Prime and maximal ideals in subrings of $C(X)$

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Abstract


The structure of ideals in the ring $C(X)$ of continuous functions on a completely regular space $X$ and its subring $C^*(X)$ consisting of the bounded functions is well known. In this paper we study the prime and maximal ideals in subrings $A(X)$ of $C(X)$ that contain $C^*(X)$. We show that many of the results known separately for $C(X)$ and $C^*(X)$, often by different methods, are true for any such $A(X)$. Our results put the problems of $C(X)$ and $C^*(X)$ in a common setting by exhibiting these as special instances of the subrings $A(X)$. We characterize prime and maximal ideals in any $A(X)$ in terms of their residue class rings and in terms of certain $z$-filters on $X$ that correspond to these ideals. We also characterize the intersection of the free ideals and the free maximal ideals in any $A(X)$.

Keywords: Rings of continuous functions, prime ideal, maximal ideal, $z$-filter.

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Introduction

The ring $C(X)$ of continuous real-valued functions on a completely regular space $X$ and its subring $C^*(X)$ of bounded functions have been studied extensively (see [1, 6]). One of the major accomplishments of this theory is the characterization of the sets of maximal ideals in these rings. In fact, the maximal ideals in $C^*(X)$ correspond in a natural way to the points of $\beta X$, the Stone-Čech compactification of $X$. In a very different way, it turns out that the maximal ideals in $C(X)$ are also in one to one correspondence with the points of $\beta X$. As pointed out in [6] it is a remarkable fact that the two distinct problems of characterizing the set of maximal ideals in these two rings have a common solution. To see why the two problems
are distinct one need only consider what it means for a function \( f \) to be invertible in each of these rings—\( f \) is invertible in \( C(X) \) if it is never zero and in \( C^*(X) \) if it is bounded away from zero. In [13] a method is given for characterizing the maximal ideals in any ring of functions \( A(X) \) on the space \( X \) such that

\[
C^*(X) \subseteq A(X) \subseteq C(X).
\]

It is shown in there that for such a ring \( A(X) \) its set of maximal ideals is in one to one correspondence with the points of \( \beta X \). This result puts the problems of \( C^*(X) \) and \( C(X) \) in a common setting.

Another focus in the study of the rings of continuous functions is the characterization of the prime ideals. Much of the work done in this direction is for \( C(X) \) only because of certain difficulties encountered in the setting of \( C^*(X) \). In this paper we show that many of the results on \( C(X) \) generalize to any subring \( A(X) \) of \( C(X) \) that contains \( C^*(X) \). (An interesting example of a ring \( A(X) \) strictly between \( C(X) \) and \( C^*(X) \) is studied in [Z].)

One of the main tools in this paper is a \( z \)-filter \( \mathcal{Z}_A(f) \) associated with each function \( f \in A(X) \) as introduced in [13]. In Section 1 we define these \( z \)-filters and show that \( \mathcal{Z}_A \) extends to a mapping from the set of ideals of \( A(X) \) into the set of \( z \)-filters on \( X \). This enables us to single out a class of ideals of \( A(X) \) we call \( \mathfrak{z} \)-ideals that includes the maximal ideals of \( A(X) \). We study the relationships between prime ideals and \( \mathfrak{z} \)-ideals and obtain as a corollary that every prime ideal is contained in a unique maximal ideal. We also prove that the ring \( A(X) \) is a lattice under its natural order, a fact used throughout the paper. In Section 2 we consider the residue class ring of \( A(X) \) modulo prime and maximal ideals and show that every prime ideal \( P \) in \( A(X) \) is absolutely convex and the ring \( A(X)/P \) is totally ordered. In Section 3 we clarify the relationship between maximal ideals in \( A(X) \) and \( z \)-ultrafilters on \( X \). In particular, we show that there is a natural one to one correspondence between the set \( \mathcal{M}(A) \) of maximal ideals in \( A(X) \) and the set \( \beta X \) of \( z \)-ultrafilters on \( X \) and that this correspondence is a homeomorphism if \( \mathcal{M}(A) \) is endowed with the hull-kernel topology and \( \beta X \) is endowed with the Stone topology. It follows that there is a natural correspondence between the maximal ideals of any two subalgebras \( A(X) \) and \( B(X) \) containing \( C^*(X) \). In Section 4 we define the ideals \( O^p_\mathcal{A} \) in any \( A(X) \) which are analogous to the ideals \( O^p \) defined in [4] for \( C(X) \). We consider \( F \)-spaces and \( P \)-spaces \( X \), and determine to what extent the algebraic conditions on \( C(X) \) that characterize these topological spaces are also satisfied by a subalgebra \( A(X) \). In Section 5 we give descriptions of the intersection of all the free ideals and the intersection of all the free maximal ideals in any \( A(X) \).

One way to view the ring \( A(X) \) is as an ordered algebra in its natural order. Indeed, \( A(X) \) is an archimedean lattice-ordered algebra over the reals with an identity \( 1 \) that is a weak order unit (i.e., \( 1 \wedge f = 0 \) implies \( f = 0 \)). Such algebras are studied in [7] and [8] where they are called \( \Phi \)-algebras. Some of the results of the present paper are valid in the more general setting of \( \Phi \)-algebras. In particular, the fact that \( \mathcal{M}(A) \) is homeomorphic to \( \beta X \) (Theorem 3.4) is a consequence of Corollary
2.8 of [8], and parts of our results on prime ideals follow from [7]. We also refer to [3, Chapter 13] for related results.

Throughout this paper $A(X)$ and $B(X)$ denote subrings of $C(X)$ that contain $C^*(X)$.

1. Prime ideals and $\mathfrak{z}$-ideals

In this section we introduce a mapping that sets up a correspondence between the set of ideals of $A(X)$ and the set of $\mathfrak{z}$-filters on $X$ and study a class of prime ideals we call $\mathfrak{z}$-ideals. The proof of Theorem 1.8 below depends on the fact that $A(X)$ is a lattice under its natural order, which we now prove.

**Theorem 1.1.** If $f \in A(X)$, then $|f| \in A(X)$.

**Proof.** Let $E = \{x \in X | f(x) \geq 1\}$ and $F = \{x \in X | f(x) \leq -1\}$. Since $E$ and $F$ are completely separated, there exists $g \in C^*(X)$ such that $g(E) = 1$ and $g(F) = -1$, and $-1 \leq g(x) \leq 1$ for all $x \in X$. Let $h = gf - |f|$. If $x \in E \cup F$, $h(x) = 0$. If $x \not\in E \cup F$, then $|h(x)| \leq |g(x)f(x)| + |f(x)| \leq 2|f(x)| \leq 2$. Hence, $h \in C^*(X) \subset A(X)$ and so $|f| = gf - h \in A(X)$. □

It is immediate from this theorem that $A(X)$ is a lattice since for any $f, g \in A(X)$,

$f \vee g - \frac{1}{2}(f + g + |f - g|) \in A(X)$.

For each $f \in A(X)$ we let $Z(f)$ denote the zero set of $f$, i.e., $Z(f) = \{x \in X | f(x) = 0\}$. The collection of all zero sets of $X$ is denoted by $Z[X] = \{Z(f) | f \in C(X)\}$. The study of the ideals in $C(X)$ depends strongly on the fact that if $I$ is an ideal in $C(X)$ then $Z[I] = \bigcup \{Z(f) | f \in I\}$ is a $\mathfrak{z}$-filter. However, this is not always true in an arbitrary $A(X)$. For example, in $C^*(\mathbb{N})$ the set $I$ of all sequences that converge to 0 is an ideal containing the sequence $(1/n)_{n \in \mathbb{N}}$ whose zero set is empty. Thus $Z[I]$ is not a $\mathfrak{z}$-filter. Another method is used for studying the ideals in $C^*(X)$ (see [6, p. 32]). Neither of these methods is applicable to the study of the ideals in an arbitrary $A(X)$ since they both depend on the simple characterizations of invertability available for $C(X)$ and $C^*(X)$.

We now describe a method applicable to an arbitrary $A(X)$. For each $f \in A(X)$ we associate a collection $\mathcal{Z}_A(f)$ of subsets of $X$ given by

$\mathcal{Z}_A(f) = \{E \in Z[X] | \exists g \in A(X) \text{ such that } fg|_E = 1\}$.

In other words, $\mathcal{Z}_A(f)$ consists of those zero sets such that $f$ is locally invertible in $A(X)$ on their complements. Following [13], for an ideal $I$ of $A(X)$ we set

$\mathcal{Z}_A[I] = \bigcup \{\mathcal{Z}_A(f) | f \in I\}$

and for a $\mathfrak{z}$-filter $\mathcal{F}$ on $X$ we set

$\mathcal{Z}_A[\mathcal{F}] = \{f \in A(X) | \mathcal{Z}_A(f) \subset \mathcal{F}\}$.

The following lemmas summarize some basic facts about the maps $\mathcal{Z}_A$ and $\mathcal{F}_A$. 
Lemma 1.2. Let $f, g \in A(X)$.
(a) $\mathcal{I}_A(fg) \subseteq \mathcal{I}_A(f) \cap \mathcal{I}_A(g)$.
(b) $\mathcal{I}_A(f^2 + g^2) \supseteq \mathcal{I}_A(f) \cup \mathcal{I}_A(g)$.
(c) If $|f| \leq |g|$, then $\mathcal{I}_A(f) \subseteq \mathcal{I}_A(g)$.

Proof. These follow from [13, Lemma 1].

Lemma 1.3. (a) If $f \in A(X)$, then $\lim_{\mathcal{I}_A(f)} fh = 0$ for $h \in A(X)$.
(b) If $\mathcal{F}$ is a $z$-filter on $X$ and if $\lim_{\mathcal{F}} fh = 0$ for all $h \in A(X)$, then $\mathcal{I}_A(f) \subseteq \mathcal{F}$.

Proof. See [13, Lemmas 2 and 3].

Lemma 1.4. Let $f \in A(X)$, I an ideal of $A(X)$, and $\mathcal{F}$ a $z$-filter on $X$.
(a) $\mathcal{I}_A(f)$ is a $z$-filter on $X$ if and only if $f$ is not invertible in $A(X)$.
(b) If $I$ is an ideal in $A(X)$, then $\mathcal{I}_A[I]$ is a $z$-filter on $X$.
(c) If $\mathcal{F}$ is a $z$-filter on $X$, then $\mathcal{I}_A[\mathcal{F}]$ is an ideal in $A(X)$.

Proof. See [13, Theorems 1, 2, and 3].

It is immediate from Lemma 1.4 that $\mathcal{I}_A$ is a mapping from the set of ideals in $A(X)$ into the set of $z$-filters on $X$, and that $\mathcal{I}_A^*$ is a mapping from the set of $z$-filters on $X$ into the set of ideals in $A(X)$. In the setting of $C(X)$, it is well known that if $M$ is a maximal ideal in $C(X)$ then $Z[M]$ is a $z$-ultrafilter on $X$, and that the mapping $Z$ is a bijection from the set of maximal ideals of $C(X)$ onto the set of $z$-ultrafilters on $X$. However, as we show in Example 1.5 below, even when $A(X) = C(X)$, $\mathcal{I}_A[M]$ may fail to be a $z$-ultrafilter for a maximal ideal $M$ in $A(X)$.

Nevertheless we show in Section 3 that $\mathcal{I}_A$ induces a bijection between the set of maximal ideals of $A(X)$ and the set of $z$-ultrafilters on $X$. The next lemma is immediate from the definitions of $\mathcal{I}_A$ and $\mathcal{I}_A^*$.

Lemma 1.5. For an ideal $I$ of $A(X)$ and a $z$-filter $\mathcal{F}$ on $X$ we have:
(a) $\mathcal{I}_A[I] \supseteq I$,
(b) $\mathcal{I}_A[\mathcal{I}_A[I]] = \mathcal{I}_A[I]$,
(c) $\mathcal{I}_A[\mathcal{I}_A^*[\mathcal{F}]] = \mathcal{F}$,
(d) $\mathcal{I}_A[\mathcal{I}_A[\mathcal{I}_A^*[\mathcal{F}]]] = \mathcal{I}_A^*[\mathcal{F}]$.

The inclusions in (a) and (c) can be proper as we show in Example 1.6 below. We first note the relationship between the filter $\mathcal{I}_A(f)$ associated with $f$ and the zero set $Z(f)$. Namely, for any $f \in A(X)$,

$Z(f) = \bigcap \mathcal{I}_A(f)$.

For, if we let $E_\varepsilon(f) = \{x \in X \mid |f(x)| \leq \varepsilon\}$, then $E_\varepsilon(f) \in \mathcal{I}_A(f)$ for every $\varepsilon > 0$. Also each element $E$ of $\mathcal{I}_A(f)$ contains some $E_\varepsilon$. Thus $\bigcap \mathcal{I}_A(f) = \bigcap \{E_\varepsilon(f) \mid \varepsilon > 0\} = Z(f)$. In the theory of $C(X)$ the zero set $Z(f)$ is a measure of where $f$ is not
invertible. The filter \( \mathcal{I}_A(f) \) plays an analogous role for an arbitrary \( A(X) \). In particular, \( \mathcal{I}_A(f) \) is a measure of where \( f \) is not invertible in \( A(X) \) when \( Z(f) = \emptyset \), a situation that does not occur in \( C(X) \).

**Example 1.6.** In the ring \( C(\mathbb{R}) \) let \( M_0 = \{ f \in C(\mathbb{R}) \mid f(0) = 0 \} \). Then \( M_0 \) is a maximal ideal in \( C(\mathbb{R}) \) and by [6, Theorem 2.5], \( Z[M_0] \) is a \( z \)-ultrafilter on \( \mathbb{R} \). Since by the preceding paragraph \( Z(f) = \bigcap \mathcal{I}_C(f) \) for any \( f \in M_0 \), it follows that \( \mathcal{I}_C(f) \subset Z[M_0] \) for all \( f \in M_0 \) and hence \( \mathcal{I}_C[M_0] \subset Z[M_0] \). On the other hand, the set \( \{ 0 \} \) is an element of \( Z[M_0] \) but not an element of \( \mathcal{I}_C[M_0] \). Thus \( \mathcal{I}_C[M_0] \neq Z[M_0] \).

Let \( \mathcal{F} = Z[M_0] \). Then \( \mathcal{I}_C[\mathcal{F}] = M_0 \) and hence \( \mathcal{I}_C[\mathcal{I}_C[\mathcal{F}]] = \mathcal{I}_C[M_0] \subset Z[M_0] = \mathcal{F} \). This is an example of proper inclusion in (c) of Lemma 1.5. For an example of proper inclusion in (a) of that lemma consider the principal ideal \( I = (i) \) in \( C(\mathbb{R}) \) where \( i \) is the identity function on \( \mathbb{R} \). Note that for any \( f \in M_0 \), \( \mathcal{I}_C(f) \subset \mathcal{I}_C(i) \) and so \( \mathcal{I}_C[M_0] \subset \mathcal{I}_C[I] \). Since the reverse inclusion is obvious we have \( \mathcal{I}_C[M_0] = \mathcal{I}_C[I] \).

Then \( M_0 = \mathcal{I}_C[\mathcal{I}_C[M_0]] \supset \mathcal{I}_C[\mathcal{I}_C[I]] \supset I \). However, \( M_0 \neq I \). (For instance, \( i^{1/3} \in M_0 \setminus I \). For if \( i^{1/3} \in I \), then \( i^{1/3} = ig \) for some \( g \in C(\mathbb{R}) \). But then \( g = i^{-2/3} \), a contradiction.)

The ideals \( I \) of \( A(X) \) that satisfy \( \mathcal{I}_A[I] = I \) play an important role in the rest of the paper. We call ideals that satisfy this property \( \mathcal{Z}_A \)-ideals. Thus \( I \) is a \( \mathcal{Z}_A \)-ideal if and only if \( \mathcal{I}_A(f) \subset \mathcal{I}_A[I] \) implies \( f \in I \).

Any maximal ideal is a \( \mathcal{Z}_A \)-ideal. In \( C(\mathbb{N}) \) every ideal is a \( \mathcal{Z}_A \)-ideal. For, suppose \( f \in C(\mathbb{N}) \) and \( \mathcal{I}_C(f) \subset \mathcal{I}_C[I] \). Since \( \mathbb{N} \) is discrete, it is easy to see that \( Z(f) \in \mathcal{I}_C(f) \) and \( \mathcal{I}_C[I] = Z[I] \). Thus \( Z(f) \in Z[I] \) and so there exists \( g \in I \) such that \( Z(f) = Z(g) \).

Define \( h \) as follows: \( h(n) = 0 \) for \( n \in Z(g) \) and \( h(n) = f(n)/g(n) \) for \( n \in Z(g) \). Since \( f = hg \), it follows that \( f \in I \).

The definition of a \( z \)-ideal in [6, 2.7] applies to \( C(X) \) only and is somewhat less restrictive than the notion of a \( \mathcal{Z}_A \)-ideal in the sense that an ideal in \( C(X) \) may be a \( z \)-ideal but may fail to be a \( \mathcal{Z}_A \)-ideal. For example, the ideal \( O_0 = \{ f \in C(\mathbb{R}) \mid Z(f) \) is a neighborhood of \( 0 \} \) is a \( z \)-ideal in \( C(\mathbb{R}) \) [6, p. 28]. Now \( \mathcal{I}_C[O_0] = \mathcal{I}_C[M_0] \), where \( M_0 \) is as in Example 1.6. Thus \( \mathcal{I}_C[\mathcal{I}_C[O_0]] = \mathcal{I}_C[\mathcal{I}_C[M_0]] = M_0 \). Since \( O_0 \neq M_0 \) it follows that \( O_0 \) is not a \( \mathcal{Z}_A \)-ideal. (We study this situation in more detail in Section 3.) Nevertheless, \( \mathcal{Z}_A \)-ideals in the setting of \( A(X) \) play an analogous role to \( z \)-ideals in \( C(X) \).

We recall that an ideal \( P \) in a commutative ring \( R \) is prime if \( ab \in P \) implies \( a \in P \) or \( b \in P \).

**Theorem 1.7.** Every \( \mathcal{Z}_A \)-ideal in \( A(X) \) is an intersection of prime ideals.

**Proof.** For every \( n \in \mathbb{N} \), \( \mathcal{I}_A(f^n) = \mathcal{I}_A(f) \). Thus, if \( I \) is a \( \mathcal{Z}_A \)-ideal, then \( f^n \in I \) implies \( f \in I \). This implies that \( I \) is the intersection of all the prime ideals which contain it (see [6, 0.18]). \( \square \)
Theorem 1.8. For any \( \mathfrak{A} \)-ideal \( I \) in \( A(\mathcal{X}) \), the following are equivalent.

(a) \( I \) contains a prime ideal.
(b) For every \( g, h \in A(\mathcal{X}) \), if \( gh = 0 \), then \( g \in I \) or \( h \in I \).
(c) For every \( f \in A(\mathcal{X}) \) and for every \( \varepsilon > 0 \) there exists a zero set in \( \mathcal{X}_A[I] \) on which \( f \leq \varepsilon \) or \( f \geq -\varepsilon \).

Proof. (a) \( \Rightarrow \) (b) Clear.
(b) \( \Rightarrow \) (c) Let \( P \) be a prime ideal contained in \( I \). If \( gh = 0 \), then \( gh \in P \). So either \( g \) or \( h \) is in \( P \) and hence in \( I \).
(c) \( \Rightarrow \) (d) Since \((f \vee 0)(f \wedge 0) = 0 \) for all \( f \in A(\mathcal{X}) \), \( f \vee 0 \in I \) or \( f \wedge 0 \in I \). If \( f \vee 0 \in I \), then \( \mathcal{X}_A[f \vee 0] \subseteq \mathcal{X}_A[I] \). In particular, \( E_\varepsilon(f \vee 0) = \{ x \in \mathcal{X} \mid |(f \vee 0)(x)| \leq \varepsilon \} \) is in \( \mathcal{X}_A(f \vee 0) \) and hence in \( \mathcal{X}_A[I] \). That is, \( f \leq \varepsilon \) on \( E_\varepsilon(f \vee 0) \) which is a zero set in \( \mathcal{X}_A(f \vee 0) \) and hence in \( \mathcal{X}_A[I] \). Similarly, if \( f \wedge 0 \in I \) then \( f \geq -\varepsilon \) on \( E_\varepsilon(f \wedge 0) \) which is a zero set in \( \mathcal{X}_A(f \wedge 0) \) and hence in \( \mathcal{X}_A[I] \).
(d) \( \Rightarrow \) (a) Let \( gh \in I \) and consider \(|g| - |h|\). For any \( \varepsilon > 0 \) there exists \( E \in \mathcal{X}_A[I] \) on which \(|g| - |h| < \varepsilon \), say. Thus for any \( \delta > \varepsilon \), \(|g| > \delta \) implies \(|h| > \delta - \varepsilon \) on \( E \). Let \( E_g = \{ x \in \mathcal{X} \mid |g(x)| \leq \delta \} \) and \( E_h = \{ x \in \mathcal{X} \mid |h(x)| \leq \delta - \varepsilon \} \). We have \( E_g \in \mathcal{X}_A(g) \), \( E_h \in \mathcal{X}_A(h) \), and \( E \cap E_h \subseteq E \cap E_g \). Now \( E_g \supset E \cap E_h \) is a \( \mathfrak{A} \)-ideal containing \( I \) and hence is prime by Theorem 1.8. But the intersection of distinct prime ideals (in any commutative ring) is never prime.

Corollary 1.9. Every prime ideal in \( A(\mathcal{X}) \) is contained in a unique maximal ideal.

Proof. Suppose a prime ideal \( P \) in \( A(\mathcal{X}) \) is contained in two distinct maximal ideals \( M \) and \( M' \). Then the ideal \( M \cap M' \) is a \( \mathfrak{A} \)-ideal containing \( P \) and hence is prime by Theorem 1.8. But the intersection of distinct prime ideals (in any commutative ring) is never prime.

Corollary 1.9 also follows from [4, Theorem 3.4]. The next theorem is a natural generalization of a result known for \( C^*(\mathcal{X}) \) and \( C(\mathcal{X}) \).

Theorem 1.10. If \( A(\mathcal{X}) \subseteq B(\mathcal{X}) \), then \( P \) is a prime ideal in \( B(\mathcal{X}) \) if and only if \( P \cap A(\mathcal{X}) \) is a prime ideal in \( A(\mathcal{X}) \).

Proof. Suppose \( P \cap A(\mathcal{X}) \) is prime and \( fg \in P \). By [6, p. 211] there exist \( u \) and \( v \) in \( C^*(\mathcal{X}) \) such that \( uf = (f \vee -1) \wedge 1 \) and \( vg = (g \vee -1) \wedge 1 \). Since \( uf \) and \( vg \) are bounded, \( ufvg \in P \cap A(\mathcal{X}) \). Thus \( uf \in P \cap A(\mathcal{X}) \), say, and so \( uf \in P \). It is clear that \( u(|f| \vee 1) = 1 \). Since \( |f| \vee 1 \in B(\mathcal{X}) \) by Theorem 1.1, it follows that \( u \) is invertible in \( B(\mathcal{X}) \). Thus \( f \in P \).
2. Maximal ideals and residue class rings

In this section we study maximal ideals in $A(X)$ in terms of their associated residue class fields. As a consequence we show that every prime ideal $P$ in $A(X)$ is absolutely convex and the residue class ring $A(X)/P$ is totally ordered.

An ideal $I$ in $C(X)$ is fixed if $\bigcap Z[I]$ is nonempty and free if $\bigcap Z[I]$ is empty. Similarly we define an ideal $I$ of $A(X)$ to be fixed if $\bigcap z[I] \neq \emptyset$ and free if $\bigcap z[I] = \emptyset$. The two definitions are actually equivalent since $\bigcap z[I] = \emptyset$ if and only if $\bigcap Z[I] = \emptyset$. Indeed, we have

$$\bigcap z[I] = \bigcap Z[I].$$

The fixed maximal ideals in $A(X)$ are easy to characterize. The proof of the following theorem follows from the proof of the corresponding theorems for $C(X)$ and $C^*(X)$ (see [6, Theorems 4.6 and 4.7]). For an ideal $I$ of $A(X)$ we write $I(a)$ for the residue class of $f$ modulo $I$.

**Theorem 2.1.** The fixed maximal ideals in $A(X)$ are precisely the sets

$$M_a^R = \{ f \in A(X) \mid f(p) = 0 \} \quad (p \in X).$$

The ideals $M_a^R$ are distinct for distinct $p$ and the mapping $M_a^R(f) \to f(p)$ is the unique isomorphism of $A(X)/M_a^R$ onto $\mathbb{R}$. If $B(X)$ is another subring of $C(X)$ such that $A(X) \subseteq B(X)$, then there is a one to one correspondence between the fixed maximal ideals in $B(X)$ and those in $A(X)$ given by

$$M_B^R \to M_a^R = M_B^R \cap A(X).$$

If the maximal ideal $M$ in $A(X)$ is fixed, then $A(X)/M = \mathbb{R}$ as in Theorem 2.1. If $M$ is free then $A(X)/M$ may properly contain an isomorphic copy of $\mathbb{R}$. We show that this latter property is a result of the presence of unbounded functions in $A(X)$ (Theorem 2.7). First we recall that an ideal $I$ in the lattice-ordered ring $A(X)$ is convex if whenever $0 \leq x \leq y$ and $y \in I$, then $x \in I$. Also $I$ is absolutely convex if whenever $|x| \leq |y|$ and $y \in I$, then $x \in I$. If $I$ is convex, the ring $A(X)/I$ is a partially ordered ring according to the definition: $I(f) \geq 0$ if there exists $g \in A(X)$ such that $g \geq 0$ and $f \equiv g \pmod{I}$. (See [6, Theorems 5.2 and 5.3] for other properties of absolutely convex ideals.) It is clear that every $\mathcal{I}$-ideal in $A(X)$ is absolutely convex and hence every maximal ideal in $A(X)$ is absolutely convex. For a maximal ideal $M$ in $C(X)$ it is known that the order in $C(X)/M$ is closely connected with the zero sets in $X$—for instance, $M(f) > 0$ if and only if $f > 0$ on some member of $Z[M]$. The following theorem contains the corresponding statement in our more general setting of $A(X)$.

**Theorem 2.2.** Let $M$ be a maximal ideal in $A(X)$ and $f \in A(X)$. Then $M(f) \geq 0$ if and only if $\lim_{x \to M} f[h] \geq 0$ for every $h \in A(X)$. Moreover, $M(f) > 0$ if and only if there exists $h \in A(X)$ such that $\lim_{x \to M} f[h] > 0$. 
Proof. The theorem follows from Lemmas 2.3 and 2.4 below. □

Lemma 2.3. For a \( \mathfrak{Z} \)-ideal \( I \) in \( A(X) \) and \( f \in A(X) \), \( I(f) \neq 0 \) if and only if \( \lim_{E \downarrow I} f|E| > 0 \) for every \( h \in A(X) \).

Proof. (⇒) Since \( I \) is absolutely convex, \( I(f) \neq 0 \) implies \( f - |f| \in I \) [6, Theorem 5.3]. Then \( f|E| - |f| |E| \in I \) for each \( h \in A(X) \) and so \( \mathcal{Z}_A(f|E| - |f| |E|) \subseteq \mathcal{Z}_A[I] \). In particular, \( E_r(f|E| - |f| |E|) \subseteq \mathcal{Z}_A[I] \) for every \( \varepsilon > 0 \) and thus \( f|E| \geq - \varepsilon \) on \( E_r(f|E| - |f| |E|) \).

(⇐) Suppose that for \( \varepsilon > 0 \) and \( h \in A(X) \), \( f|E| \geq - \varepsilon \) on some member \( E \) of \( \mathcal{Z}_A[I] \). Then \( -2\varepsilon \leq (f|E| - |f| |E|) \leq 0 \) on \( E \) and so \( E \subseteq E_{2\varepsilon}(f|E| - |f| |E|) \). Thus \( E_{2\varepsilon}(f|E| - |f| |E|) \subseteq \mathcal{Z}_A[I] \) for all \( \varepsilon > 0 \). By Lemma 1.3(b) we obtain \( \mathcal{Z}_A(f - |f|) \subseteq \mathcal{Z}_A[I] \). Since \( I \) is a \( \mathfrak{Z} \)-ideal it follows that \( f - |f| \in I \). Thus \( I(f) \neq 0 \). □

Lemma 2.4. Let \( I \) be a \( \mathfrak{Z} \)-ideal of \( A(X) \) and \( f \in A(X) \). If there exists \( h \in A(X) \) such that \( \lim_{E \downarrow I} f|E| > 0 \), then \( I(f) > 0 \). If \( I \) is a maximal ideal the converse holds as well.

Proof. Suppose that there exists \( h \in A(X) \) and \( \varepsilon > 0 \) such that \( f|E| > \varepsilon \) on \( E \subseteq \mathcal{Z}_A[I] \). First note that \( f > 0 \) on \( E \) and so \( I(f) \neq 0 \) by Lemma 2.3. On the other hand, since \( f|E| > \varepsilon \) on \( E \subseteq \mathcal{Z}_A[I] \), some member of \( \mathcal{Z}_A(f|E|) \) does not meet \( E \). Since \( \mathcal{Z}_A(f|E|) \subseteq \mathcal{Z}_A(f) \), some member of \( \mathcal{Z}_A(f) \) does not meet \( E \). So \( f \in I \) and thus \( I(f) > 0 \). Now suppose that \( I \) is a maximal ideal and \( I(f) > 0 \). Since \( (I,f) = A(X) \) there exists \( g \in I \) and \( h \in A(X) \) such that \( g + fh = 1 \). Fix \( \varepsilon < \frac{1}{2} \). Then \( E_r(fh) \) is disjoint from \( E_r(g) \subseteq \mathcal{Z}_A[I] \). Also, by Lemma 2.3 there exists \( E' \subseteq \mathcal{Z}_A[I] \) on which \( f|E'| > - \varepsilon \). Then \( |f| > \varepsilon \) on \( E' \cap E_r(g) \). Thus \( f|E'| > \varepsilon \) on \( E' \cap E_r(g) \subseteq \mathcal{Z}_A[I] \).

Theorem 2.5. Every prime ideal \( P \) in \( A(X) \) is absolutely convex and the residue class ring \( A(X)/P \) is totally ordered. Moreover, the mapping \( r \mapsto P(r) \) is an order-preserving isomorphism of the real field \( \mathbb{R} \) onto \( A(X)/P \).

Proof. To show that \( P \) is absolutely convex, let \( 0 < |f| \leq |g| \) with \( g \in P \). Then \( 0 < f^2 \leq g^2 \) with \( g \in P \). Now it is easy to see that there is a unit \( u \) of \( C(X) \) such that \( g^2 \wedge 1 = ug^2 \) and \( u \in C^*(X) \). Thus \( ug^2 \in C^*(X) \subseteq A(X) \) and \( 0 \leq ug^2 \leq u \) with \( ug^2 \in P \). Define \( h \) as follows:

\[
h(x) = \begin{cases} u(x)f^4(x)/g^2(x), & \text{if } x \notin Z(g), \\ 0, & \text{if } x \in Z(g). \end{cases}
\]

Then \( h(x) \in C^*(X) \subseteq A(X) \). Now \( uf^4 = hg^2 \) and hence \( uf^4 \in P \). Since \( P \) is prime, either \( u \in P \) or \( f \in P \). We show that \( u \notin P \). For if \( u \notin P \), then \( u + ug^2 \in P \). Now if \( g^2(x) < 1 \), then \( u(x) = 1 \) and hence \( (u + ug^2)(x) \geq 1 \). If \( g^2(x) \geq 1 \) then \( u(x)g^2(x) = 1 \) and hence \( (u + ug^2)(x) \geq 1 \). It follows that \( u + ug^2 \) is invertible in \( A(X) \), a contradiction. Thus \( u \notin P \) and so \( f \in P \). Therefore \( P \) is absolutely convex.
Since \((f - |f|)(f + |f|) = 0\), either \(f = |f| \mod I\) or \(f = -|f| \mod I\). That is, \(I(f) \geq 0\) or \(I(f) \leq 0\) [6, Theorem 5.3]. Thus \(A(X)/P\) is totally ordered. The final statement in the theorem is clear. 

**Theorem 2.6.** If \(I\) is a \(\mathfrak{A}\)-ideal in \(A(X)\), then \(I\) is prime if and only if \(A(X)/I\) is totally ordered.

**Proof.** Necessity is a special case of Theorem 2.5. Conversely, if \(A(X)/I\) is totally ordered then either \(I(f) \geq 0\) or \(I(f) \leq 0\) for every \(f \in A(X)\). If \(I(f) > 0\) then by Lemma 2.3 for every \(\varepsilon > 0\), \(f \geq -\varepsilon\) on some member of \(\mathcal{D}_A[I]\). Similarly, if \(I(f) \leq 0\) then for every \(\varepsilon > 0\), \(f \leq \varepsilon\) on some member of \(\mathcal{D}_A[I]\). It follows from Theorem 1.8 that \(I\) is prime. 

Every residue class field of \(A(X)\) modulo a maximal ideal \(M\) contains a canonical copy of the real field \(\mathbb{R}\): the set of images of the constant functions under the canonical homomorphism. As in the case of \(C(X)\) [6, p. 701], when the canonical copy of \(\mathbb{R}\) is the entire field \(A(X)/M\), we call \(M\) a real ideal. When \(A(X)/M\) is not isomorphic to \(\mathbb{R}\) it is called hyper-real and \(M\) is called a hyper-real ideal. The residue class field is hyper-real if and only if it is nonarchimedean. A nonarchimedean totally ordered field is characterized by the presence of infinitely large elements, that is, elements \(a\) such that \(a > n\) for all \(n \in \mathbb{N}\) [6, p. 70].

**Theorem 2.7.** Let \(f \in A(X)\). For a given maximal ideal \(M\) in \(A(X)\), the following are equivalent.

(a) \(|M(f)|\) is infinitely large.

(b) \(f\) is unbounded on every member of \(\mathcal{D}_A[M]\).

(c) For each \(n \in \mathbb{N}\), the zero set \(Z_n = \{x | |f(x)| \geq n\}\) belongs to \(\mathcal{D}_A[M]\).

**Proof.** (b) \(\Rightarrow\) (a) Suppose \(|M(f)| \leq n\). Then Theorem 2.2 implies that for \(\varepsilon > 0\), \(n - |f| \geq -\varepsilon\) on some member of \(\mathcal{D}_A[M]\). Thus \(|f| \leq n + \varepsilon\) on some member of \(\mathcal{D}_A[M]\).

(a) \(\Rightarrow\) (c) Since \(|M(f)|\) is infinitely large, \(|M(f)| > n\) for all \(n\). By Theorem 2.2, for every \(n \in \mathbb{N}\) there is some \(E_n \in \mathcal{D}_A[M]\) on which \(|f| - n > 0\). Since \(E_n \subset Z_n\), \(Z_n \in \mathcal{D}_A[M]\).

(c) \(\Rightarrow\) (b) Let \(Z \in \mathcal{D}_A[M]\). Since \(Z \cap Z_n \neq \emptyset\) for each \(n \in \mathbb{N}\), \(f\) is clearly unbounded on \(Z\). 

**Theorem 2.8.** \(|M(f)|\) is infinitely large for some maximal ideal \(M\) in \(A(X)\) if and only if \(f\) is unbounded on \(X\).

**Proof.** The necessity follows from Theorem 2.7. Conversely, if \(f\) is unbounded then the family of sets \(Z_n\) has the finite intersection property, and hence is embeddable in a \(z\)-ultrafilter \(\mathcal{F}\). The ideal \(M = \mathcal{D}_A[\mathcal{F}]\) is maximal (see Theorem 3.2(b)) and so the result follows from Theorem 2.7. 

Corollary 2.9. Every maximal ideal in \( A(X) \) is real when and only when \( A(X) = C^*(X) \).

3. Maximal ideals and \( z \)-ultrafilters

In this section we consider how the maps \( \mathcal{L}_A \) and \( \mathcal{L}_A^+ \) relate ideals in \( A(X) \) and filters on \( X \). We show that for each maximal ideal \( M \) of \( A(X) \) the \( z \)-filter \( \mathcal{L}_A[M] \) is (possibly properly) contained in a unique \( z \)-ultrafilter. In particular, our result corrects that in [13, p. 765] where it is assumed that \( \mathcal{L}_A[M] \) is always a \( z \)-ultrafilter. Indeed, we show in Section 4 that the map \( \mathcal{L}_A \) does not distinguish between prime ideals contained in a given maximal ideal. Nevertheless, we prove that the map \( \mathcal{L}_A \) induces a bijection between the set of maximal ideals in \( A(X) \) and the set of \( z \)-ultrafilters on \( X \). We also study some relationships between maximal ideals in different subrings of \( C(X) \).

Lemma 3.1. Let \( A(X) \) and \( B(X) \) be subrings of \( C(X) \) such that \( B(X) \subset A(X) \). Then for any ideal \( I \) of \( A(X) \), \( \mathcal{L}_A[I] = \mathcal{L}_B[I \cap B(X)] \).

Proof. It is enough to show that \( \mathcal{L}_A[I] = \mathcal{L}_B[I \cap C^*(X)] \). For \( f \in I \) there exists \( u \in C^*(X) \) such that \( fu = (f \wedge 1) \vee (-1) \). Since \( fu \in I \cap C^*(X) \), \( \mathcal{L}_B(f) = \mathcal{L}_B(fu) \). Thus \( \mathcal{L}_A[I] \subset \mathcal{L}_B[I \cap C^*(X)] \). The other inclusion is obvious. \( \square \)

Theorem 3.2. (a) If \( M \) is a maximal ideal in \( A(X) \), then \( \mathcal{L}_A[M] \) is contained in a unique \( z \)-ultrafilter.

(b) If \( \mathcal{F} \) is a \( z \)-ultrafilter on \( X \), then \( \mathcal{L}_A[\mathcal{F}] \) is a maximal ideal in \( A(X) \).

In particular, the map \( \mathcal{L}_A \) induces a bijection from the set of maximal ideals of \( A(X) \) onto the set of \( z \)-ultrafilters on \( X \).

Proof. (a) Let \( M \) be a maximal ideal in \( A(X) \). Then there exists a \( z \)-ultrafilter \( \mathcal{F} \) on \( X \) such that \( \mathcal{L}_A[M] \subset \mathcal{F} \). As noted above, there exists a maximal ideal \( M' \) of \( C(X) \) such that \( \mathcal{F} = Z[M'] \). It follows that

\[
M = \mathcal{L}_A[\mathcal{L}_A[M]] \subset \mathcal{L}_A[Z[M']].
\]

Now

\[
M = \mathcal{L}_A[Z[M']] \supset \mathcal{L}_A[\mathcal{L}_C[M']] = \mathcal{L}_A[\mathcal{L}_A[M' \cap A(X)]] = M' \cap A(X),
\]

where the second equality follows from Lemma 3.1. Thus \( \mathcal{L}_C[M'] \subset \mathcal{L}_A[M] \subset Z[M'] \).

Now, suppose that there exists another maximal ideal \( N \) of \( C(X) \) such that \( \mathcal{L}_A[M] \subset Z[N] \). By the same argument as in the previous paragraph \( \mathcal{L}_C[N] \subset \mathcal{L}_A[M] \subset Z[N] \). So, any zero set \( E \in Z[N] \) intersects every member of \( \mathcal{L}_A[M] \) and hence also intersects every member of \( \mathcal{L}_C[M'] \). Thus \( E \) intersects with every member
of \( Z[M'] \) (see, for example, the comments following Lemma 1.5), and therefore \( E \in Z[M'] \) [6, Theorem 2.6]. Thus \( Z[N] = Z[M'] \). It follows that \( \mathcal{I}_A[M] \) is contained in a unique \( z \)-ultrafilter.

(b) Since \( \mathcal{F} = Z[M'] \) for some maximal ideal \( M' \) of \( C(X) \), we have
\[
\mathcal{I}_A\left[\mathcal{F}\right] = \mathcal{I}_A[Z[M']] = \mathcal{I}_A[\mathcal{I}_c[M']]
\]
\[= \mathcal{I}_A[M' \cap A(X)] = M' \cap A(X).
\]

Let \( M \) be a maximal ideal of \( A(X) \) such that \( M' \cap A(X) \subseteq M \). Then \( \mathcal{I}_c[M'] = \mathcal{I}_A[M' \cap A(X)] \subseteq \mathcal{I}_A[M] \). Now, let \( N \) be a maximal ideal of \( C(X) \) such that \( \mathcal{I}_A[M] \subseteq Z[N] \). Then as before, \( \mathcal{I}_c[N] \subseteq \mathcal{I}_A[M] \) and \( Z[N] = Z[M'] \). Therefore \( M \subseteq \mathcal{I}_A[Z[M']] \). Since \( M \) is maximal, \( M = \mathcal{I}_A[Z[M']] = \mathcal{I}_A[\mathcal{F}] \).

It follows from (a) and (b) that \( \mathcal{I}_A \) defines a function from the set of maximal ideals of \( A(X) \) onto the set of \( z \)-ultrafilters on \( X \). Moreover, \( \mathcal{I}_A \) is one to one since \( \mathcal{I}_A[\mathcal{I}_A[M]] = M \) for a maximal ideal \( M \) of \( A(X) \) (see Lemma 1.5). \( \square \)

The set of \( z \)-ultrafilters on \( X \) endowed with the Stone topology is \( \beta X \), the Stone–Čech compactification of \( X \). Thus it is immediate from Theorem 3.2 that the set \( M(A) \) of maximal ideals of \( A(X) \) is in one to one correspondence with the points of \( \beta X \). If \( M(A) \) is equipped with the hull kernel topology then it is homeomorphic to \( \beta X \). The proof is the same as in [6, Chapter 6]. We write \( M_A^p \) for the maximal ideal of \( A(X) \) corresponding to \( p \in \beta X \) under this homeomorphism. We note that for \( p \in \beta X \), \( \lim \mathcal{I}_A[M_A^p] = p \).

For a filter \( \mathcal{F} \) on \( X \) we write \( S[\mathcal{F}] \) for the set of cluster points of \( \mathcal{F} \) in \( \beta X \). In other words \( S[\mathcal{F}] \) is the subset of \( \beta X \) defined by
\[
S[\mathcal{F}] = \left\{ \{E \in \beta X \mid E \in \mathcal{F}\} \right\}.
\]

We have the following analogue of the Gelfand–Kolmogoroff Theorem [6, Theorem 7.3].

**Theorem 3.3.** Let \( M_A^p \) be the maximal ideal of \( A(X) \) corresponding to the point \( p \) of \( \beta X \). Then
\[
M_A^p = \{ f \in A(X) \mid p \in S[\mathcal{I}_A(f)] \}.
\]

**Proof.** Since \( M_A^p \) is a \( 3 \)-ideal, \( f \in M_A^p \) if and only if \( \mathcal{I}_A(f) \subseteq \mathcal{I}_A[M_A^p] \). Since \( \mathcal{I}_A[M_A^p] \) converges to \( p \), the last statement is true if and only if \( p \) is a cluster point of \( \mathcal{I}_A(f) \). \( \square \)

Another characterization of \( M_A^p \) follows from Lemma 1.3. We have
\[
M_A^p = \{ f \in A(X) \mid \lim_{\mathcal{I}_A[M_A^p]} fh = 0 \text{ for every } h \in A(X) \}.
\]

For if the condition is satisfied for every \( h \in A(X) \), then by Lemma 1.3(b) \( \mathcal{I}_A(f) \subseteq \mathcal{I}_A[M_A^p] \) and since \( M_A^p \) is a \( 3 \)-ideal, it follows that \( f \in M_A^p \). Conversely, if \( f \in M_A^p \), then \( fh \in M_A^p \) for every \( h \in A(X) \). Thus by Lemma 1.3(a) \( \lim \mathcal{I}_A(fh)fh = 0 \). But since \( \mathcal{I}_A(fh) \subseteq \mathcal{I}_A[M_A^p] \) it follows that \( \lim \mathcal{I}_A[M_A^p]fh = 0 \) for \( h \in A(X) \).
Let $A(X)$ and $B(X)$ be subrings of $C(X)$ that contain $C^*(X)$. By the remarks following Theorem 3.2 the sets $M(A)$ and $M(B)$ are in one to one correspondence with the points of $\beta X$. It follows that the sets $M(A)$ and $M(B)$ are in one to one correspondence with each other. Clearly such a correspondence is given by

$$M_A^p \leftrightarrow M_B^p \quad (p \in \beta X).$$

The following theorem gives an explicit description of this correspondence which shows that the correspondence is a natural one. Writing $\mathbb{F}_A[M]$ for the unique $z$-ultrafilter containing $\mathbb{F}_A[M]$ we have the following.

**Theorem 3.4.** Let $A(X)$ and $B(X)$ be subrings of $C(X)$ that contain $C^*(X)$ and let $M$ be a maximal ideal in $A(X)$. The map

$$M \to \mathbb{F}_A[\mathbb{F}_A[M]]$$

gives a homeomorphism between the spaces $M(A)$ and $M(B)$ with the hull-kernel topology. Moreover, under this map free ideals map to free ideals and fixed ideals to fixed ideals.

**Proof.** The map $\mathbb{F}_A$ is a homeomorphism of $M(A)$ onto $\beta X$ and the map $\mathbb{F}_B$ is a homeomorphism of $\beta X$ onto $M(B)$. \□

To study further the relationship between $M_A^p$ and $M_B^p$ for different subrings $A(X)$ and $B(X)$, we consider the extension of functions to $\beta X$. Every $f \in A(X)$ may be regarded as a map into the one point compactification $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$ and so has a Stone extension

$$f^*: \beta X \to \mathbb{R}^*.$$

We recall also that an element $a$ in a totally ordered field is infinitely small if $a < 1/n$ for all $n \in \mathbb{N}$.

**Theorem 3.5.** Let $f \in A(X)$.

(a) $f^*(p) = \infty$ if and only if $|M_A^p(f)|$ is infinitely large.

(b) $f^*(p) = r \in \mathbb{R}$ if and only if $|M_A^p(f) - r|$ is either infinitely small or zero.

**Proof.** If $f^*(p) = \infty$, then for each $n \in \mathbb{N}$, $p$ is in the closure of the sets $E_n = \{x \in X ||f(x)|| \geq n\}$. Now, $E_n = Z(g_n)$ where $g_n = (|f(x)| - n) \wedge 0$. Thus $p$ is a cluster point of $\mathbb{F}_A(g_n)$. By Theorem 3.3, $g_n \in M_A^p$ and so $\mathbb{F}_A(g_n) \subseteq \mathbb{F}_A[M_A^p]$. Clearly each $E_n \subseteq \mathbb{F}_A(g_{n-1})$ for $n \geq 2$ and thus $E_n \subseteq \mathbb{F}_A[M_A^p]$ for all $n \geq 2$. Theorem 2.7 implies that $|M_A^p(f)|$ is infinitely large. Similarly, if $f^*(p) = r$, then $|M_A^p(f) - r| \leq 1/n$ for each $n \in \mathbb{N}$ so that $|M_A^p(f) - r|$ is infinitely small or zero. The converses follow from the fact that the possibilities considered are mutually exclusive and exhaustive. \□
We now state results that relate the ideals $M_\alpha$ and $M_\beta$. First, for $B(X) \subset A(X)$ if $f \in M_\alpha$, then $f^*(p) = 0$ by Theorem 3.5. It follows that $M_\alpha \cap B(X) \subset M_\beta$. The two are not in general the same as shown in [6, 4.7] for the case of $C(X)$ and $C^*(X)$. We have the following corollary.

**Corollary 3.6.** Let $B(X) \subset A(X)$. Then $M_\beta$ is the set of all $f \in B(X)$ for which $|M_\alpha(f)|$ is either infinitely small or zero.

**Proof.** This is immediate from Theorem 3.5(b).

**Corollary 3.7.** Let $B(X) \subset A(X)$. Then $M_\alpha$ is hyper-real if and only if $M_\beta$ contains a unit of $A(X)$.

**Proof.** If $M_\alpha$ is hyper-real, there exists $g \in A(X)$ such that $g > 1$ and $|M_\alpha(g)|$ is infinitely large (Theorem 2.7). Thus $|M_\alpha(g^{-1})|$ is infinitely small. By Corollary 3.6, $g^{-1} \in M_\beta$. The converse follows directly from Corollary 3.6.

**Corollary 3.8.** Let $B(X) \subset A(X)$. Then, $M_\alpha \cap B(X) = M_\beta$ if and only if $M_\beta$ is real.

**Proof.** Necessity follows from Corollary 3.7 and sufficiency from Corollary 3.6.

4. The ideals $O_\alpha$

In this section we consider the analogue in $A(X)$ of the ideals $O^\alpha$ defined in [6] for $C(X)$. For each $p \in \beta X$ we define $O_\alpha^p$ as follows:

$$O_\alpha^p = \{ f \in A(X) | p \in \text{int } S([\mathcal{A}(f)]) \}.$$

This is a reformulation of the definition of $O^\alpha_p$ in [12, p. 48] using the notation of this paper. Each $O_\alpha^p$ is an ideal of $A(X)$ and $O_\alpha^p \subset M_\alpha.$

The next theorem shows that $\mathcal{A}(O_\alpha^p) = \mathcal{A}(M_\alpha)$ for all $p \in \beta X$. In particular, for prime ideals $P$ contained in $M_\alpha$, $\mathcal{A}(P) = \mathcal{A}(M_\alpha)$ and so every prime $\beta$-ideal is maximal. Thus the map $\mathcal{A}$ is not so sensitive as far as distinguishing prime ideals is concerned.

Before proving the next theorem we note a fact about the extension of functions. If $f$ is a bounded function on $X$ then $f$ has a continuous extension $f^\beta$ to $\beta X$. Thus, from the continuity of $f$ and the fact that $\lim_{\mathcal{A}[M_\alpha]} = p$ we have

$$f^\beta(p) = \lim_{\mathcal{A}[M_\alpha]} f \quad (p \in \beta X).$$

In particular, if $f \in M_\alpha$, then $f^\beta(p) = 0$. 
Theorem 4.1. \( \mathcal{L}_A[O^p_A] = \mathcal{L}_A[M^p_A] \) for all \( p \in \beta X \).

**Proof.** Clearly \( \mathcal{L}_A[O^p_A] \) is contained in \( \mathcal{L}_A[M^p_A] \). We show \( \mathcal{L}_A[O^p_A] \supseteq \mathcal{L}_A[M^p_A] \). It is enough to show that \( f \in M^p_A \) implies that \( \mathcal{L}_A(f) \subseteq \mathcal{L}_A[O^p_A] \). Let \( f \in M^p_A \) and without loss of generality assume that \( f \) is bounded. Thus \( f \) has a continuous extension \( f^\beta \) to \( \beta X \) with \( f^\beta(p) = 0 \) by the remarks preceding this theorem. Now, let \( E \in \mathcal{L}_A(f) \). Then there exists \( g \in A(X) \) such that \( fg = 1 \) on \( E^c \). Thus the point \( p \) and \( \text{cl}_E X \) are completely separated in \( \beta X \) and so there exists a neighborhood \( V \) in \( \beta X \) of \( p \) and a function \( h_0 \in C(\beta X) \) such that \( h_0(x) = 0 \) on \( V \) and \( h_0(x) = 1 \) on \( \text{cl}_E X \). Let \( h \) denote the restriction of \( h_0 \) to \( X \). Then \( Z(h) \supseteq V \cap X \) and so \( \text{cl}_E X Z(h) \supseteq \text{cl}_E X Z(h) \supseteq V \). Since every member of \( \mathcal{L}_A(h) \) contains \( Z(h) \) we have \( V = \bigcap \{ \text{cl}_E X F \mid F \subseteq \mathcal{L}_A(h) \} \). Thus \( h \in O^p_A \). Since \( h(x) = 1 \) on \( E^c \) it follows that \( E \subseteq \mathcal{L}_A(h) \). Thus \( \mathcal{L}_A(f) \subseteq \mathcal{L}_A(h) \subseteq \mathcal{L}_A[O^p_A] \).

A question considered in the case of \( C(X) \) is which spaces \( X \) have the property that every ideal \( O^p \) is prime. Another question is which spaces \( X \) have the property that every ideal \( O^p \) is maximal. We investigate the corresponding questions for any \( A(X) \).

A completely regular space \( X \) is called a \( P \)-space if every prime ideal in \( C(X) \) is maximal. This definition is motivated by the fact that if \( X \) is discrete then \( C(X) \) has this property, as first pointed out by Kaplansky [9]. We show that if every prime ideal in any \( A(X) \) is maximal, then \( X \) is a \( P \)-space.

**Theorem 4.2.** The following are equivalent for any subring \( A(X) \) of \( C(X) \).

\begin{itemize}
  \item[(a)] \( M^p_A = O^p_A \) for \( p \in \beta X \).
  \item[(b)] Every prime ideal in \( A(X) \) is maximal.
\end{itemize}

**Proof.** It is noted in [12, p. 48] that it follows from [4] that each \( C^*_A \) is an intersection of prime ideals in \( A(X) \) and a prime ideal \( P \) in \( A(X) \) is contained in \( M^p_A \) if and only if \( P \) contains \( O^p_A \). The theorem is an immediate consequence of this.

**Lemma 4.3.** \( f^\beta(p) = 0 \) for every \( p \in S[\mathcal{L}_A(f)] \).

**Proof.** Let \( p \in S[\mathcal{L}_A(f)] \). Then \( \mathcal{L}_A(f) \subseteq \mathcal{L}_A[M^p_A] \), and so
\[
0 = \lim_{f \in S[\mathcal{L}_A(f)]} f = \lim_{f \in \mathcal{L}_A[M^p_A]} f = f^\beta(p).
\]

**Theorem 4.4.** If \( M^p_A = O^p_A \) for every \( p \in X \), then every zero set of \( X \) is open.

**Proof.** We show that \( f \in A(X) \) vanishes on a neighborhood of \( p \) whenever \( f \) vanishes at \( p \in X \). If \( f(p) = 0 \), then \( f \in M^p_A = O^p_A \). Thus \( p \in \text{int} S[\mathcal{L}_A(f)] \). The result now follows from Lemma 4.3.
It is shown in [5, Theorem 5.3] that $X$ is a $P$-space if and only if every zero set of $X$ is open. From this fact and Theorems 4.2 and 4.4 we obtain the following.

**Theorem 4.5.** Let $A(X)$ be any subring of $C(X)$. If every prime ideal in $A(X)$ is maximal, then $X$ is a $P$-space.

The converse is not true, however, since for the $P$-space $\mathbb{N}$ and $A(X) = C^*(\mathbb{N})$, $M_p^* \neq O^*_p$ for $p \in \beta\mathbb{N}\setminus\mathbb{N}$ [12, p. 48].

The space $X$ is called an $F$-space if every finitely generated ideal in $C(X)$ is principal. This is equivalent to the condition that every $O^p$ in $C(X)$ be prime. Several other equivalent formulations of this concept are given in [6, 14.25]. It is easily seen that a parallel result to Theorem 14.25 of [6] is valid for any $A(X)$. Indeed, the equivalences (1)-(9) of that theorem are valid with nearly identical proofs if $O^p$ is replaced by $O^*_p$ and $C(X)$ is replaced by $A(X)$. In particular, (6) of that theorem is a purely topological characterization (independent of $A(X)$) of $F$-spaces, namely that every cozero set is $C^*$-embedded. This leads to the following result.

**Theorem 4.6.** Let $A(X)$ be any subring of $C(X)$. Then, $X$ is an $F$-space if and only if every finitely generated ideal in $A(X)$ is principal.

5. Intersections of free maximal ideals

Writing $A_F(X)$ for the intersection of the free maximal ideals in $A(X)$, we have the following generalization of a result known for $C(X)$ and $C^*(X)$.

**Theorem 5.1.** If $B(X) \subset A(X)$, then $A_F(X) \subset B_F(X)$.

**Proof.** If $f \in A_F(X)$, then $\mathcal{I}_A(f) \subset \mathcal{I}_A[M]$ for every free maximal ideal $M \subset A(X)$. Thus $\mathcal{I}_A(f)$ is contained in every free $z$-ultrafilter on $X$. But $\mathcal{I}_B(f) \subset \mathcal{I}_A(f)$, and so $\mathcal{I}_B(f) \subset \mathcal{F}$ for every free $z$-ultrafilter $\mathcal{F}$ on $X$. Thus $f \in \mathcal{I}^*_B[\mathcal{F}]$ for every free $z$-ultrafilter $\mathcal{F}$. But by Theorem 3.4, every free maximal ideal in $B(X)$ is of the form $\mathcal{I}^*_B[\mathcal{F}]$ for a free $z$-ultrafilter $\mathcal{F}$. □

The support of a function $f \in C(X)$ is the set $\overline{X - Z(f)}$, the closure of $X - Z(f)$ in $X$. We write $C_c(X)$ for the collection of all functions in $C(X)$ with compact support. We show below that the intersection of all the free ideals in any $A(X)$ is precisely the ideal $C_c(X)$. This result is well known for $C(X)$ and $C^*(\mathbb{N})$. We need some preliminary lemmas.
Lemma 5.2. Let \( I \) be an ideal in \( A(X) \). Then \( I \) is free if and only if for every compact set \( K \subseteq X \) there exists \( f \in I \) such that \( Z(f) \cap K = \emptyset \).

**Proof.** Suppose there exists a compact set \( K \) such that for every \( f \in I \), \( Z(f) \cap K \neq \emptyset \). Since \( Z(f) = \bigcap \mathcal{L}_A(f) \), it follows that \( E \cap K \neq \emptyset \) for every \( E \in \mathcal{L}_A[I] \). Since \( \mathcal{L}_A[I] \) is a \( z \)-filter (Lemma 1.4), \( \mathcal{F} = \{ E \cap K | E \in \mathcal{L}_A[I] \} \) is a collection of closed sets in \( K \) with the finite intersection property. Since \( K \) is compact, \( \bigcap \mathcal{F} \neq \emptyset \). The result now follows from the following inclusions: \( \bigcap \mathcal{F} = \bigcap \mathcal{L}_A[I] = \bigcap Z[I] \).

Conversely, if \( I \) is fixed then any compact subset of the nonempty set \( \bigcap \{ Z(f) | f \in I \} \) is a compact set which fails to satisfy the condition in the lemma. □

Lemma 5.3. \( C_c(X) = \{ f \in C^*(X) | Z_{\beta X}(f^\beta) \) is a neighborhood of \( \beta X \setminus X \}. \)

**Proof.** See [6, p. 109]. □

Theorem 5.4. The intersection of the free ideals in any \( A(X) \) is \( C_c(X) \).

**Proof.** Suppose \( f \in C(X) \) has compact support and let \( I \) be a free ideal in \( A(X) \). By Lemma 5.2 there exists \( g \in I \) such that \( Z(g) \cap (X - Z(f)) = \emptyset \). Thus \( Z(f) \) is a neighborhood of \( Z(g) \). It follows that there exists \( h \in C^*(X) \) such that \( f = hg \) and so \( f \in I \). Thus \( f \) belongs to every free ideal.

Conversely, suppose \( f \) belongs to every free ideal. Then in particular, \( f \in O_{\beta X}^p \) for every \( p \in \beta X \setminus X \) and so \( \beta X \setminus X \subseteq \text{int} S[\mathcal{L}_A(f)] \). It follows from Lemma 4.3 that \( f^\beta(p) = 0 \) on a neighborhood of \( \beta X \setminus X \). Lemma 5.3 now implies that \( f \) has compact support. □

We now consider the intersection of the free maximal ideals in \( A(X) \). (See [6] and [10] for the cases of \( C(X) \) and \( C^*(X) \).) Following [13] we call a set \( E \subset X \) small if every zero set contained in \( E \) is compact. We denote \( \mathcal{K} = \{ E \subset Z(X) | E \subset \text{is small} \} \) and \( A_K(X) = \{ f \in A(X) | \mathcal{L}_A(f) \subset \mathcal{K} \} \).

Theorem 5.5. The intersection of the free maximal ideals in any \( A(X) \) is \( A_K(X) \).

**Proof.** See [13, Theorem 6]. □

Note that there is no restriction on the topology of \( X \) in Theorem 5.5. On the other hand, if \( X \) is realcompact it is shown in [6, Theorem 8.19] that the intersection of the free maximal ideals in \( C(X) \) is \( C_c(X) \). We give below a generalization of this result for any \( A(X) \) in terms of the notion of \( A \)-compactness (see [13]). A space \( X \) is called \( A(X) \)-compact (or simply \( A \)-compact) if every real maximal ideal in \( A(X) \) is fixed. With this terminology a compact space is \( C^* \)-compact while a realcompact space is \( C \)-compact. It is possible to give a characterization of \( A \)-compactness analogous to the beautiful characterization of realcompactness given by Mandelker [11]. We call a collection \( \mathcal{F} \) of subsets of \( X \) \( A \)-stable if every \( f \in A(X) \) is bounded on some member of \( \mathcal{F} \).
Theorem 5.6. The space $X$ is $A$-compact if and only if every $A$-stable $z$-ultrafilter on $X$ converges.

Proof. Suppose $X$ is $A$-compact and $\mathcal{F}$ is an $A$-stable $z$-ultrafilter that does not converge. Then $\{\text{cl}_{\beta X} F | F \in \mathcal{F} \}$ is a $z$-ultrafilter on $\beta X$ and hence there exists $p \in \beta X \setminus X$ such that $p$ is the limit of $\mathcal{F}$. So $\mathcal{F} \not\supseteq \mathcal{I}_A[M^*_A]$. Since $X$ is $A$-compact, the ideal $M^*_A$ is hyper-real and so there exists $f \in M^*_A$ such that $|M^*_A(f)|$ is infinitely large. By Theorem 2.7(c), $Z_n \in \mathcal{I}_A[M^*_A]$ for all $n$. Thus for $F \in \mathcal{F}$, $F \cap Z_n \neq \emptyset$ for all $n$ and hence $f$ is unbounded on each such $F$. Therefore $\mathcal{F}$ is not $A$-stable.

Conversely, if $X$ is not $A$-compact then there exists $p \in \beta X \setminus X$ such that $M^*_p$ is real. Thus by Theorem 2.7 every $f \in A(X)$ is bounded on some member of $\mathcal{I}_A[M^*_p]$ and hence bounded on some member of $\mathcal{I}_A[M^*_A]$, the unique $z$-ultrafilter containing $\mathcal{I}_A[M^*_A]$. Thus this $z$-ultrafilter is $A$-stable but does not converge in $X$. □

Lemma 5.7. If $X$ is $A$-compact, then $f \in A(X)$ has compact support if and only if $\mathcal{I}_A(f) \subseteq \mathcal{K}$.

Proof. Suppose $f$ has compact support. Let $E \in \mathcal{I}_A(f)$. Then there exists $g \in A(X)$ such that $fg = 1$ on $E^c$. It follows that $E^c$ is contained in the support of $f$ and hence $F^c$ is compact. Thus any zero set contained in $F^c$ is compact, whence $F^c$ is small. Therefore $E \in \mathcal{K}$ and so $\mathcal{I}_A(f) \subseteq \mathcal{K}$.

Conversely, if $f$ has noncompact support then there exists $p \in \text{cl}_{\beta X}(X - Z(f))$ such that $p \not\in X$. Since $X$ is $A$-compact the ideal $M^*_p$ is hyper-real and so there exists $g \in A(X)$ such that $g^*(p) = \infty$ (Theorem 3.5(a)). Thus $g$ is unbounded on $X - Z(f)$. By [6, Theorems 1.20 and 1.18] $X - Z(f)$ contains a noncompact zero set $E$ such that $E$ and $Z(f)$ are completely separated. Thus $E$ and $Z(f)$ are contained in disjoint zero set neighborhoods $U$ and $V$, respectively. Clearly $V \in \mathcal{I}_A(f)$ but $V \not\in \mathcal{K}$. □

Theorem 5.8. If $X$ is $A$-compact, then the intersection of the free maximal ideals in $A(X)$ is $C_\infty^*(X)$.

Proof. This follows from Theorem 5.6 and Lemma 5.7. □

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References