Real Analysis Comprehensive Examination
Sept. 17, 2016

Choose six of the nine problems. On the first page of your work, please write the numbers of the problems that you want graded.

Notations: for $A$ a measurable subset of $\mathbb{R}^d$, $m(A)$ is the Lebesgue measure of $A$. The characteristic function $\chi_A(x)$ is the function that equals 1 for $x \in A$ and equals zero for $x \notin A$.

1. Given $\epsilon > 0$ show that there exists a closed subset $E$ of the interval $[0, 1]$ containing no open intervals but whose measure $m(E)$ exceeds $1 - \epsilon$.

2. A function $f(x)$ on $\mathbb{R}^d$ taking values in the extended real line $\mathbb{R} \cup \{\pm \infty\}$ is measurable if the set $\{x : f(x) < a\}$ is a measurable subset of $\mathbb{R}^d$ for every real number $a$. Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions on $\mathbb{R}^d$. Show the following:
   (i) $\sup_{1 \leq n < \infty} f_n$ is measurable;
   (ii) $\inf_{1 \leq n < \infty} f_n$ is measurable;
   (iii) Assuming (i) and (ii) show that $\limsup_{n \to \infty} f_n(x)$ is measurable.

3. Given a Lebesgue measurable set $E$ in $\mathbb{R}^d$ show the following: for any $\epsilon > 0$ there exist an open set $G$ and a closed set $F$, $F \subset E \subset G$, with $m(G \setminus F) < \epsilon$. You may assume the definition of Lebesgue measurability, that the collection of measurable sets is a $\sigma$–algebra, and that Lebesgue measure is subadditive.

4. Suppose $f$ and $g$ are continuous functions on $\mathbb{R}^d$ which are equal almost everywhere. Prove that they are equal everywhere.

5. Prove that a sequence of functions which converges in $L^1(\mathbb{R}^d)$ must converge in measure, but that the converse is false.

6. Suppose that $f$ is a non–negative measurable function on the interval $[0, 1]$, and suppose that
   $$\int_0^1 x^{\epsilon - 1} f(x) \, dx \leq 10,$$
   for all $\epsilon \in (0, 1)$. Show that $\int_0^1 x^{-1} f(x) \, dx \leq 10$

7. Let $E_j$, $j = 1, 2, \ldots$, be disjoint Lebesgue measurable subsets of $\mathbb{R}^d$, and let $E$ be their union. Let $f$ be a measurable function satisfying
   $$\sum_{j=1}^\infty \int_{E_j} |f(x)| \, dm$$
   is finite. Show the following:
   (i) $\chi_E(x) f(x)$ is integrable;
   (ii) $\int_E f \, dm = \sum_{j=1}^\infty \int_{E_j} f \, dm$.

8. Prove that if $f \in L^1(\mathbb{R})$ and
   $$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) \exp(-2\pi i x \xi) \, dx,$$
   then $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

9. Let $f$ be a continuous function of $\mathbb{R}$ with support contained in the interval $(-1, 1)$. Show that $f(\delta x)$ converges in the $L^1(\mathbb{R})$ norm to $f$ as $\delta \to 1$. 