Do any six (6) problems. Indicate which six you want graded.

1. Show that an infinite dimensional Banach space is not the union of countably many finite-dimensional subspaces.

2. State and prove the parallelogram identity for Hilbert spaces. Then use this to show that \( C[0,1] \) with the uniform metric is not a Hilbert space.

3. An operator on a Hilbert space is skew-symmetric if \( S^* = -S \), where \( S^* \) is the Hilbert-space adjoint of \( S \). Show that if \( S \) is skew symmetric and \( Sx = \lambda x \) for some constant \( \lambda \) and non-zero \( x \), then \( \lambda \) must be pure imaginary. Give an example of a skew-symmetric operator.

4. (a) Give accurate statements of the Open Mapping Theorem, and the Closed Graph Theorem.
(b) Prove the Closed Graph Theorem from the Open Mapping Theorem.

5. Show that the matrix for the Hilbert-space adjoint of a linear operator on a finite-dimensional space is the conjugate transpose of the matrix for the operator.

6. (a) Define bounded linear operator.
(b) Prove that the left-shift operator \( S : \ell^2 \rightarrow \ell^2 \) defined by
\[
S(\alpha_1, \alpha_2, \alpha_3, \ldots) = (\alpha_2, \alpha_3, \ldots)
\]
is bounded and find its norm.

7. (a) State the Riesz Representation Theorem for linear functionals on a Hilbert space \( H \).
(b) Consider the linear functional \( f : \ell^2 \rightarrow \mathbb{R} \) defined for \( x = (\alpha_k) \in \ell^2 \) by
\[
f(x) = \sum_{k=1}^{\infty} \frac{\alpha_k}{k}
\]
Find \( z \in \ell^2 \) so that \( f(x) = \langle x, z \rangle \) for all \( x \in \ell^2 \) and find \( \|f\| \).

8. (a) Let \( M \) be a subset of a Hilbert space \( H \). Define the orthogonal complement \( M^\perp \).
(b) Prove that \( M^\perp \) is a closed subspace of \( H \).

9. (a) State the Hahn-Banach Theorem for real normed vector spaces.
(b) Prove that if \( X \) is a normed vector space and \( x_0 \in X \), \( x_0 \neq 0 \), then there exists \( f \in X' \) such that \( \|f\| = 1 \) and \( f(x_0) = \|x_0\| \)

10. (a) State the Uniform Boundedness Principle.
(b) Prove that if \( X, Y \) are Banach spaces and \( T_n : X \rightarrow Y \) is a sequence of bounded linear operators such that for each \( x \in X \) the sequence \( (T_n x) \) converges in \( Y \), then there exists \( M > 0 \) such that \( \|T_n\| < M \) for all \( n \).