Do six of the following eight problems. Clearly mark the 6 problems you wish to be graded.

1. Solve the following initial value problem

\[ xu_x + u_y^2 = 2u \]

with the initial condition \( u(x, 0) = x^2 + a \) using the method of characteristics. For what constant \( a \) is there a solution? Is the solution unique?

2. The equation below models the change of the density of cars on a one-way highway:

\[ (c\rho(1 - \rho/\rho_{\text{max}}))_x + \rho_y = 0, \]

where the constant \( \rho_{\text{max}} \) denotes the maximum density of cars on the highway (i.e., under bumper-to-bumper conditions), and the constant \( c \) is the speed limit of the highway. Suppose the initial density is

\[ \rho(x, 0) = \begin{cases} \frac{1}{2}\rho_{\text{max}} & \text{for } x < 0 \\ \rho_{\text{max}} & \text{for } x > 0. \end{cases} \]

Find the shock curve, and describe the weak solution. Interpret your result for the traffic flow.

3. Solve the following initial–boundary value problem for the unknown function \( u(x, t) \). State your answer as explicitly as possible, and show all details in the derivation.

\[
\begin{align*}
  u_t(x, t) &= u_{xx}(x, t) \quad \text{for } 0 < x < \pi \text{ and } t > 0, \\
  u(0, t) &= 0, \quad \text{for } t > 0, \\
  u(\pi, t) &= 0, \quad \text{for } t > 0, \\
  u(x, 0) &= f(x), \quad \text{for } 0 < x < \pi.
\end{align*}
\]

4. (a) Write the two-dimensional Laplace operator \( \Delta = D_{xx} + D_{yy} \) in polar coordinates \((r, \theta)\). Derive this by applying the chain rule.

(b) Find all radial harmonic functions, that is, all functions \( u \) for which \( \Delta u = 0 \) in the complement of the origin and \( u(x, y) = f(r) \), \( r = \sqrt{x^2 + y^2} \).
5. (a) Verify that \( u(x, t) = F(x + ct) + G(x - ct) \), \( F \) and \( G \) twice differentiable, is a solution of the wave equation 
\[ u_{tt} = c^2 u_{xx}. \]
Use this to solve the initial-value problem for the wave equation with initial conditions 
\[ u(x, 0) = f(x), \quad \text{for } x \in \mathbb{R}, \]
\[ u_t(x, 0) = g(x), \quad \text{for } x \in \mathbb{R}. \]
Verify your solution.

(b) Solve the initial-boundary problem for the wave equation on the quarter–plane \( \{(x, t) : x > 0, \ t > 0\} \) with general initial conditions, as above, but for \( x > 0 \), and boundary condition \( u(0, t) = 0, \) for \( t > 0 \).

6. Suppose \( u(x, y) \) is harmonic in the upper half–disk \( \{(x, y) : x^2 + y^2 < 1, \ y > 0\} \), continuous on \( \{(x, y) : x^2 + y^2 < 1, \ y \geq 0\} \), and equals zero on \( \{(x, y) : y = 0, ; -1 < x < 1\} \). Show that \( u \) can be extended to a harmonic function on the unit disk \( x^2 + y^2 < 1 \). You may assume that a function that satisfies the mean–value property is harmonic.

7. Solve the following initial value problem of the first order system with unknown functions \( u(x, t) \) and \( v(x, t) \):
\[
\begin{align*}
  u_t - 4u_x - 6v_x & = 1, \\
  v_t + 3u_x + 5v_x & = -1, \\
  u(x, 0) & = x, \\
  v(x, 0) & = 0.
\end{align*}
\]

8. State and prove Duhamel’s principle for the one–dimensional heat equation. That is, assume that the solution of the homogeneous heat equation \( u_t(x, t) = u_{xx}(x, t) \) with general non–homogeneous initial condition \( u(x, 0) = f(x) \) is known. Use this to solve the non-homogeneous heat equation \( u_t(x, t) = u_{xx}(x, t) + h(x, t) \) with homogeneous initial condition \( u(x, 0) = 0 \).