MAXIMAL IDEALS IN SUBALGEBRAS OF $C(X)$
LOTHAR REDLIN AND SALEEM WATSON

ABSTRACT. Let $X$ be a completely regular space, and let $A(X)$ be a subalgebra of $C(X)$ containing $C^*(X)$. We study the maximal ideals in $A(X)$ by associating a filter $Z(f)$ to each $f \in A(X)$. This association extends to a one-to-one correspondence between $\mathcal{M}(A)$ (the set of maximal ideals of $A(X)$) and $\beta X$. We use the filters $Z(f)$ to characterize the maximal ideals and to describe the intersection of the free maximal ideals in $A(X)$. Finally, we outline some of the applications of our results to compactifications between $vX$ and $\beta X$.

1. Introduction. The algebra $C(X)$ of continuous real-valued functions on a completely regular space $X$ and its subalgebra $C^*(X)$ of bounded functions have been studied extensively (see Gillman and Jerison [3], and Aull [1]). One of the interesting problems considered in [3] is that of characterizing the maximal ideals in these two algebras. It is a remarkable fact that the distinct problems of identifying the maximal ideals in $C(X)$ and $C^*(X)$ have a common solution—the maximal ideals are in one-to-one correspondence with the points of $\beta X$ in a natural way. The methods of achieving this correspondence, however, are quite different in the two cases. In this paper we consider this problem for subalgebras $A(X)$ of $C(X)$ that contain $C^*(X)$. We show that for such algebras the maximal ideals are in one-to-one correspondence with $\beta X$. The correspondence we construct reduces to that in [3] for the cases of $C(X)$ and $C^*(X)$. Thus our result puts in a common setting these apparently distinct problems.

A function is invertible in $C(X)$ if it is never zero, and in $C^*(X)$ if it is bounded away from zero. In an arbitrary $A(X)$, of course, there is no such description of invertibility which is independent of the structure of the algebra. Thus in §2 we associate to each noninvertible $f \in A(X)$ a $z$-filter $Z(f)$ that is a measure of where $f$ is “locally” invertible in $A(X)$. This correspondence extends to one between maximal ideals of $A(X)$ and $z$-ultrafilters on $X$. In §3 we use the filters $Z(f)$ to describe the intersection of the free maximal ideals in any algebra $A(X)$. Finally, our main result allows us to introduce the notion of $A(X)$-compactness of which compactness and realcompactness are special cases. In §4 we show how the Banach-Stone theorem extends to $A(X)$-compact spaces.

2. The structure space. Throughout this paper $X$ will denote a completely regular Hausdorff space and $A(X)$ a subalgebra of $C(X)$ containing $C^*(X)$. In this section we construct the correspondence mentioned in the introduction.

A zero set in $X$ is a set of the form $Z(f) = \{x \in X : f(x) = 0\}$ for some $f \in C(X)$. The complement of a zero set is a cozero set. $Z[X]$ will denote the

Received by the editors April 18, 1986.
1980 Mathematics Subject Classification (1985 Revision). Primary 54C40; Secondary 46E25.
Key words and phrases. Algebras of continuous functions, maximal ideal, compactifications.

©1987 American Mathematical Society
0002-9939/87 $1.00 +$.25 per page

763
collection of all zero sets in \( X \). If \( E \) is a cozero set in \( X \) we will say that \( f \in A(X) \) is \( E\text{-regular} \) if there exists \( g \in A(X) \) such that \( fg|_E = 1 \).

**Lemma 1.** Let \( f, g \in A(X) \) and let \( E, F \) be cozero sets in \( X \).

(a) If \( f \) is \( E\text{-regular} \) and \( F \subseteq E \), then \( f \) is \( F\text{-regular} \).

(b) If \( f \) is \( E\text{-regular} \) and \( F \subseteq E \), then \( f \) is \( E \cup F\text{-regular} \).

(c) If \( f(x) \geq c > 0 \) for all \( x \in E \), then \( f \) is \( E\text{-regular} \).

(d) If \( 0 < f(x) \leq g(x) \) for all \( x \in E \) and \( f \) is \( E\text{-regular} \), then \( g \) is \( E\text{-regular} \).

(e) If \( f \) is \( E\text{-regular} \) and \( g \) is \( F\text{-regular} \), then \( fg \) is \( E \cap F\text{-regular} \) and \( f^2 + g^2 \) is \( E \cup F\text{-regular} \).

**Proof.** (a) Obvious.

(b) Let \( h, k \in A(X) \) satisfy \( hf|_E = 1 \) and \( kf|_F = 1 \). Let \( w = h + k - fhk \). Then \( fw|_{E \cup F} = 1 \).

(c) Let \( h = \max\{c, f\} \). Then \( h|_E = f|_E \) and \( h \geq c \). So \( 0 < h^{-1} \leq c^{-1} \). Hence \( h^{-1} \in C^+(X) \subseteq A(X) \), and \( h^{-1}f|_E = 1 \).

(d) Let \( h \in A(X) \) satisfy \( hf|_E = 1 \). For \( x \in E \), \( h(x) > 0 \), so \( h(x)g(x) \geq h(x)f(x) = 1 \). Thus by (c), there exists \( k \in A(X) \) such that \( khg|_E = 1 \).

(e) If \( hf|_E = 1 \) and \( kg|_E = 1 \), then \( hkg|_{E \cap F} = 1 \). Now \( f^2 + g^2 \geq f^2 \), so by (d), \( f^2 + g^2 \) is \( E\text{-regular} \). Similarly, it is \( F\text{-regular} \), and so the result follows by (b).

For \( f \in A(X) \), we define

\[
Z(f) = \{ E \in Z[X] : f \text{ is } E^c\text{-regular} \},
\]

and for \( S \subseteq A(X) \), \( Z[S] = \bigcup_{f \in S} Z(f) \). We recall that a \textit{z-filter} is a nonempty collection \( \mathcal{F} \) of zero sets in \( X \) such that \( \mathcal{F} = \mathcal{G} \cap Z[X] \), for some filter \( \mathcal{G} \) on \( X \).

**Theorem 1.** If \( f \) is not invertible in \( A(X) \), then \( Z(f) \) is a z-filter on \( X \), and conversely.

**Proof.** If \( f \) is not invertible, \( \emptyset \notin Z(f) \). Moreover, if \( E, F \in Z(f) \), then by Lemma 1(b), \( E \cap F \in Z(f) \). If \( G \) is a zero set containing \( E \in Z(f) \), then \( G \in Z(f) \) by Lemma 1(a). Hence \( Z(f) \) is a z-filter.

The converse is obvious.

**Theorem 2.** If \( I \) is an ideal in \( A(X) \), then \( Z[I] \) is a z-filter on \( X \).

**Proof.** Clearly \( \emptyset \notin Z[I] \). If \( E, F \in Z[I] \), there exist \( f, g \in I \) such that \( f \) is \( E^c\text{-regular} \) and \( g \) is \( F^c\text{-regular} \). Then \( f^2 + g^2 \in I \), and by Lemma 1(e), \( f^2 + g^2 \) is \( (E \cap F)^c\text{-regular} \). Thus \( E \cap F \in Z[I] \). Finally, if \( F \) is a zero set and \( F \supseteq E \in Z[I] \), then \( E \in Z(f) \) for some \( f \in I \), and so \( F \in Z(f) \subseteq Z[I] \) by Theorem 1.

Using the notation of [3], we write \( Z^{-1}[\mathcal{F}] = \{ f \in A(X) : Z(f) \subseteq \mathcal{F} \} \) for the inverse of the set function \( Z \). We will show that if \( \mathcal{F} \) is a z-filter, then \( Z^{-1}[\mathcal{F}] \) is an ideal in \( A(X) \), giving a converse to the above theorem. We need two preliminary lemmas.

**Lemma 2.** If \( f \in A(X) \), then \( \lim_{Z(f)} fh = 0 \) for any \( h \in A(X) \).

**Proof.** We claim \( \lim_{Z(f)} fh = 0 \). The result will follow from this claim and Lemma 1(e), since then \( \lim_{Z(fh)} fh = 0 \) and \( Z(fh) \subseteq Z(f) \). So let \( V = (-\varepsilon, \varepsilon) \) be a neighborhood of zero in \( \mathbb{R} \) and let \( E = f^{-1}(V) \). Clearly \( f \) is \( E^c\text{-regular} \) (Lemma 1(b) and (c)). Thus \( f^{-1}(V) \in Z(f) \) and so \( fh \) converges to zero on \( Z(f) \).
LEMMA 3. Let $\mathcal{F}$ be a z-filter on $X$. If $\lim_{F} fh = 0$ for all $h \in A(X)$, then $Z(f) \subseteq \mathcal{F}$.

PROOF. For $E \in Z(f)$ we show that there is an $F \in \mathcal{F}$ such that $F \subseteq E$. Suppose not. Then $F \cap E^c \neq \emptyset$ for every $F \in \mathcal{F}$. Let $h \in A(X)$ satisfy $fh|_{E^c} = 1$. It follows that 1 is a cluster point of $\{fh(F) : F \in \mathcal{F}\}$, contradicting our hypothesis.

THEOREM 3. For any z-filter $\mathcal{F}$ on $X$, $I = Z^{-}[\mathcal{F}]$ is an ideal in $A(X)$.

PROOF. If $f \in I$ and $g \in A(X)$, then $Z(fg) \subseteq Z(f)$ (Lemma 1(e)), so $fg \in I$. Now if $f, g \in I$, then by Lemma 2, $\lim_{F} fh = \lim_{F} gh = 0$ for every $h \in A(X)$. So $\lim_{F} fh + \lim_{F} gh = \lim_{F} (f + g)h = 0$ for all $h \in A(X)$, and hence by Lemma 3, $Z(f + g) \subseteq \mathcal{F}$. Finally, we note that since $\emptyset \notin \mathcal{F}$, $I$ consists of noninvertible elements only.

Both $Z$ and $Z^{-}$ preserve inclusion and so they map maximal elements to maximal elements. Hence $Z$ provides a one-to-one correspondence between $\beta X$ and the set $M(A)$ of maximal ideals of $A(X)$. If $M(A)$ is equipped with the hull-kernel topology, then as in [3] in the cases of $C^*(X)$ and $C(X)$, we have the following theorem (see [6] for a different method of arriving at this result).

THEOREM 4. The maximal ideal space $M(A)$ of $A(X)$ equipped with the hull-kernel topology is homeomorphic to $\beta X$.

3. Free maximal ideals. Let $M^p$ be the maximal ideal corresponding to $p \in \beta X$ and $\mathcal{U}^p$ the z-ultrafilter on $X$ that converges to $p$, so that $Z(M^p) = \mathcal{U}^p$. Using our filter $Z(f)$ we see immediately that for $f \in A(X)$, $f \in M^p$ if and only if $Z(f) \subseteq \mathcal{U}^p$. Thus we have the following analogue of the Gelfand-Kolmogoroff theorem [3, Theorem 7.3] for an arbitrary $A(X)$.

THEOREM 5. For the maximal ideals in $A(X)$, we have

$$M^p = \{f \in A(X) : p \text{ is a cluster point of } Z(f) \text{ in } \beta X\}.$$ 

We now describe the intersection of all the free maximal ideals in $A(X)$. An ideal $I$ is free if $\cap Z[I] = \emptyset$, otherwise it is fixed. Note that a maximal ideal is free if and only if it is of the form $M^p$ for some $p \in \beta X \setminus X$. We call a set $E \subseteq X$ small if every zero set contained in $E$ is compact. Let $K = \{E \in Z[X] : E^c \text{ is small}\}$, and let $A_K(X) = \{f \in A(X) : Z(f) \subseteq K\}$.

THEOREM 6. $A_K(X) = \cap \{M^p : p \in \beta X \setminus X\}$.

PROOF. Let $f \in A_K(X)$. If $\mathcal{U}$ is any z-ultrafilter on $X$ such that $Z(f) \notin \mathcal{U}$, then there exist disjoint zero sets $E \in Z(f)$ and $F \subseteq \mathcal{U}$. But then $F \subseteq E^c$, so $F$ is compact and $\mathcal{U}$ is fixed. It follows that $Z(f)$ is contained in every free z-ultrafilter, and so $f$ belongs to every free maximal ideal. Conversely, if $f$ is in every free maximal ideal, then $Z(f)$ belongs to every free z-ultrafilter. Suppose $E \in Z(f)$ is not in $K$. Then $E^c$ must contain a noncompact zero set $F$. Since $E \cup F \supseteq E \in Z(f)$, $E \cup F$ belongs to every free z-ultrafilter, and hence $F$ belongs to no free z-ultrafilter. But clearly every noncompact zero set must belong to some free z-ultrafilter. Thus $E$ is in $K$ and $f \in A_K(X)$.

We note that if $X$ is realcompact and $A(X) = C(X)$, then $A_K(X)$ is the family of functions on $X$ of compact support and Theorem 8.19 of [3] follows from our Theorem 6. If $A(X) = C^*(X)$, then $A_K(X)$ is the family of functions on $X$ that vanish at infinity and Lemma 3.2 in [4] is a special case of Theorem 6.
4. **A-compactness.** It is well known that $C^*$ distinguishes among compact spaces (the Banach-Stone theorem) and that $C$ distinguishes among realcompact spaces (Hewitt's isomorphism theorem). Theorem 4 allows us to define the notion of $A$-compactness which will enable us to place both of these theorems in a common setting (Theorem 7).

A maximal ideal $M$ in $A(X)$ is *real* if $A(X)/M$ is isomorphic to $\mathbb{R}$. Every fixed maximal ideal is real. If every real maximal ideal is fixed, we will say that $X$ is $A(X)$-compact (or simply $A$-compact). With this definition, a compact space is one that is $C^*$-compact while a realcompact space is $C$-compact.

**THEOREM 7.** Let $X$ be $A$-compact and $Y$ be $B$-compact. If $A(X)$ is isomorphic to $B(Y)$, then $X$ is homeomorphic to $Y$.

**PROOF.** Since $X$ is $A$-compact its points correspond to the real maximal ideals of $A(X)$ under the homeomorphism described in Theorem 4. Thus we can recover $X$ from the ring structure of $A(X)$. Since this can be done in the same way for $Y$, the result follows.

Although the converse of the above theorem is trivial if $A$ and $B$ are $C$ or $C^*$, in this more general setting the converse is not even true. For a given $X$ there can exist nonisomorphic algebras $A(X)$ and $B(X)$ for which $X$ is both $A$-compact and $B$-compact. For example, let $H(N)$ be the algebra of sequences which occur as the coefficients of the Taylor series representation of functions holomorphic on the open unit disc. Then $N$ is both $H$-compact (see [2]) and $C$-compact, but $H(N)$ is obviously not isomorphic to $C(N)$. Indeed, it is clear from the definition that if $X$ is $A$-compact and $B(X) \supseteq A(X)$, then $X$ is $B$-compact. This raises the question: Does there exist in some sense a "minimal" algebra $A_m(X)$ for which $X$ is $A_m$-compact, at least up to isomorphism?

We conclude by noting that another characterization of $A$-compactness follows from Mandelker [5]. We call a family $S$ of closed sets in $X$ $A$-stable if every $f \in A(X)$ is bounded on some member of $S$. Then one can show (as in [5]) that a space is $A$-compact if and only if every $A$-stable family of closed sets with the finite intersection property has nonempty intersection.

**REFERENCES**


**PENNSYLVANIA STATE UNIVERSITY, OGONTZ CAMPUS, ABINGTON, PENNSYLVANIA 19001**

**PENNSYLVANIA STATE UNIVERSITY, DELAWARE COUNTY CAMPUS, MEDIA, PENNSYLVANIA 19063**