Visualizing a Nonmeasurable Set

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In most real analysis textbooks, the standard example of a nonmeasurable set is a subset of the real line that is due to Vitali [3]. We describe a similar nonmeasurable subset of the torus (and hence the plane), where we can more easily visualize the set. In the process of constructing the set, students get an opportunity to experience how topics they learned in algebra and topology can be used in analysis.

The idea of Vitali’s example is to express the unit interval $I$ as a disjoint union of countably many mutually congruent sets $A_k$. The nonmeasurability of each $A_k$ follows from the observation that $I = \bigcup_{k \in \mathbb{Z}} A_k$ and that countable additivity of measure implies that $1 = m(I) = \sum_{k \in \mathbb{Z}} m(A_k)$. Since each set $A_k$ must have the same measure, the last equation shows that no nonnegative value can be assigned as the measure of each $A_k$. We will use this same idea with the square $[0, 1] \times [0, 1]$ in the plane $\mathbb{R}^2$. The advantage is that we will now have a more visual object than that of Vitali’s example because the example will appear as a subset of a torus.

The torus

In order to understand the example that we will eventually construct, we need to consider different ways of describing the torus. We will exploit topological and group theoretic properties associated with two different representations of a torus to obtain information that we can piece together to construct an interesting example of a nonmeasurable set.

Begin by considering the square $[0, 1] \times [0, 1]$ as a topological subspace of $\mathbb{R}^2$ endowed with the usual topology. After identifying opposite edges of the square, we obtain (via the identification topology) a space called the torus, denoted by $T$. A convenient way to visualize the torus is as the surface of a doughnut. In fact, the mapping $\Omega : [0, 1] \times [0, 1] \to \mathbb{R}^3$ given by

$$\Omega(r, s) = \left(\left[2 + \cos(2\pi s)\right] \cos(2\pi r), \left[2 + \cos(2\pi s)\right] \sin(2\pi r), \sin(2\pi s)\right)$$

renders a concrete parametrization of the torus as a subspace of $\mathbb{R}^3$. The mapping $\Omega$ identifies the pair of edges $[0, 1] \times \{0\}$ and $[0, 1] \times \{1\}$ as well as the pair of edges $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$, as in Figure 1. This latter pair of edges of the square, labelled $M$ in the figure, correspond to a circle on the torus called a meridian.

Another way to visualize the torus is as the topological product of two circles $S^1 \times S^1$ where $S^1 = \{e^{2\pi ir} \mid 0 \leq r \leq 1\}$ is the unit circle in the complex plane $\mathbb{C}$. Viewed in this way, the torus is a topological group under componentwise multiplication. The mapping

$$\Psi : \mathbb{R}^2 \to T$$

given by $\Psi(r, s) = (e^{2\pi ir}, e^{2\pi is})$ satisfies

$$\Psi((a, b) + (c, d)) = \Psi((a, b))\Psi((c, d))$$
A torus is a square with edges identified

where \((a, b)\) and \((c, d)\) are points in \(\mathbb{R}^2\). Indeed, \(\Psi\) is a continuous surjective group homomorphism (actually it is a covering map) from the additive group \(\mathbb{R}^2\) onto the multiplicative group \(\mathbb{T}\). Moreover, the points \((a, b)\) and \((c, d)\) are identified via the mapping \(\Psi\) if \((e^{2\pi ia}, e^{2\pi ib}) = (e^{2\pi ic}, e^{2\pi id})\). Thus each unit square in \(\mathbb{R}^2\) is wrapped once around the torus by \(\Psi\). Note also that each of the vertical lines labelled \(M_k\) in **Figure 2** corresponds to the meridian \(M\).

The mapping \(\Psi\) suggests yet another way to describe the torus. It is the quotient space of \(\mathbb{R}^2\) relative to the following equivalence relation: two points \((a, b)\) and \((c, d)\) in \(\mathbb{R}^2\) are identified if \(c = a + k\) and \(d = b + l\) for some integers \(k\) and \(l\). When this is the case we will write \((a, b) \equiv (c, d) \mod 1\).

**One-parameter subgroups of \(\mathbb{T}\)** Our goal in this section is to describe a family of continuous group homomorphisms from \(\mathbb{R}\) into the torus \(\mathbb{T}\). Such a map would wrap the real line on the torus. To visualize such a map, we will first send \(\mathbb{R}\) into \(\mathbb{R}^2\) and then identify \(\mathbb{R}^2\) with the torus via the map \(\Psi\).

Let \(\alpha\) and \(\beta\) be fixed real numbers. A mapping \(\varphi: \mathbb{R} \to \mathbb{R}^2\) given by \(\varphi(t) = (\alpha t, \beta t)\) is a continuous group homomorphism between the additive groups \(\mathbb{R}\) and \(\mathbb{R}^2\). The image of \(\varphi\) is called a *one-parameter subgroup* of \(\mathbb{R}^2\). This image is the line whose Cartesian equation is \(y = (\beta/\alpha)x\). It is easy to see that if two points \((\alpha s, \beta s)\)
and \((\alpha t, \beta t)\) corresponding to \(s \neq t\) in \(\mathbb{R}\) are equivalent mod 1, then \(\beta/\alpha\) is a rational fraction.

Now let’s suppose that the fixed real numbers \(\alpha\) and \(\beta\) have an irrational ratio. In this case, if \((\alpha s, \beta s) \equiv (\alpha t, \beta t)\) mod 1, then \(s = t\). Thus the mapping

\[
\Psi \circ \varphi : \mathbb{R} \to \mathbb{T}
\]

is injective, it is also a continuous group homomorphism whose image is a one-parameter subgroup of \(\mathbb{T}\). Let \(L\) denote the image of \(\varphi\) in \(\mathbb{R}^2\). Then \(L\) is a line in the plane. We can visualize the image \(\Psi(L)\) as a coil on the torus, a piece of which is shown in bold in FIGURE 3. Proving that this coil is dense in the torus makes a nice exercise, though we will not do so here.

![Figure 3](image)

The line \(L\) and its translates correspond to parallel coils on the torus

The line \(L\) is a subgroup of the additive group \(\mathbb{R}^2\). For \((a, b) \in \mathbb{R}^2\) the coset \((a, b) + L\) is a line parallel to \(L\), often referred to as a translate of \(L\). Since \(\Psi\) is a homomorphism the coil \(\Psi(L)\) is a subgroup of the multiplicative group \(\mathbb{T}\). For \(p \in \mathbb{T}\), the coset \(p\Psi(L)\) is a coil parallel to \(\Psi(L)\), because it is the image of a line \((a, b) + L\) parallel to \(L\). Indeed, if \(p = \Psi((a, b))\), then

\[
\Psi((a, b) + L) = \Psi((a, b))\Psi(L) = p\Psi(L).
\]

In other words, under the map \(\Psi\), parallel lines of the form \((a, b) + L\) in \(\mathbb{R}^2\) correspond to parallel coils of the form \(p\Psi(L)\) in \(\mathbb{T}\). FIGURE 3 shows three such lines and the corresponding coils. It is important to note that the lines \((a, b) + L\) and \((c, d) + L\) correspond to the same coil on the torus if and only if \((a, b) \equiv (c, d)\) mod 1.

Now, each coil \(p\Psi(L)\) intersects the meridian \(M\) in \(\mathbb{T}\) infinitely many times; this is easily seen in each of the representations of the torus. So by the Axiom of Choice there is a subset \(\Lambda\) of \(M\) such that each coset of \(\Psi(L)\) is represented by a unique point in \(\Lambda\). Thus the sets \(p\Psi(L), p \in \Lambda\), form a complete set of cosets of \(\Psi(L)\) in \(\mathbb{T}\), so we obtain the disjoint union

\[
\bigcup_{p \in \Lambda} (p\Psi(L)) = \mathbb{T}.
\]

In other words, we can visualize the torus as the disjoint union of uncountably many parallel coils, each one a coset of \(\Psi(L)\).
A nonmeasurable subset of $\mathbb{T}$

Now we will construct a nonmeasurable set by partitioning the torus into a disjoint union of countably many geometrically congruent sets $A_k$. We begin with the subsets in the plane that consist of those parts of the translates (cosets) of $L$ that lie in the strip between $M_k$ and $M_{k+1}$. The sets $A_k$ are the corresponding sets in the torus. They can be visualized as the portions of each coil $p\Psi(L)$ starting and ending at $M$ (see Figure 3). The details are as follows.

For each integer $k$ in $\mathbb{Z}$, define the set

$$L^{(k)} = \{ (\alpha t, \beta t) \mid k \alpha \leq t < (k + 1) \alpha \}.$$

Observe that the sets $L^{(k)}$ are pairwise disjoint; they are merely half-open intervals on the line $L$. In fact, they are the parts of the line $L$ between consecutive vertical lines $M_k$ in Figure 3. The corresponding sets on the torus are $\Psi(L^{(k)})$. Now for each integer $k$ in $\mathbb{Z}$, set $A_k = \bigcup_{p \in A} p\Psi(L^{(k)})$ and observe that

$$\mathbb{T} = \bigcup_{k \in \mathbb{Z}} A_k.$$

Finally the subsets $A_k$ of the torus are pairwise disjoint by construction, pairwise congruent via translation (a multiplication in $\mathbb{T}$), and there are countably many of them. The Lebesgue measure of the torus $\mathbb{T}$ is its surface area, a positive number. Recall that Lebesgue measure is translation invariant and countably additive. Therefore, if the sets $A_k$ are measurable, then they have the same positive measure. Since the torus $\mathbb{T}$ is the countable union of such sets, the sets $A_k$ cannot be measurable.

REFERENCES


Nondifferentiability of the Ruler Function

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A fixture of any introductory course in mathematical analysis is the pathological function, one whose intuition-defying behavior serves to crystallize our understanding of analytic concepts. Among the more accessible of these is the so-called ruler function, defined on $(0, 1)$ by

$$r(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ (lowest terms)} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$