California State University, Long Beach
Department of Mathematics and Statistics

Complex Analysis Comprehensive Examination
September 10th, 2016

On the first page of your work, please write the numbers of the problems that you want graded.

Do three problems from each part, for a total of six problems.

**PART A**

A1. Let $\rho > 0$, $a \in \mathbb{R}$, $\rho \neq |a|$. Compute $\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$.

A2. (a) Show that if $z + w$ and $zw$ are both real then either $z, w$ are both real or $z = \bar{w}$.

(b) Simplify $\sin \phi + \sin 2\phi + \cdots + \sin n\phi$ into a closed form.

A3. Let $n$ be a positive integer and $w = e^{i2\pi/n} = \cos(2\pi/n) + i \sin(2\pi/n)$.

(a) Show that $1 + w^k + w^{2k} + \cdots + w^{(n-1)k} = 0$ for any integer $k$ which is not a multiple of $n$.

(b) Define an $n \times n$ matrix $A = (a_{pq})$ by $a_{pq} = w^{pq} = e^{i2\pi pq/n}, 1 \leq p, q \leq n$. Find $A^{-1}$ if it exists.

A4. Compute the integral $\int_0^{2\pi} \frac{d\theta}{1 + a \cos \theta + b \sin \theta}$ for $a, b \in \mathbb{R}$ and $a^2 + b^2 < 1$.

A5. A function has Laurent series

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-1)^n}{3^n} + 2 \sum_{n=1}^{\infty} \frac{2^n}{(z-1)^n}.$$  

Determine the set of $z$ for which the series converges, and the sum of the series in closed form.
PART B

B1. Let \( p(z) = \sum_{k=0}^{n} a_k z^k, a_n \neq 0 \), be a polynomial of degree \( n \) such that \( |p(z)| \leq M \) for \( |z| \leq 1 \).

Show that \( |p(z)| \leq M |z|^n \) for \( |z| \geq 1 \).

B2. Suppose that \( f \) is entire and \( \lim_{z \to \infty} \frac{f(z)}{z} = c \), where \( c \) is a constant.

(a) Show that the function \( g \) defined by \( g(z) = \frac{f(z) - f(0)}{z} \) can be extended to an entire function and is bounded.

(b) Show that \( f(z) = cz + b \) for some constant \( b \).

B3. Prove that for a fixed \( w \) in the unit disk \( D = \{z : |z| < 1\} \), the fractional linear transformation \( F(z) = \frac{w - z}{1 - wz} \) satisfies the following properties:

(a) If \( |z| = 1 \) then \( |F(z)| = 1 \).

(b) \( F \) maps the unit disk \( D \) to itself and is analytic.

B4. (a) Carefully state Schwarz Lemma.

(b) Let \( f : D \to D \) be analytic, where \( D \) is the open unit disk. Show that if \( f(a) = a \) and \( f(b) = b \) for two distinct points \( a, b \in D \) then \( f(z) = z \) for all \( z \in D \).

B5. Let \( f(z) \) be an analytic function on the open unit disk \( D = \{z : |z| < 1\} \). Suppose there is an annulus \( U = \{z : r < |z| < 1\} \) such that the restriction of \( f(z) \) to \( U \) is one-to-one (injective). Show that \( f(z) \) is one-to-one on \( D \).