Math 479  Linear Programming Homework part 2

1. Given a linear system $Ax = b$ with nonsingular matrix $A$,

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

let $x_0$ be a solution of the linear system and

$$A^{-1} = [d_1, d_2, \ldots, d_n].$$

For any column $d_k$ of $A^{-1}$, prove that for any positive $\sigma$, $x_1 = x_0 - \sigma d_k$ satisfies

$$a_1^T x_1 = b_1$$
$$\vdots$$
$$a_{k-1}^T x_1 = b_{k-1}$$
$$a_k^T x_1 < b_k$$
$$a_{k+1}^T x_1 = b_{k+1}$$
$$\vdots$$
$$a_n^T x_1 = b_n$$

2. Given $x_0 \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $d \in \mathbb{R}^n$, $\sigma > 0$, prove that

a. if $c^T d > 0$, then $c^T (x_0 - \sigma d) < C^T x_0$;

b. if $c^T d < 0$, then $c^T (x_0 - \sigma d) > C^T x_0$.

For the following questions, consider the LP problem (I)

$$\min c^T x$$
$$a_1^T x \leq b_1$$
$$a_2^T x \leq b_2$$
$$\vdots$$
$$a_m^T x \leq b_m$$

where $c, a_i \in \mathbb{R}^n$, $i = 1, \ldots, m$.

3. Let $x_0$ be a nondegenerate extreme point of the LP problem (I) and the first $n$ constraints are active at $x_0$. Let

$$D = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix}, \quad D^{-1} = [d_1, \ldots, d_n].$$
Let $d_k$ be the next search direction and $x = x_0 - \sigma d_k$. Explain why

$$\sigma_0 = \min \left\{ \frac{a_i^T x_0 - b_i}{a_i^T d_k} \mid a_i^T d_k < 0 \right\}$$

is the largest possible step size to make $x_1 = x_0 - \sigma_0 d_k$ be still feasible.

4. Assume that the active constraints of the LP problem (I) at an extreme point $x_0$ are the first $k$ constraints. If there are nonnegative numbers $u_1, \ldots, u_k$ such that

$$-c = u_1 a_1 + \cdots + u_k a_k,$$

then $x_0$ is an optimal solution of the LP.

5. Assume that the active constraints of the LP problem (I) at an extreme point $x_0$ are the first $n$ constraints. $D$ and $D^{-1} = [d_1, \ldots, d_n]$ are same as in Question 3. Prove that if $c^T d_i \leq 0$ for all $i = 1, \ldots, n$, then there are nonnegative numbers $u_1, \ldots, u_n$ such that

$$-c = u_1 a_1 + \cdots + u_n a_n.$$

(Thus $x_0$ is an optimal solution of the LP according to Question 4.)

6. Prove that the LP problem (I) above has no more than

$$\frac{m!}{(m-n)!n!}$$

number of extreme points.

7. Let $(x^*, \epsilon^*)$ be an optimal solution of the new LP problem

$$\begin{array}{lll}
\min & \epsilon \\
& a_1^T x - \epsilon & \leq b_1 \\
& a_2^T x - \epsilon & \leq b_2 \\
& \vdots \\
& a_m^T x - \epsilon & \leq b_m \\
& -\epsilon & \leq 0
\end{array}$$

where $c, a_i \in \mathbb{R}^n, i = 1, \ldots, m$ as before and $\epsilon \in \mathbb{R}$. Prove that

a. if $\epsilon^* = 0$, then $x^*$ is a feasible solution of the LP problem (I) above;

b. if $\epsilon^* > 0$, then the LP problem (I) above has no feasible solution.
8. Solve the following LP by using the algorithm discussed in class, starting from 
\( x_0 = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \).

\[
\begin{align*}
\text{min} & \quad -5x_1 + 2x_2 \\
-2x_1 + x_2 & \leq 2 \\
x_1 + 2x_2 & \leq 14 \\
4x_1 + 3x_2 & \leq 36 \\
x_1 & \geq 0 \\
x_2 & \geq 0 
\end{align*}
\]