilon}{2 \Re \mu_\pm(\xi^2)} \int_0^{\infty} |U_d(x)|^2 \, dx. \tag{238}
\]

However, it even fails to show the \( L^2 \) integrability of \( \hat{w}(\alpha + i\beta) \) for fixed \( \varepsilon > 0 \). Surprisingly enough, the only part involving \( \xi \) (or \( \beta \)) in the last term of (238), namely, \( \frac{\varepsilon}{2 \Re \mu_\pm(\xi^2)} \), does not even decay as \( \beta \to \pm \infty \), see (87).

We will take a different approach to prove (233). The strategy is to go back to the original IBVP (205) and prove the boundary estimate directly by energy method. The energy method has its own limitations. We certainly wouldn’t expect it to work for the whole class of boundary conditions satisfying our SKC. But, if it works, even for just one boundary condition (which obviously has to satisfy SKC), that will be enough for us since the
estimate (233) is independent of any particular boundary condition and is always equivalent to (235), or
\[
\int_0^\infty |U'(0, t)|^2 e^{-2\mu} dt \lesssim O(1) \int_0^\infty |U_d(x)|^2 dx \tag{239}
\]
as long as the boundary condition satisfies SKC.

5.3. Weighted \(L^2\) Energy Estimate

The idea [7] is to find a suitable symmetrizer, a symmetric (or Hermitian) positive definite matrix \(H\), such that \(HA\) is symmetric, \(HS\) (or its symmetric part) is negative definite, and the boundary integral has the proper sign.

Therefore, we choose
\[
H = \begin{pmatrix} a & -\lambda \\ -\lambda & 1 \end{pmatrix}. \tag{240}
\]

Now, multiply (205) by \(e^{-2\mu(t)} U'\) and integrate over \([0, T] \times [0, \infty)\), we get
\[
\frac{1}{2} \int_0^\infty (U', HU')(x, T) e^{-2\mu t} dx
+ \frac{1}{2} \int_0^T \int_0^\infty (U', HU')(x, t) e^{-2\mu t} dx dt
+ \frac{1}{2} \int_0^T \int_0^\infty (v', \lambda^2 - \lambda a')^2 (x, t) e^{-2\mu t} dx dt
+ \frac{1}{2} \int_0^T \int_0^\infty (a\lambda'(u')^2 - 2au'v' + \lambda(v')^2)(0, t) e^{-2\mu t} dt
= \frac{1}{2} \int_0^\infty (U_d(x), HU_d(x)) dx. \tag{241}
\]

The first three terms in (241) are all non-negative. The crucial part is the boundary integral term
\[
\frac{1}{2} \int_0^T (a\lambda'(u')^2 - 2au'v' + \lambda(v')^2)(0, t) e^{-2\mu t} dt. \tag{242}
\]
In order for the energy method to work, the boundary condition has to satisfy

\[ a \lambda \dot{u}(0, t)^2 - 2a \dot{u}(0, t) \ v(0, t) + \lambda v(0, t)^2 \geq c \ |U^\prime(0, t)|^2 \]  

(243)

for some positive constant \( c \) whenever

\[ B_u \dot{u}(0, t) + B_v v(0, t) = 0. \]  

(244)

For such boundary conditions, the boundary estimate (239) follows then easily from (241) by taking the limit \( T \to \infty \).

There are plenty of boundary conditions satisfying the above requirement

\[ \frac{B_u}{B_v} > \frac{a}{\lambda} \left[ 1 - \sqrt{1 - \frac{\lambda^2}{a}} \right] \quad \text{or} \quad \frac{B_u}{B_v} < -\frac{a}{\lambda} \left[ 1 + \sqrt{1 - \frac{\lambda^2}{a}} \right] \quad \text{if} \quad \lambda > 0, \]  

(245)

\[ -\frac{a}{\lambda} \left[ 1 - \sqrt{1 - \frac{\lambda^2}{a}} \right] < \frac{B_u}{B_v} < -\frac{a}{\lambda} \left[ 1 + \sqrt{1 - \frac{\lambda^2}{a}} \right] \quad \text{if} \quad \lambda < 0, \]  

(246)

and

\[ \frac{B_u}{B_v} > 0 \quad \text{if} \quad \lambda = 0. \]  

(247)

These are enough for us to prove (233), though they are only a subclass of the SKC. By our previous argument, the boundary estimate (239) now holds for all boundary conditions satisfying SKC.

5.4. **Stiff Well-Posedness**

Next, we show the uniform estimate on \( \tilde{U}_\eta^\prime \):

\[ \left( \int_0^\infty \int_{-\infty}^\infty |\tilde{U}_\eta^\prime(x, \xi)|^2 \, dx \, d\eta \right)^{1/2} \leq O(1) \]  

(248)

First, we consider the case \( \dot{\lambda} \neq 0 \). Note that with \( \dot{\lambda} \neq 0 \), we have

\[ \left| \frac{1}{g(\xi) - k(\xi)} \right| = \left| \frac{1 + \xi}{\sqrt{\lambda^2 + 4a\xi(1 + \xi)}} \right| \leq \frac{1 + |\xi|}{\sqrt{\lambda^2 + 4a |\xi|^2}} \leq O(1). \]  

(249)
Therefore, by (233), we get

\[ \left| \mathcal{U}(x, t) \right|^2 \leq O(1) \sup_{\beta} \frac{e}{-2 \, \text{Re} \, \mu_{-}(\xi)} \int_{-\infty}^{\infty} |\mathcal{W}(\xi)|^2 \, d\beta \]

\[ \leq (O(1) \varepsilon) \int_{0}^{\infty} |U_{d}(x)|^2 \, dx \quad \lambda < 0, \]

\[ (O(1) \varepsilon) \int_{0}^{\infty} |U_{d}(x)|^2 \, dx \quad \lambda > 0. \]  

(250)

The case \( \lambda = 0 \) can be treated slightly differently. Instead of using (233) and estimating \( \mathcal{U} \) directly, we make use of the energy equality (241) one more time. Since the matrix \( \mathcal{H} \) is now strictly positive definite (\( \sigma > 0 \)) and we have already shown the boundary estimate (239) for all boundary conditions satisfying SKC, (241) now yields

\[ \left| \mathcal{U}(x, t) \right|^2 \leq O(1) \alpha^{-1} \int_{0}^{\infty} |U_{d}(x)|^2 \, dx, \]

(251)

or equivalently,

\[ \int_{0}^{\infty} \int_{-\infty}^{\infty} |\mathcal{U}^{\varepsilon}(x, \eta)|^2 \, dx \, d\beta \leq O(1) \alpha^{-1} \int_{0}^{\infty} |U_{d}(x)|^2 \, dx. \]  

(252)

On the other hand, the same estimate holds for \( \mathcal{U}^{\varepsilon}_1 \)

\[ \int_{0}^{\infty} \int_{-\infty}^{\infty} |\mathcal{U}^{\varepsilon}_1(x, \xi)|^2 \, dx \, d\beta \leq O(1) \alpha^{-1} \int_{0}^{\infty} |U_{d}(x)|^2 \, dx \]  

(253)

independently of any particular boundary condition.

Since \( \mathcal{U}^{\varepsilon}_1 = \mathcal{U}^{\varepsilon} - \mathcal{U}^{\varepsilon}_1 \), we get from the above

\[ \int_{0}^{\infty} \int_{-\infty}^{\infty} |\mathcal{U}^{\varepsilon}_1(x, \xi)|^2 \, dx \, d\beta \leq O(1) \alpha^{-1} \int_{0}^{\infty} |U_{d}(x)|^2 \, dx \]

(254)

for all boundary conditions satisfying SKC.

5.5. Asymptotic Convergence

The goal of this subsection is to establish the expected convergence of \( U^{\varepsilon}_n \rightarrow 0 \) (or equivalently, \( \mathcal{U}^{\varepsilon}_n \rightarrow 0 \) as \( \varepsilon \downarrow 0 \)) and justify the boundary layer given in (231) for the case \( \lambda < 0 \).

The estimate in (250) already shows the the convergence of \( \mathcal{U}^{\varepsilon}_n \rightarrow 0 \) in the case \( \lambda < 0 \). For \( \lambda \geq 0 \), the uniform estimate for \( \mathcal{U}^{\varepsilon}_n \) which we proved in
the last subsection can be improved by a simple integration by part (assuming $U_0 \in H^l(\mathbb{R}^+)$ and $U_0(0) = 0$)

$$\hat{w}(\xi) = \int_0^\infty e^{-\mu_+(\frac{\xi}{\varepsilon}) y^\varepsilon} \left( u_0(y) - \frac{1}{\alpha} g(\xi y) v_0(y) \right) dy$$

$$= \frac{e}{\mu_+(\frac{\xi}{\varepsilon})} \int_0^\infty e^{-\mu_+(\frac{\xi}{\varepsilon}) y^\varepsilon} \left( u_0(y) - \frac{1}{\alpha} g(\xi y) v_0(y) \right) dy.$$  \hspace{1cm} (255)

The desired convergence comes from the integrated factor $\frac{e}{\mu_+(\frac{\xi}{\varepsilon})}$ since

$$\left| \frac{e}{\mu_+(\frac{\xi}{\varepsilon})} \right| \leq \sup_{\beta} \frac{e}{\Re \mu_+(\frac{\xi}{\varepsilon})} \leq \frac{O(1)}{e} \left( \lambda > 0 \right),$$

$$\leq \frac{O(1)}{\sqrt{e}} \left( \lambda = 0 \right).$$  \hspace{1cm} (256)

All other parts in $\hat{w}$ or $\hat{U}_H$ remain the same except that $U_0$ is replaced by $U_0$. Therefore, we have

$$\int_0^\infty \int_{-\infty}^{\infty} |\hat{U}_H|^2 d\beta dx = \begin{cases} O(1) e^2 \int_0^\infty |U_0^\prime(x)|^2 dx & (\lambda > 0), \\ O(1) \varepsilon \int_0^\infty |U_0^\prime(x)|^2 dx & (\lambda = 0). \end{cases}$$  \hspace{1cm} (257)

Arbitrary convergence rate can be achieved at the cost of higher Sobolev norms and stronger compatibility assumptions. This is not surprising since no boundary layer develops when $\lambda \geq 0$.

Finally by noticing

$$\left| \frac{e}{\mu_+(\frac{\xi}{\varepsilon})} \right| \leq \frac{1}{k(\xi)} \leq O(1) \left| \frac{1}{\xi} \right|^{-1},$$  \hspace{1cm} (258)

we can get more from the equation (255):

$$\int_{-\infty}^{\infty} |\hat{w}(\xi)|^2 d\beta \leq O(1) \int_0^\infty |U_0^\prime(x)|^2 dx.$$  \hspace{1cm} (259)

This is closely related to the Laplace transform of a derivative and is essential in proving the boundary layer estimate

$$\int_0^\infty \int_{-\infty}^{\infty} |\hat{U}_H^\prime(x, \xi) - \hat{U}_H(x, \xi)|^2 dx d\beta \leq O(1) \varepsilon^2 \|U_0\|_{L^2}^2.$$  \hspace{1cm} (260)

in the case $\lambda < 0$. Details are omitted.

5.6. Case $n > 1$

We now turn to prove Theorem 1.2 for the IBVP (205) in the general case of $n > 1$. As before, we assume (149) and (150).
Again, the solution $U^*$ of the IBVP (205) consists of two parts

$$U^* = U^*_1 + U^*_2,$$

where $U^*_1$ solves the following extended Cauchy problem

$$\partial_t U^*_1 + A \partial_x U^*_1 = \frac{1}{\varepsilon} SU^*_1,$$  \hspace{1cm} (262)$$

$$U^*_1(x, 0) = \begin{cases} U_0(x) & x \geq 0, \\ 0 & x < 0, \end{cases}$$

and $U^*_2$ solves the following IBVP with homogeneous initial data

$$\partial_t U^*_2 + A \partial_x U^*_2 = \frac{1}{\varepsilon} SU^*_2,$$  \hspace{1cm} (263)$$

$$U^*_2(x, 0) = 0, \hspace{1cm} BU^*_2(0, t) = - BU^*_1(0, t).$$

The uniform $L^2$ estimates, the corresponding asymptotic convergence and initial layer property for the Cauchy problem (262) are all direct consequences of Theorem 1.3, see also Section 5.2. The main difficulty is again with the IBVP (263).

Both (262) and (263) can be solved by Laplace transform. In particular, we have

$$\tilde{U}^*_1(x, \xi) = - \left( I_n \begin{pmatrix} G(\xi) \end{pmatrix} \right) e^{\nu(\xi) \xi/2} (B_u + \nu_x(G(\xi)))^{-1} (B_u + \nu_x(K(\xi)))^{-1} \tilde{W}(\xi),$$

where

$$G(\xi) = \text{diag}\{g_1(\xi), g_2(\xi), \ldots, g_n(\xi)\},$$

$$K(\xi) = \text{diag}\{k_1(\xi), k_2(\xi), \ldots, k_n(\xi)\},$$

$$\tilde{W}(\xi) = \text{diag}\{\tilde{w}_1(\xi), \tilde{w}_2(\xi), \ldots, \tilde{w}_n(\xi)\},$$

$$g_j(\xi) = \frac{\lambda_j + \sqrt{\lambda_j^2 + 4a(1 + \xi)}}{2(1 + \xi)},$$

$$k_j(\xi) = \frac{\lambda_j - \sqrt{\lambda_j^2 + 4a(1 + \xi)}}{2(1 + \xi)},$$

$$\tilde{w}_j(\xi) = \int_0^{\infty} e^{-\nu^*(\xi) y} \left( u_{0,j}(y) - \frac{1}{a} g_j(\xi) v_{0,j}(y) \right) dy.$$
Using the matrix identity

\[(B_u + B_v G)^{-1} (B_u + B_v K)(G - K)^{-1} = (G - K)^{-1} - (B_u + B_v G)^{-1} B_v,\] (271)

we can rewrite (264) as

\[\tilde{U}_n^r(x, \xi) = -\left( I_n \right) \left( \frac{e^{\nu \cdot (\xi \cdot x) / \nu G(\xi \cdot \xi) - K(\xi \cdot \xi)^{-1} \tilde{W}^n(\xi)}}{G(\xi \cdot \xi)} \right) e^{\nu \cdot (\xi \cdot x) / \nu G(\xi \cdot \xi) - K(\xi \cdot \xi)^{-1} B_v \tilde{W}^n(\xi)}. \] (272)

Now the uniform $L^2$ estimate on $\tilde{U}_n^r$ in the present case of $n > 1$

\[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\tilde{U}_n^r(x, \xi)|^2 \, dx \, d\beta \leq O(1) \int_{0}^{\infty} |U_0(x)|^2 \, dx \] (273)

follows from similar estimates for $n = 1$, see (233) and (248).

Combining the estimates on $\tilde{U}_n^r$ and $\tilde{U}_n^l$, we obtain

\[\int_{0}^{\infty} \int_{0}^{\infty} |U^r(x, t)|^2 \, e^{-2\beta t} \, dx \, dt = \frac{1}{2\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} |\tilde{U}_n^r(x, \xi)|^2 \, dx \, d\beta \leq O(1) \int_{0}^{\infty} |U_0(x)|^2 \, dx. \] (274)

On the other hand, there is no difficulty in obtaining the following uniform boundary estimate

\[\int_{0}^{\infty} |U_0(0, t)|^2 \, e^{-2\beta t} \, dt \leq O(1) \int_{0}^{\infty} |U_0(x)|^2 \, dx \]

since

\[\tilde{U}_n^a(0, \xi) = \left( (B_u + B_v G(\xi \cdot \xi))^{-1} B_v \right) (G(\xi \cdot \xi))^{-1} (B_u + B_v G(\xi \cdot \xi))^{-1} B_v \tilde{W}^n(\xi). \] (275)

This finishes the proof of stiff well-posedness.

The remaining issues (asymptotic convergence, boundary layer, etc.) can be studied in the same way as in the case $n = 1$. However, there are striking differences between the general case of $n > 1$ and the simplest case of $n = 1$ which we studied earlier in this section. First, due to the mixing of the
boundary conditions, the adjusted IBVP (263) will also produce a non-trivial equilibrium limit $U_{II}$ as $\epsilon \downarrow 0$. This part is not reflected in the equilibrium limit $U_I$ of the Cauchy problem (262). Therefore the boundary effect $U_{Ii}$ is no longer negligible and the equilibrium limit $U$ in Theorem 1.2 has to be replaced by $U_I + U_{II}$. Secondly, the uniformly characteristic boundary layer modes may also be excited.

The equilibrium limit $U_{II}$, the non-characteristic boundary layer $U^{b.l.}$ and the uniformly characteristic boundary layer $U^{u.c.}$ can then be determined explicitly (and uniquely) by the same matched asymptotic expansion method we used in Section 4. The only difference is that the boundary data $b(t)$ is now replaced by

$$ b_{II}(t) = -BU_I(0, t) = -(B_u + B_v A_{-}) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ u_{0,q+1}(-\lambda_{q+1} t) \\ \vdots \\ u_{0,d}(-\lambda_d t) \end{pmatrix} \quad (276) $$

with

$$ \tilde{b}_{II}(\xi) = -(B_u + B_v A_{-}) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{\lambda_{q+1}} \int_{0}^{\infty} e^{\sqrt{\gamma} \xi + i y} u_{0,q+1}(\xi) \, dy \\ \vdots \\ \frac{1}{\lambda_d} \int_{0}^{\infty} e^{\sqrt{\gamma} \xi + i y} u_{0,d}(\xi) \, dy \end{pmatrix} \quad (277) $$

Therefore, we have

$$ \tilde{a}_{II,j}(x, \xi) = \begin{cases} e^{-\xi \lambda_j} e_j(B_u + B_v A_{-})^{-1} \tilde{b}_{II}(\xi) & (\lambda_j > 0), \\ 0 & (\lambda_j \leq 0), \end{cases} \quad (278) $$

$$ \tilde{a}_{II,j}(x, \xi) = \begin{cases} e^{i \sqrt{\gamma} \xi + i y_j} e_j(B_u + B_v A_{-})^{-1} \tilde{b}_{II}(\xi) & (\lambda_j < 0), \\ 0 & (\lambda_j \geq 0), \end{cases} \quad (279) $$

$$ \tilde{a}_{II,\lambda}(x, \xi) = \begin{cases} e^{-\sqrt{\gamma} \xi + i y_\lambda} e_j(B_u + B_v A_{-})^{-1} \tilde{b}_{II}(\xi) & (\lambda_j = 0), \\ 0 & (\lambda_j \neq 0). \end{cases} \quad (280) $$

All these are consistent with the solution representation of $\tilde{U}_{II}$ in (264) and can be rigorously justified in the same way as before. Details are omitted.
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