TRIANGULAR AND COTRIANGULAR BASES
IN TOPOLOGICAL ALGEBRAS

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Abstract. Topological algebras with bases have been studied by a number of authors. In this paper we introduce the concepts of triangular and cotriangular bases in topological algebras. We describe the relationship between orthogonal bases and triangular and cotriangular bases, and we characterize $c_0$ and $s$ in terms of the types of bases they possess. We also show that Fréchet algebras with triangular or cotriangular bases are functionally continuous and semisimple.

Schauder bases in topological vector spaces have been studied extensively since the appearance of Schauder's treatise [12]. Several papers have appeared recently in which bases are studied in the context of topological algebras [4], [5], [11], [14], [15]. It is clear that the concept of a basis is a topological vector space concept, so that the mere existence of a basis in a topological algebra yields no new information about the structure of the algebra. On the other hand, in a topological algebra $A$ with a basis the multiplication is completely determined by its action on the basis elements. Thus if the action of the multiplication on the basis is described in a simple way, then the existence of a basis satisfying this action yields valuable information about the algebraic structure of $A$. Perhaps the simplest interaction between the bases and the multiplication is when the basis $\{x_n\}$ (together with $0$, if necessary) is a closed set with respect to the multiplication. In other words, it forms a semigroup under the algebra multiplication. Actually only special types of such bases have been considered, including orthogonal [5], cyclic [15], and finitely generated [14].

In this paper we consider topological algebras with two related types of bases that we call triangular and cotriangular. We show that a triangular (or cotriangular) basis and an orthogonal basis in the same topological algebra are always related in a definite way. We give characterizations of some sequence algebras in terms of the types of bases they possess. For instance, a Banach algebra with an unconditional orthogonal basis and a triangular basis must be $c_0$, while a Fréchet algebra with an orthogonal basis and cotriangular basis must be $s$. Triangular bases and cotriangular bases are complementary concepts, as our results show.

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Topological algebras with bases satisfying the condition we call "cotriangular" are considered in [3], where it is shown that a Fréchet algebra having such a basis and satisfying another condition is functionally continuous; that is, every multiplicative linear functional on such an algebra is continuous. We show that an algebra satisfying the conditions in [3] is actually algebraically and topologically identical with $\mathcal{A}$, the Fréchet algebra of all sequences. In fact, a much weaker condition yields this result. We point out in Section 4 that a Fréchet algebra with a triangular or cotriangular basis is always functionally continuous. Finally, in Section 5 we provide some examples of algebras to which our results apply.

For notation and results on bases, we refer to Singer [13], and on topological algebras, we refer to Zelazko [17].

1. Preliminaries. A basis in a topological vector space $E$ is a sequence $\{x_n\}$ such that every $x \in E$ has a unique representation $x = \sum_{k=1}^{\infty} \alpha_k x_k$, where $\alpha_k \in \mathbb{C}$ and the series converges in the topology of $E$. The coefficient functionals associated to the basis $\{x_n\}$ are the functionals $x_n^*$ defined by $x_n^*(x) = \alpha_n$. A basis for which all the coefficient functionals are continuous is called a Schauder basis. It is well known that a basis in a Fréchet space is always a Schauder basis. A sequence $\{x_n, f_n\}, \{x_n\} \subseteq E, \{f_n\} \subseteq E'$ (the continuous dual of $E$), with $f_n(x_m) = \delta_{nm}$ is called biorthogonal. For a given biorthogonal system $\{x_n, f_n\}$ we define the partial sum operators $S_n, n = 1, 2, \cdots$, by $S_n(x) = \sum_{k=1}^{n} f_k(x) x_k$. The sequence $\{x_n\}$ is a basis in $E$ if and only if for every $x \in E, \lim_{n \to \infty} S_n(x) = x$.

Let $\{x_n\}$ be a basis in $E$. Following Kalton [6], we call the basis bounded if the set $\{x_n\}$ is a bounded set in $E$, regular if it "bounded away from zero" in the sense that there exists a neighborhood $V$ of zero such that $x_n \notin V$ for each $n$, and normalized if it is both bounded and regular. We state without proof two results that we use later. The first is from [6], and the second is proved in [13] for Banach spaces, but its generalization to Fréchet spaces is straightforward.

**Theorem 1.** Let $E$ be a Fréchet space with a basis $\{x_n\}$.

(i) The basis $\{x_n\}$ is regular if and only if the convergence of $\sum_{n=1}^{\infty} \alpha_n x_n$ in $E$ implies that $\alpha_n \to 0$.

(ii) The basis $\{x_n\}$ is bounded if and only if $\sum_{n=1}^{\infty} |\alpha_n| < \infty$ implies $\sum_{n=1}^{\infty} \alpha_n x_n$ converges.

**Theorem 2.** Let $E$ be a Fréchet space with a normalized basis $\{x_n\}$. Then the sequence $\{y_n\}$ defined by $y_n = \sum_{k=1}^{n} \alpha_k x_k$ is a basis for $E$ if and only if $\{\frac{y_n}{|\alpha_n+1|}\}$ is a bounded set in $E$.

A sequence $\{x_n\}$ in a topological algebra $A$ is called orthogonal if

$$x_n x_m = \delta_{mn} x_m \quad (n, m = 1, 2, \cdots)$$
An orthogonal basis is a basis that forms an orthogonal sequence. For the main results on orthogonal bases, see [5]. We point out here that an orthogonal basis in a topological algebra is unique (up to permutation) and that such a basis is always a Schauder basis. In the rest of this paper, we consider only Schauder bases, and the word "basis" will mean Schauder basis.

2. Topological algebras with triangular bases.

A sequence \( \{y_n\} \) in a topological algebra \( A \) is triangular if

\[
y_{n+m} = y_ny_m = y_m \quad m \leq n \quad (n, m = 1, 2, \ldots)
\]

A triangular basis is a basis that is a triangular sequence. We consider here conditions under which an algebra with a triangular basis has an orthogonal basis. An orthogonal sequence \( \{x_n\} \) can be obtained from the triangular sequence \( \{y_n\} \) by setting

\[
x_1 = y_1
\]

\[
x_n = y_n - y_{n-1} \quad (n \geq 2)
\]

We begin by noting that a triangular basis in a \( B_0 \)-algebra is necessarily regular.

**Theorem 3.** If \( A \) is a \( B_0 \)-algebra then any triangular basis for \( A \) is regular.

**Proof.** Let \( \{y_n\} \) be a triangular basis for \( A \), and let \( y \in A, y = \sum_{k=1}^{\infty} \alpha_k y_k \). Then \( yy_1 = (\sum_{k=1}^{\infty} \alpha_k y_k)y_1 = (\sum_{k=1}^{\infty} \alpha_k)y_1 \), and so \( \sum_{k=1}^{\infty} \alpha_k \) converges. It follows that \( \alpha_k \to 0 \). Thus \( \{y_n\} \) is regular by Theorem 1. \( \square \)

**Theorem 4.** A topological algebra with a bounded triangular basis has a bounded orthogonal basis.

**Proof.** Let \( A \) be a topological algebra with bounded triangular basis \( \{y_n\} \). Clearly the orthogonal sequence \( \{x_n\} \) defined above is also bounded. We show that this sequence is a basis for \( A \).

For each \( y \in A, \sum_{k=1}^{\infty} y_k^*(y) \) converges in \( C \), as in the proof of Theorem 3. Hence the sequence of functionals

\[
f_n(y) = \sum_{k=n}^{\infty} y_k^*(y)
\]

is well defined. Clearly \( f_n(x_m) = \delta_{nm}, (n, m = 1, 2, \ldots) \) and so \( \{x_n, f_n\} \) is a biorthogonal
system. We have
\[ \sum_{i=1}^{k} f_i(y) x_i = \left( \sum_{j=1}^{\infty} y_j f_j(y) \right) y_1 + \sum_{i=2}^{k} \left( \sum_{j=1}^{\infty} y_j f_j(y) \right) (y_i - y_{i-1}) \\
= \sum_{i=1}^{k-1} y_i f_i(y) y_i + \left( \sum_{j=k}^{\infty} y_j f_j(y) \right) y_k \\
= \sum_{i=1}^{k-1} y_i f_i(y) y_i + f_k(y) y_k \] (1)

Since \( \lim_{i \to \infty} \sum_{k=1}^{\infty} y_k^*(y) = 0 \) and since \( \{y_i\} \) is bounded, \( f_i(y) y_i \to 0 \) for all \( y \in A \). Thus by (1), \( \sum_{i=1}^{\infty} f_i(y) x_i = y \). \( \square \)

**Theorem 5.** A \( B_0 \)-algebra with a bounded triangular basis has a normalized orthogonal basis.

**Proof.** Let \( \{y_n\} \) be a bounded triangular basis for the \( B_0 \)-algebra \( A \) and let \( \{x_n\} \) be the (bounded) orthogonal basis obtained from this triangular basis as in Theorem 4. It remains to show that the basis \( \{x_n\} \) is regular. Let \( x \in A \), where \( x = \sum_{k=n}^{\infty} \alpha_n x_n \). From the proof of Theorem 4 we have \( \alpha_n = \sum_{k=n}^{\infty} y_k^*(x) \). Thus \( \alpha_n \to 0 \) as \( n \to \infty \), and so \( \{x_n\} \) is regular by Theorem 1. \( \square \)

A natural question is: under what conditions does an algebra \( A \) with an orthogonal basis \( \{x_n\} \) have a triangular basis? Consider the sequence \( \{y_n\} \) defined by
\[ y_n = \sum_{k=1}^{n} x_k \]
A simple calculation show that this sequence is triangular. We call this sequence the triangular sequence obtained from the orthogonal sequence \( \{x_n\} \). The next result is an immediate consequence of Theorem 2.

**Theorem 6.** Let \( A \) be a \( B_0 \)-algebra with a normalized orthogonal basis \( \{x_n\} \). Then the triangular sequence \( \{y_n\} \) obtained from \( \{x_n\} \) as above is a basis for \( A \) if and only if \( \{y_n\} \) is a bounded set in \( A \). \( \square \)

We now give a characterization of the Banach algebra \( c_0 \) in terms of the existence of triangular bases.

**Theorem 7.** Let \( A \) be a Banach algebra with a bounded unconditional orthogonal basis \( \{x_n\} \). Then the triangular sequence \( \{y_n\} \) obtained from \( \{x_n\} \) is a basis for \( A \) if and only if \( A \) is algebraically and topologically isomorphic to \( c_0 \).

**Proof.** \( (\Rightarrow) \) Suppose that \( \{y_n\} \) is a basis for \( A \). Since \( \{x_n\} \) is the orthogonal sequence obtained from \( \{y_n\} \), it follows from the proof of Theorem 4 that if \( x \in A \), \( x = \sum_{k=1}^{\infty} \alpha_k x_k \),
then \(\alpha_k = \sum_{i=k}^{\infty} y_i(x)\). Thus \(\alpha_k \to 0\), so that \((\alpha_k) \in c_0\). Define the map \(T : A \to c_0\) by

\[
\sum_{n=1}^{\infty} \alpha_n x_n \mapsto (\alpha_n)
\]

This map is clearly an algebra homomorphism and is an injection because of the uniqueness of the basis representation. We show that \(T\) is onto \(c_0\). First, let \(M > 0\) be such that \(|\|y_n\| \leq M, n = 1, 2, \cdots\) (Theorem 6). Now, let \((\beta_n) \in c_0\). Let \(\epsilon > 0\) and choose \(N > 0\) such that \(m > N\) implies \(|\beta_m| < \epsilon/M\). Then, for \(n \geq m > N\), and using an equivalent norm \(\|\cdot\|\) (Singer [13], Theorem 16.1, part 23), we have

\[
\left\| \sum_{i=m}^{n} \beta_i x_i \right\| \leq \sup_{f \in E^*} \sum_{i=m}^{n} |\beta_i| |f(x_i)| \leq \frac{\epsilon}{2M} \sum_{i=m}^{n} |f(x_i)|
\]

\[
\leq \frac{\epsilon}{2M} \|y_n - y_{m-1}\| \leq \epsilon
\]

Thus the series \(\sum_{k=1}^{\infty} \beta_k x_k\) is Cauchy and so converges in \(A\). This shows that \(T\) is onto. Now, since the coefficient functionals \(e_n^*\) associated with the unit vector basis \(\{e_n\}\) (see Section 5) are a separating family of continuous linear functionals on \(c_0\), and since \(e_n^* \circ T = x_n^*\), it follows by the Closed Graph Theorem that \(T\) is continuous. Since \(T\) is onto, \(T\) is open by the Open Mapping Theorem.

\((\Leftarrow\Rightarrow)\) Conversely, suppose that there exists an algebraic and topological isomorphism \(T : A \to c_0\). Then \((T(x_n))\) is an orthogonal basis in \(c_0\). Since an orthogonal basis in a topological algebra is unique (up to a permutation) [5], it follows that \(\{T(x_n)\} = \{e_n\}\) where \(\{e_n\}\) is the unit vector basis of \(c_0\). Since the triangular sequence obtained from \(\{e_n\}\) is a basis for \(c_0\) (Section 5), the same must be true for the triangular sequence \(\{y_n\}\) obtained from \(\{x_n\}\).

In the next theorem we show that if a topological algebra has an orthogonal basis as well as a triangular basis, then these bases are always related to each other. In fact, the triangular basis is the triangular sequence obtained from the orthogonal basis (up to a permutation of the orthogonal basis).

**Lemma 8.** Let \(A\) be a topological algebra with a triangular basis \(\{y_n\}\). Then \(\{y_n\}\) is a maximal triangular sequence.

**Proof.** Suppose that \(\{y_n\}\) is not maximal (i.e., it is properly contained in another triangular sequence). Let \(y \in A, y \neq y_k, k = 1, 2, \cdots\), such that \(\{y_1, y_2, \cdots, y_{n_0}, y, y_{n_0+1}, \cdots\}\) is a triangular sequence. In particular, \(y^2 = y, y y_k = y_k (k \leq n_0)\), and \(y y_k = y (k > n_0)\).

Writing \(y = \sum_{k=1}^{\infty} \alpha_k y_k\), we have

\[
y_{n_0} = y y_{n_0} = \sum_{k=1}^{n_0-1} \alpha_k y_k + \left( \sum_{k=n_0}^{\infty} \alpha_k \right) y_{n_0}
\]  

(2)
\[ y = y^2 = \sum_{k=1}^{n_0} \alpha_k y_k + \left( \sum_{k=n_0+1}^{\infty} \alpha_k \right) y \]  \hspace{1cm} (3)

\[ y = y y_{n+1} = \sum_{k=1}^{n_0} \alpha_k y_k + \left( \sum_{k=n_0+1}^{\infty} \alpha_k \right) y_{n+1} \]  \hspace{1cm} (4)

From (2) and the uniqueness of the basis representation it follows that \( \alpha_1 = \alpha_2 = \cdots = \alpha_{n_0-1} = 0 \) and \( \sum_{k=n_0}^{\infty} \alpha_k = 1 \). Thus from (3) and (4), we have \( (1 - \alpha_{n_0}) y = (1 - \alpha_{n_0}) y_{n+1} \), and since \( y \neq y_{n+1} \), it follows that \( \alpha_{n_0} = 1 \). Then (4) simplifies to \( y = y_{n+1} \), a contradiction.  \[ \blacksquare \]

**Theorem 9.** Let \( A \) be a topological algebra with an orthogonal basis \( \{ x_n \} \) and a triangular basis \( \{ y_n \} \). Then there exists a permutation \( \pi \) of \( \mathbb{N} \) such that \( y_n = \sum_{k=1}^{n} x_{\pi(k)} \).

**Proof.** Since \( \{ x_n \} \) is a basis each \( y_n \) has a representation \( y_n = \sum_{k=1}^{\infty} \alpha_k x_k \) and since \( y_n^2 = y_n \) it follows that each \( \alpha_k \) is either 0 or 1. Thus we can write

\[ y_n = \sum_{k \in J_n} x_k \]

where each \( J_n \) is a nonempty subset of \( \mathbb{N} \). Set \( J_0 = \emptyset \). Since \( y_m y_n = y_m \) for \( m \leq n \) it follows that \( J_m \subset J_n \) for \( m < n \). We claim that \( J_{n+1} \setminus J_n \) is a singleton set, \( n = 0, 1, \cdots \). For if not, then there exists \( J' \) such that \( J_n \not\subset J' \not\subset J_{n+1} \). Then with \( y' = \sum_{k \in J'} x_k, \{ y_1, y_2, \cdots, y_n, y', y_{n+1}, \cdots \} \) is a triangular sequence properly containing the sequence \( \{ y_i \} \), contradicting Lemma 8. It follows that the cardinality of each \( J_n \) is \( n \).

Since each \( y_n = \sum_{k \in J_n} x_k \), the result follows.  \[ \blacksquare \]

**3. Topological algebras with contriangular bases.**

We say that a sequence \( \{ z_n \} \) in an algebra \( A \) is cotriangular if

\[ z_m z_n = z_n z_m = z_n, \hspace{1cm} n \geq m \hspace{1cm} (n, m = 1, 2, \cdots) \]

A basis \( \{ z_n \} \) is called a cotriangular basis if the sequence \( \{ z_n \} \) is cotriangular.

Let \( P = \{ p_n \} \) be a sequence of seminorms defining the topology of the Fréchet algebra \( A \). Husain and Liang [3] consider algebras with cotriangular bases satisfying the additional property

\[ p_n(z_n) \neq 0 \hspace{1cm} \text{and} \hspace{1cm} p_n(z_{n+1}) = 0 \hspace{1cm} (n = 1, 2, \cdots) \]  \hspace{1cm} (5)

They show that a Fréchet algebra with a cotriangular basis that satisfies (5) is functionally continuous. In fact, a Fréchet algebra satisfying these conditions is algebraically and topologically isomorphic to the algebra \( s \) of sequences as shown in [1]. This last result is actually a simple consequence of the Closed Graph Theorem, as we now show. It is immediate that \( p_i(z_j) = 0 \) for \( i < j \). Thus for every sequence of scalars \( (\alpha_n), \sum_{n=1}^{\infty} \alpha_n z_n \) converges
(absolutely) in $A$. Now the algebra $s$ of sequences has the basis $\{w_n\}$ (see Section 5). The
map $T: A \rightarrow s$ defined by
\[
\sum_{n=1}^{\infty} \alpha_n z_n \mapsto \sum_{n=1}^{\infty} \alpha_n w_n
\]
is an algebraic isomorphism onto $s$. Since $w_n^* \circ T = z_n^*$ and $z_n^*$ is continuous (because $A$
is a Fréchet space), it follows by the Closed Graph Theorem that $T$ is continuous, and so $T$
is open by the Open Mapping Theorem.

In the above argument it is sufficient to assume that for each $p \in P, p(z_n) = 0$ for
sufficiently large $n$. In fact, we show that the same result is true under the weaker assumption
that for every $p \in P, p(z_n) \to 0$ as $n \to \infty$, i.e., $z_n \to 0$ as $n \to \infty$. Indeed, this
condition is both necessary and sufficient for a Fréchet algebra with a cotriangular basis
to be isomorphic to $s$ (Theorem 11 below). This result is a consequence of a theorem on
orthogonal bases, so we first note some relationships between cotriangular and orthogonal
bases.

If $\{z_n\}$ is a cotriangular sequence in $A$, then a simple calculation shows that the sequence
$\{x_n\}$ defined by
\[
x_n = z_n - z_{n+1} \quad (n = 1, 2, \cdots)
\]
is an orthogonal sequence in $A$. The following theorem gives conditions under which the
sequence $\{x_n\}$ obtained from the cotriangular basis $\{z_n\}$ in this way is actually a basis for
$A$.

**Theorem 10.** Let $A$ be a topological algebra with a cotriangular basis $\{z_n\}$. Then
the orthogonal sequence $\{x_n\}$ obtained from $\{z_n\}$ is a basis in $A$ if and only if $z_n \to 0$ as $n \to \infty$.

**Proof.** ($\Longleftarrow$) For $x \in A$, define
\[
f_n(x) = \sum_{k=1}^{n} z_k^*(x)
\]
Clearly $f_n(x_m) = \delta_{nm}, (n, m = 1, 2, \cdots)$ and so $\{x_n, f_n\}$ is a biorthogonal system. We have
\[
\sum_{i=1}^{k} f_i(x)x_i = \sum_{i=1}^{k} \left( \sum_{j=1}^{i} z_j^*(x) \right) (z_i - z_{i+1})
\]
\[
= \sum_{i=1}^{k} z_i^*(x)z_i - \left( \sum_{i=1}^{k} z_i^*(x) \right) z_{k+1}
\]
\[
= \sum_{i=1}^{k} z_i^*(x)z_i - \left( \sum_{i=1}^{k} z_i^*(x)z_i \right) z_{k+1}
\]
\[
= \left( \sum_{i=1}^{k} z_i^*(x)z_i \right) (z_1 - z_{k+1})
\]
Since by hypothesis $z_k \to 0$ as $k \to \infty$, it follows that
\[
\lim_{k \to \infty} \sum_{i=1}^{k} f_i(x)z_i = \lim_{k \to \infty} \left( \sum_{i=1}^{k} z_i^*(x)z_i \right) (z_i - z_{i+1}) = \left( \sum_{i=1}^{\infty} z_i^*(x)z_i \right) z_1 = \sum_{i=1}^{\infty} z_i^*(x)z_i = z.
\]

(\rightarrow) Conversely, suppose that the orthogonal sequence $\{z_n\}$ is a basis for $A$. Then, since $z_1$ is an identity for $A$, $z_1 = \sum_{i=1}^{\infty} x_i$ (Husain and Watson [5]). We have
\[
z_1 = \sum_{i=1}^{\infty} x_i = \lim_{n \to \infty} \sum_{i=1}^{n} (z_i - z_{i+1}) = \lim_{n \to \infty} (z_1 - z_{n+1}).
\]
Thus $z_n \to 0$ as $n \to \infty$.

Theorem 11. Let $A$ be a Fréchet algebra with a cotriangular basis $\{z_n\}$ such that $z_n \to 0$ as $n \to \infty$. Then $A$ is algebraically and topologically isomorphic to $s$.

Proof. By Theorem 10, $A$ has an orthogonal basis. Since $z_1$ is an identity for $A$, the result follows from [5, Theorem 3.4].

Remark. For locally $m$-convex algebras $A$ the condition in Theorem 10 that $z_n \to 0$ is equivalent to the condition that for every $p \in \mathcal{P}$, $p(z_n) = 0$ for sufficiently large $n$. To see this, let $p \in \mathcal{P}$. Since $p(z_n - z_{n+1}) \leq p(z_n) + p(z_{n+1})$ and since $p(z_n) \to 0$, it follows that there exists $N > 0$ such that
\[
p(z_n - z_{n+1}) < 1 \quad (n > N)
\]
Now $p(z_n - z_{n+1}) \leq p(z_n - z_{n+1})^2$ and so $p(z_n - z_{n+1})[1 - p(z_n - z_{n+1})] \leq 0$. It follows from (6) that for $n > N$, $p(z_n - z_{n+1}) \leq 0$ and so $p(z_n - z_{n+1}) = 0$. But then $|p(z_n - p(z_{n+1}))| \leq p(z_n - z_{n+1}) = 0$, so $p(z_n) = p(z_{n+1})$ for all $n > N$. Since $p(z_n) \to 0$ as $n \to \infty$, we have that $p(z_n) = 0$ for $n > N$.

We now consider under what conditions a topological algebra $A$ with an orthogonal basis $\{z_n\}$ has a cotriangular basis. Since an algebra with a cotriangular basis always has an identity, this condition is certainly necessary. Thus $e = \sum_{i=1}^{\infty} z_i$ converges in $A$ [5]. This means that the natural cotriangular sequence $\{z_n\}$ obtained from the basis $\{x_n\}$, where
\[
z_n = \sum_{k=n}^{\infty} x_k,
\]
is well-defined. The next theorem gives a condition under which this cotriangular sequence is actually a basis in $A$.

Theorem 12. Let $A$ be a topological algebra with identity $e$ and with an orthogonal basis $\{x_n\}$. Then the cotriangular sequence $\{z_n\}$ defined above is a basis in $A$ if and only if for each $x \in A$, $\lim_{n \to \infty} x_n^*(x)z_n = 0$.

Proof. Define the functionals $\{f_n\}$ on $A$ by
\[
f_1(x) = z_1^*, \quad f_n(x) = z_n^* - z_{n-1}^* \quad (n = 2, 3, \ldots)
\]
The sequence \( \{z_n, f_n\} \) is a biorthogonal sequence. We have (using the convention that \( z_0 = 0 \)):

\[
\sum_{i=1}^{k} f_i(x) z_i = \sum_{i=1}^{k} \left( (z_i^n(x) - z_{i-1}^n(x)) \sum_{j=1}^{\infty} x_j \right) \\
= \sum_{j=1}^{k-1} \sum_{i=1}^{j} (z_i^n(x) - z_{i-1}^n(x)) x_j + \sum_{j=k}^{\infty} \sum_{i=1}^{k} (z_i^n(x) - z_{i-1}^n(x)) x_j \\
= \sum_{j=1}^{k-1} z_j^n(x) x_j + \sum_{j=k}^{\infty} z_k^n(x) x_j \\
= \sum_{j=1}^{k-1} z_j^n(x) x_j + z_k^n(x) z_k
\]  

(7)

If \( \{z_n\} \) is a basis, then taking the limit as \( k \to \infty \) in (7) above shows that \( z_k^n(x) z_k \to 0 \) if and only if \( \sum_{j=1}^{\infty} f_j(x) z_j = x \). 

The following theorem is a characterization of \( s \) similar to that of \( c_0 \) in Theorem 7.

**Theorem 13.** Let \( A \) be a Fréchet algebra with an orthogonal basis \( \{x_n\} \). Then \( A \) has a cotriangular basis if and only if \( A \) is algebraically and topologically isomorphic to \( s \).

**Proof.** If \( A \) has a cotriangular basis \( \{z_n\} \), then \( z_1 \) is an identity for \( A \) and the result follows by [5, Theorem 3.4]. Conversely, if \( A \) is isomorphic to \( s \), then \( A \) has a cotriangular basis since \( s \) has such a basis (Section 5).

We now describe the relationship between a cotriangular basis and an orthogonal basis in the same topological algebra. The proofs of the lemma and theorem that follow are entirely complementary to the proofs of Lemma 8 and Theorem 9.

**Lemma 14.** Let \( A \) be a topological algebra with a cotriangular basis \( \{z_n\} \). Then \( \{z_n\} \) is a maximal cotriangular sequence.

**Theorem 15.** Let \( A \) be a topological algebra with an orthogonal basis \( \{x_n\} \) and a cotriangular basis \( \{z_n\} \). Then there exists a permutation \( \pi \) of \( \mathbb{N} \) such that \( z_n = \sum_{k=n}^{\infty} x_{\pi(k)} \).

It remains to consider the possibility of an algebra simultaneously having a triangular and cotriangular basis. This however is impossible since an algebra with a cotriangular basis \( \{z_n\} \) has \( z_1 \) as identity, whereas an algebra with a triangular basis cannot have an identity, as we now show.

**Theorem 16.** If the topological algebra \( A \) has a triangular basis, then \( A \) does not have an identity element.

**Proof.** Let \( \{y_n\} \) be a triangular basis for \( A \). Suppose that \( A \) has an identity \( e = \sum_{i=1}^{\infty} \alpha_i y_i \). We have \( y_2 = ey_2 = (\sum_{i=1}^{\infty} \alpha_i y_i) y_2 = \alpha_1 y_1 + (\sum_{i=2}^{\infty} \alpha_i) y_2 \). Thus \( \alpha_1 = 0 \).
Now suppose that $\alpha_i = 0$ for $i < n$. Then $y_{n+1} = \varepsilon y_{n+1} = (\sum_{i=n}^\infty \alpha_i) y_{n+1} + (\sum_{i=n+1}^\infty \alpha_i) y_{n+1}$, so that $\alpha_n = 0$. By induction, $\alpha_i = 0$ for all $i$, which is absurd. ■


We have just seen that the classes of algebras that have triangular bases and that have cotriangular bases are mutually exclusive. Nevertheless, parallel arguments show that Fréchet algebras with triangular or cotriangular bases are functionally continuous. These results are consequences of a theorem of Zelazko [17]: A Fréchet algebra with countable maximal ideal space is functionally continuous. We write $\mathcal{M}(A)$ for the collection of continuous multiplicative linear functionals on $A$.

Lemma 17. Let $A$ be a topological algebra.

(a) If $A$ has a triangular basis $\{x_n\}$ then for $f \in \mathcal{M}(A)$ there exists $n_0 \in \mathbb{N}$ such that $f = f^{n_0}$, where

$$f^{n_0}(x_k) = \begin{cases} 0 & \text{if } k < n_0 \\ 1 & \text{if } k \geq n_0 \end{cases}$$

(b) If $A$ has a cotriangular basis $\{x_n\}$ then for $f \in \mathcal{M}(A)$ there exists $n_0 \in \mathbb{N}$ such that $f = f_{n_0}$, where

$$f_{n_0}(x_k) = \begin{cases} 1 & \text{if } k \leq n_0 \\ 0 & \text{if } k > n_0 \end{cases}$$

Proof. In either case, each $x_n$ is idempotent, so $f(x_n) = f(x_n) f(x_n)$, and hence $f(x_n)$ has value 0 or 1. Now if the basis $\{x_n\}$ is triangular and if $f(x_m) = 1$, then for $m < n$ we have $f(x_m) f(x_n) = f(x_n)$, so $f(x_n) = 1$. This proves part (a). The complementary proof for a cotriangular basis yields part (b). ■

Theorem 18. A Fréchet algebra with a triangular basis or a cotriangular basis is functionally continuous.

Proof. Since a continuous linear functional is completely determined by its values on the basis elements, it follows by Lemma 17 that such an algebra has only countably many continuous linear functionals. The result then follows from the theorem of Zelazko [17] cited above. ■

For each $n$, the functionals $f^n$ and $f_n$ in Lemma 17 are well-defined and multiplicative, and hence continuous by Theorem 18. Thus if $A$ is a Fréchet algebra with a triangular (respectively, cotriangular) basis, then $\mathcal{M}(A) = \{f^n : n \in \mathbb{N}\}$ (respectively, $\mathcal{M}(A) = \{f_n : n \in \mathbb{N}\}$).

Theorem 19. A Fréchet algebra with a triangular basis or a cotriangular basis is semisimple.

Proof. Let $A$ be a Fréchet algebra with basis $\{x_n\}$, and let $x = \sum_{i=1}^\infty \alpha_i x_i \in \text{Rad}(A)$, so that $f(x) = 0$ for every $f \in \mathcal{M}(A)$. If the basis is triangular, then for every $n \in$
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N, \( f^n(x) = \sum_{i=n}^{\infty} \alpha_i = 0 \), so that each \( \alpha_i \) is 0. Thus \( x = 0 \). Similarly, if the basis is cotriangular, then for each \( n, f_n(x) = \sum_{i=1}^{n} \alpha_i = 0 \), so again each \( \alpha_i \) is 0, whence \( x = 0 \).

Theorems 18 and 19, together with [9, Theorem 14.2], show that a Fréchet algebra with a triangular or cotriangular basis has unique Fréchet algebra topology.

5. Examples.

In this section we give some examples of algebras with triangular and with cotriangular bases. Two obvious examples are \( c_0 \) and \( s \), each of which has a basis consisting of the sequence of unit vectors \( \{ e_n \} \), where \( e_n = (\delta_{ni})_{i=1}^{\infty} \). This is clearly an orthogonal basis under pointwise operations, and is unconditional in the case of \( c_0 \). If \( \pi \) is a permutation of \( \mathbb{N} \), and \( y_n = \sum_{k=1}^{n} e_{\pi(k)} \), then by Theorem 6, \( \{ y_n \} \) is a triangular basis for \( c_0 \). By Theorem 9, every triangular basis for \( c_0 \) is obtained in this way via some permutation \( \pi \).

Similarly, by Theorem 12, \( s \) has the cotriangular basis \( \{ w_n \} \), where \( w_n = \sum_{k=n}^{\infty} e_k \).

The space \( \ell^p(\mathbb{N}) (1 < p < \infty) \) of complex sequences \( x = (\alpha_i) \) satisfying \( ||x||_p = \left( \sum_{i=1}^{\infty} |\alpha_i|^p \right)^{1/p} < \infty \) is a Banach algebra under pointwise operations. The unit vectors \( \{ e_n \} \) form an unconditional orthogonal basis in \( \ell^p \). However, Theorems 9 and 6, together with a simple calculation, show that \( \ell^p \) does not have a triangular basis.

The Banach spaces \( L^p(T) (1 < p < \infty) \) of the circle group \( T \) with the norm \( ||x||_p = \left( \frac{1}{2\pi} \int_0^{2\pi} |x(e^{-i\phi})|^p d\phi \right)^{1/p} \) are Banach algebras with the convolution product

\[
xy(\theta) = \frac{1}{2\pi} \int_0^{2\pi} x(\theta - \phi)y(\phi) d\phi \quad (x, y \in L^p(T))
\]

(see [16]). Define \( x_{2k+1}(t) = t^k(= e^{ikt}) \) and \( x_{2k+2}(t) = t^{-k}(= e^{-ikt}), t \in T, k = 0, 1, \ldots \). Then \( \{ x_n \} \) is a basis for \( L^p(T) \) [8, page 51], which is orthogonal and normalized. It is unconditional for \( p = 2 \). Now \( L^p(T) \) has no triangular basis. For if \( \{ y_n \} \) were a triangular basis, then by Theorem 9 there would exist a permutation \( \pi \) of \( \mathbb{N} \) such that \( y_n = \sum_{k=1}^{n} x_{\pi(k)} \). Now by the Hausdorff-Young Theorem [18], if \( 1 < p \leq 2 \), then \( y_n \in \ell^2(\mathbb{N}) \) (where \( 1/p + 1/q = 1 \)), and \( ||y_n||_q = n^{1/q} \leq ||y_n||_p \). If \( p > 2 \), note that \( ||y_n||_p \geq ||y_n||_2 \).

Thus in either case, \( \{ y_n \} \) is not a bounded set, and so by Theorem 6, \( L^p(T) \) has no triangular basis.

For \( 1 < p < \infty, H^p(D) \) is the Hardy \( p \)-space on the open unit disc \( D \): that is, the Banach space of all functions \( f \) holomorphic in \( D \) and satisfying \( ||f||_p < \infty \), where \( ||f||_p = \lim_{r \to 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} \). Let \( x_n(z) = z^n, n = 0, 1, \ldots \). Then \( \{ x_n \} \) forms an unconditional basis for \( H^p(D) \), since the Maclaurin series for \( f \in H^p(D) \) converges to \( f \) in the \( H^p \) norm [10, Theorem 3.12(b)]. With the convolution product

\[
f * g(x) = \frac{1}{2\pi i} \int_{|z|=r} f(z)g(xz^{-1})z^{-1} dz \quad (|x| < r < 1),
\]

\( H^p(D) \) is a Banach algebra in which the basis \( \{ x_n \} \) is orthogonal. By Theorem 7, \( H^p(D) \) cannot have a triangular basis, since clearly \( H^p(D) \) is not isomorphic to \( c_0 \).
Let $H(D)$ be the space of all functions holomorphic on the open unit disc $D$, equipped with the compact-open topology. This topology is generated, for example, by the seminorms

$$p_r(f) = \sup_k \frac{|f^{(k)}(0)|}{r^k k!} \quad (f \in H(D), r > 1) \quad (8)$$

(see [7]). With the convolution product $\ast$ defined above for $H^p(D)$, the space $H(D)$ becomes a topological algebra with identity $i(z) = (1 - z)^{-1}$. The set of functions $\{x_n\}$, with $x_n(z) = z^n (n = 0, 1, 2, \ldots)$ is an orthogonal basis for $H(D)$. Let $z_n = \sum_{k=1}^\infty x_k$. A straightforward calculation shows that $\lim_{n \to \infty} (f^{(n)}(0)/n!) z_n = 0$ for every $f \in H(D)$, so using (8) and Theorem 12, $\{z_n\}$ is a cotriangular basis in $H(D)$.

The space $bv_0$ is the space of sequences that converge to 0 and have bounded variation: that is, sequences $x = (\alpha_i)$ for which $\sum |\alpha_i - \alpha_{i+1}| < \infty$; with norm $\|x\| = \sup i_{\geq 1} \sum |\alpha_i - \alpha_{i+1}|$. Under pointwise operations, $bv_0$ is a Banach algebra with normal orthogonal basis $\{e_n\}$ [8, pages 101 and 108]. Set $y_n = \sum_{k=1}^n e_k$. Then $\|y_n\| = 2, n = 1, 2, \ldots$, so by Theorem 6, $\{y_n\}$ is a triangular basis for $bv_0$. However, the basis $\{e_n\}$ is not unconditional, so Theorem 7 does not apply.

Finally, we mention the James space $J$, which often appears in the literature as a counterexample. In our context, though, the algebra $J$ is very well behaved, having both an orthogonal and a triangular basis. Precisely, $J$ consists of those sequences $x = \{x_n\}$ that converge to 0 and for which the norm

$$\|x\| = \sup \left\{ \left( \sum_{i=2}^n |x_p(i) - x_p(i-1)|^2 \right)^{1/2} : 1 \leq p(1) < \cdots < p(n) \right\}$$

is finite. Under pointwise operations, $J$ is a Banach algebra with the basis of unit vectors $\{e_n\}$ [2]. By Theorem 6, the triangular sequence obtained from $\{e_n\}$ is a basis, since $\|\sum_{k=1}^n e_k\| = 1$ for all $n$.

References


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