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1. Introduction

If $X$ is a completely regular space then the collection $C(X)$ of continuous real-valued functions on $X$ is a ring under pointwise operations. The ring $C(X)$ and its subring $C^*(X)$ consisting of the bounded functions have been the objects of considerable study and the theory is beautifully presented in the book by Gillman and Jerison [4]. In many respects these two rings of continuous functions are radically different. However, it is not hard to see that any purely algebraic property which holds for every $C(X)$ must also hold for every $C^*(X)$. This is because every $C^*(X)$ is algebraically isomorphic to a ‘$C^*$’, that is, is isomorphic to $C(Y)$ for some $Y$ (in this case $Y = \beta X$). For instance, a property of ideals valid in every $C(X)$ would be valid in $C^*(X)$. Of course, to study ideals in any ring one needs to know when an element is invertible. In both $C(X)$ and $C^*(X)$ there are simple criteria for invertibility—a function is invertible in $C(X)$ if it never vanishes, and in $C^*(X)$ if it is bounded away from zero.

In this paper we consider the structure of the prime ideals in any ring of continuous functions $A(X)$ that lies between $C^*(X)$ and $C(X)$, that is,

$$C^*(X) \subset A(X) \subset C(X).$$

The difficulty in studying ideals in $A(X)$ is that there is no simple criterion for invertibility like those available for $C(X)$ and $C^*(X)$. To overcome this obstacle a form of ‘local invertibility’ of elements of $A(X)$ was introduced in [8]. The idea is to associate to each $f \in A(X)$ a collection of sets $\mathcal{Z}_A(f)$ which consists of those zero
sets for which \( f \) is locally invertible on their complement. The usefulness of this
definition stems from the following surprising criterion for invertibility: \( f \) is
noninvertible in \( A(X) \) if and only if \( \mathcal{Z}_A(f) \) is a \( z \)-filter on \( X \). Moreover, for an
ideal \( I \) in \( A(X) \), \( \mathcal{Z}_A[I] \) is again a \( z \)-filter on \( X \). (Plank [7] uses a different method
which associates \( z \)-ultrafilters to maximal ideals.) This assignment of \( z \)-filters to
ideals serves as a bridge between the algebraic structure of \( A(X) \) and the topology of
\( X \). Thus, for a given \( A(X) \) we are able to define, in a natural way, an \( A \)-
compactification of \( X \). This concept generalizes the usual notions of
compactifications as follows: the \( C^* \)-compactification is the Stone-Čech
compactification and the \( C \)-compactification is the Hewitt realcompactification.
We use the \( A \)-compactifications of \( X \) to characterize those rings of continuous
functions between \( C^*(X) \) and \( C(X) \) which are \( C \)'s. Finally, as a simple application
of \( A \)-compactness, we generalize a theorem concerning prime ideals [6].

2. **A-Compactifications**

Let \( X \) be a completely regular space. *Throughout this paper \( A(X) \) denotes a ring
of continuous real-valued functions on \( X \) such that \( C^*(X) \subset A(X) \subset C(X) \).*

A *zero set* in \( X \) is a set of the form \( Z(f) = \{ x \in X : f(x) = 0 \} \) for some \( f \in C(X) \);
the collection of all zero sets in \( X \) is denoted by \( Z(X) \). The complement of a zero
set is called a *cozero set*. If \( f \in A(X) \) and \( E \) is a cozero set in \( X \), then we say that
\( f \) is E-*regular* if there exists \( g \in A(X) \) such that \( fg \mid E \equiv 1 \); that is, \( f \) is locally
invertible on \( E \). For each \( f \in A(X) \) we set

\[
\mathcal{Z}_A(f) = \{ E \in Z(X) : f \text{ is } E^c \text{-regular} \}.
\]

It is shown in [8] that \( f \) is noninvertible in \( A(X) \) if and only if \( \mathcal{Z}_A(f) \) is a \( z \)-filter on
\( X \). Moreover, if \( I \) is an ideal in \( A(X) \) then \( \mathcal{Z}_A[I] = \bigcup \{ \mathcal{Z}_A(f) : f \in I \} \) is again
always a \( z \)-filter on \( X \). For a \( z \)-filter \( \mathcal{F} \) on \( X \) we define

\[
\mathcal{Z}_A(\mathcal{F}) = \{ f \in A(X) : \mathcal{Z}_A(f) \subset \mathcal{F} \}.
\]

Indeed, \( \mathcal{Z}_A(\mathcal{F}) \) is always an ideal in \( A(X) \), and if \( \mathcal{U} \) is a \( z \)-ultrafilter, \( \mathcal{Z}_A^{-1}[\mathcal{U}] \) is a
maximal ideal. This correspondence between maximal ideals and ultrafilters is
one-to-one for the type of rings we are considering here ([8], [3]).
In the Stone construction of $\beta X$ the points of $\beta X$ are the $z$-ultrafilters on $X$ [4, pp. 83-89]. We write $p, q, \ldots$ for the points of $\beta X$, but when we wish to emphasize that these are $z$-ultrafilters, we use the notation $\mathcal{U}_p, \mathcal{U}_q, \ldots$. We also write $M^P_A$ for the maximal ideal corresponding to $\mathcal{U}_p$ under the map $\mathbb{Z}_A^{-};$ that is $M^P_A = \mathbb{Z}_A^{-}[\mathcal{U}_p]$.

**Theorem 1.** $M^P_A = \{f \in A(X) : \lim_{\mathcal{U}_p} fh = 0 \text{ for all } h \in A(X)\}$.

*Proof.* If $f \in M^P_A$ then $\mathbb{Z}_A(f) \subset \mathcal{U}_p$. By [8, Lemma 2], $\lim_{\mathcal{U}_p} fh = 0$ for all $h \in A(X)$. Conversely, if $f \in A(X)$ satisfies the condition $\lim_{\mathcal{U}_p} fh = 0$ for all $h \in A(X)$, then by [8, Lemma 3], $\mathbb{Z}_A(f) \subset \mathcal{U}_p$. Thus $f \in M^P_A$. \qed

If $M$ is a maximal ideal in $A(X)$ then the field $A(X)/M$ contains a canonical copy of $\mathbb{R}$: the image of the constant functions under the quotient map. When $A(X)/M$ consists only of this copy of $\mathbb{R}$ we say that $M$ is a real maximal ideal.

It is well known that a completely regular space is compact if and only if every $z$-ultrafilter on $X$ converges. This characterization has a counterpart in terms of the notion of $A$-stable collections of sets introduced in [3]. A collection $\mathcal{C}$ of closed sets in $X$ is called $A$-stable if every $f \in A(X)$ is bounded on some member of $\mathcal{C}$. We describe the relationship between $A$-stable $z$-ultrafilters and real maximal ideals.

**Theorem 2.** $M^P_A$ is a real maximal ideal if and only if $\mathcal{U}_p$ is $A$-stable.

*Proof.* If $\mathcal{U}_p$ is $A$-stable then $\lim_{\mathcal{U}_p} f$ is a real number for every $f \in A(X)$. Thus the map

$$\psi_p : A(X) \to \mathbb{R},$$

$$f \mapsto \lim_{\mathcal{U}_p} f$$

is a well-defined algebra homomorphism. By Theorem 1, its kernel is $M^P_A$, and hence $M^P_A$ is real. Conversely, if $M^P_A$ is real, then for every $f \in A(X)$ there exists a constant function $r \in \mathbb{R}$ such that $f - r \in M^P_A$. Thus by Theorem 1, using $h \equiv 1$, we have $\lim_{\mathcal{U}_p} (f - r) = 0$, or $\lim_{\mathcal{U}_p} f = r$. Hence $f$ must be bounded on some member of $\mathcal{U}_p$, and so $\mathcal{U}_p$ is $A$-stable. \qed

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An ideal \( I \) is fixed if \( \cap \{ \mathcal{Z}_A(f) : f \in I \} \neq \emptyset \). The fixed maximal ideals in \( A(X) \) are precisely the ideals \( M^p_A \) where \( p \in X \) [3, Theorem 2.1]. A completely regular space \( X \) is realcompact if every real maximal ideal in \( C(X) \) is fixed, and is compact if every real maximal ideal (they are all real) in \( C^*(X) \) is fixed. This leads to the following definition [8]: \( X \) is \( A \)-compact if every real maximal ideal in \( A(X) \) is fixed. Thus a \( C^* \)-compact space is compact and a \( C \)-compact space is realcompact.

It is shown in [3] that \( X \) is \( A \)-compact if and only if every \( A \)-stable \( z \)-ultrafilter on \( X \) converges. The following theorem generalizes the well-known fact that a compact space is realcompact.

**Theorem 3.** If \( X \) is \( A \)-compact and \( A(X) \subset B(X) \), then \( X \) is \( B \)-compact.

**Proof.** Let \( M^p_B \) be a real maximal ideal in \( B(X) \). By Theorem 2, \( U_p \) is \( B \)-stable, and hence \( A \)-stable. Applying Theorem 2 again, we have that \( M^p_A \) is a real maximal ideal in \( A(X) \). Since \( X \) is \( A \)-compact, \( M^p_A \) is fixed—that is, \( p \in X \). Thus \( M^p_B \) is also fixed, and so \( X \) is \( B \)-compact. \( \square \)

The maximal ideal space \( \mathcal{M}(A) \) is the set of maximal ideals in \( A(X) \) equipped with the hull-kernel topology. A base for the closed sets in \( \mathcal{M}(A) \) is given by the sets \( N_f = \{ M \in \mathcal{M}(A) : f \in M \}, f \in A(X) \). A base for the closed sets in the Stone topology on \( \beta X \) is given by the sets \( Z = \{ p \in \beta X : Z \in U_p \}, Z \in Z[X] \).

**Definition.** Let \( A(X) \) be a ring of continuous functions. We define a subspace \( v_A X \) of \( \mathcal{M}(A) \) by

\[
v_A X = \{ M : M \text{ is a real maximal ideal in } A(X) \}.
\]

The space \( v_A X \) is called the \( A \)-compactification of \( X \). We also distinguish a subspace \( \mathcal{F}_A \) of \( \beta X \) by

\[
\mathcal{F}_A = \{ p \in \beta X : U_p \text{ is an } A \text{-stable } z \text{-ultrafilter on } X \}.
\]

The space \( X \) is embedded in \( v_A X \) by \( x \mapsto M = \{ f \in A(X) : f(x) = 0 \} \), and it is embedded in \( \mathcal{F}_A \) by \( x \mapsto U_p \), where \( U_p \) is the principal \( z \)-ultrafilter generated by \( \{ x \} \).

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LEMMA 4. A base for the closed sets in the Stone topology on \( \beta X \) is given by the sets \( Z_f = \{ p \in \beta X : \mathcal{Z}_A(f) \subseteq \mathcal{U}_p \} \), \( f \in A(X) \).

Proof. Each \( Z_f \) is a closed set in the Stone topology because it is an intersection of the basic closed sets \( \bigcap \{ E : E \in \mathcal{Z}_A(f) \} \). To complete the proof, we show that each basic set \( Z \) is an intersection of sets of the form \( Z_f \). To this end let \( Z \in \mathcal{Z}[X] \), and let \( I_Z \) be the ideal in \( A(X) \) defined by \( I_Z = \{ f \in A(X) : f(Z) = 0 \} \).
We show that
\[
\mathcal{Z} = \bigcap \{ Z_f : f \in I_Z \}.
\]
If \( p \in \mathcal{Z} \), then \( Z \subseteq \mathcal{U}_p \). So for every \( f \in I_Z \), \( \mathcal{Z}_A(f) \subseteq \mathcal{U}_p \), and hence \( p \in Z_f \). This shows that \( \mathcal{Z} \subseteq \bigcap \{ Z_f : f \in I_Z \} \). For the other containment, let \( p \in \bigcap \{ Z_f : f \in I_Z \} \) then \( E \in \mathcal{U}_p \), then \( E \) meets every element of \( \mathcal{Z}_A[I_Z] \). We claim that \( E \) meets \( Z \). For suppose that \( E \cap Z = \emptyset \). Then there exists \( h \in A(X), 0 \leq h \leq 1 \), such that \( h(E) = 1 \) and \( h(Z) = 0 \) [4, p. 17]. Clearly \( h \in I_Z \). The set \( F = \{ x \in X : h(x) \leq \frac{1}{2} \} \) is a zero set that contains \( Z \); moreover, \( F \in \mathcal{Z}_A(h) \) since \( h \) is bounded away from zero on \( F^c \). But \( E \cap F = \emptyset \), and this contradicts the fact that \( E \) meets every element of \( \mathcal{Z}_A[I_Z] \). Thus \( E \cap Z \neq \emptyset \). Since \( E \) is an arbitrary element of the ultrafilter \( \mathcal{U}_p \), this proves that \( Z \in \mathcal{U}_p \). Thus \( p \in \mathcal{Z} \).

\[\square\]

THEOREM 5. \( \mathcal{M}(A) \) is homeomorphic to \( \beta X \) by a homeomorphism that keeps \( X \) pointwise fixed. Moreover, \( v_A X \) is homeomorphic to \( \mathcal{F}_A \) by a restriction of the same homeomorphism.

Proof. The map \( \mathcal{Z}_A^\sim : \mathcal{U}_p \rightarrow \mathcal{M}_A^P \) is a bijection from \( \beta X \) to \( \mathcal{M}(A) \) ([8], [3]). This map obviously keeps \( X \) pointwise fixed. Moreover, by Theorem 2, \( \mathcal{Z}_A^\sim |_{\mathcal{F}_A} \) is a bijection from \( \mathcal{F}_A \) to \( v_A X \). We show that \( \mathcal{Z}_A^\sim \) is a homeomorphism from \( \beta X \) to \( \mathcal{M}(A) \) by showing that it maps the base \( \{ Z_f : f \in A(X) \} \) onto the base \( \{ N_f : f \in A(X) \} \). If \( p \in Z_f \), then \( \mathcal{Z}_A(f) \subseteq \mathcal{U}_p \), and so \( f \in \mathcal{Z}_A^\sim[q] \). Thus \( \mathcal{Z}_A^\sim[Z_f] \subseteq N_f \). To show equality, let \( M \in N_f \) so that \( \mathcal{Z}_A(f) \subseteq \mathcal{Z}_A[M] \). By [3, Theorem 3.2], \( \mathcal{Z}_A[M] \) is contained in a unique \( z \)-ultrafilter \( \mathcal{U}_q \), say. Clearly \( \mathcal{Z}_A(f) \subseteq \mathcal{U}_q \), so that \( q \in Z_f \). But \( f \in \mathcal{Z}_A^\sim[q] = M \). This shows that \( \mathcal{Z}_A^\sim[Z_f] = N_f \). Thus the map \( \mathcal{Z}_A^\sim \) maps the basic closed set \( Z_f \) to the basic closed set \( N_f \), which completes the proof.

\[\square\]
We have the following version of the Banach-Stone theorem.

**Theorem 6.** If \( A(X) \) and \( B(Y) \) are isomorphic rings then \( v_A X \) and \( v_B Y \) are homeomorphic topological spaces.

**Proof.** If \( A(X) \) and \( B(Y) \) are isomorphic then their sets of real maximal ideals have the same structure. The topological spaces \( v_A X \) and \( v_B Y \) are therefore constructed in the same way, so they are homeomorphic.

We now consider the extension of functions in \( A(X) \) to \( v_A X \). For every \( p \in v_A X \), \( \mathcal{U}_p \) is \( A \)-stable (Theorem 2) and so \( \lim_{q \in \mathcal{U}_p} f \) is a real number. Thus for every \( f \in A(X) \) we may define a function \( f_A : v_A X \to \mathbb{R} \) by \( f_A(p) = \lim_{q \in \mathcal{U}_p} f \). The function \( f_A \) agrees with \( f \) on \( X \), and is easily seen to be continuous. Thus every \( f \in A(X) \) has a continuous extension \( f_A \) to \( v_A X \). We write \( A(v_A X) = \{ f_A : f \in A(X) \} \). Clearly \( A(v_A X) \supseteq C^*(v_A X) \), since \( A(X) \supseteq C^*(X) \).

3. Which \( A(X) \) is a C?

We say \( A(X) \) is a C if there exists a completely regular space \( Y \) such that \( A(X) \) is isomorphic (as a ring) to \( C(Y) \). Clearly every \( C^*(X) \) is a C since \( C^*(X) \cong C(\beta X) \). We say that \( A(X) \) is inverse closed if every \( f \in A(X) \) which never vanishes is invertible.

**Theorem 7.** \( A(X) \) is a C if and only if \( A(v_A X) \) is inverse closed. In this case \( A(X) \) is isomorphic to \( C(v_A X) \).

**Proof.** Suppose \( A(X) \) is isomorphic to \( C(Y) \) for some completely regular space \( Y \). Since \( A(X) \) is isomorphic to \( A(v_A X) \) and \( C(Y) \) is isomorphic to \( C(vY) \), it follows that \( A(v_A X) \) is isomorphic to \( C(vY) \). So by Theorem 6 there is a homeomorphism \( \varphi : vY \to v_A X \). If \( f \in A(v_A X) \) and \( f \) never vanishes then \( f \circ \varphi \in C(vY) \) never vanishes, so has an inverse \( 1/f \circ \varphi \). Now \( (1/f \circ \varphi) \circ \varphi^{-1} = 1/f \in A(v_A X) \).

Conversely, let \( f \in C(v_A X) \). Then \( f/(1 + f^2) \in C^*(v_A X) \subseteq A(v_A X) \). Also 135
1 + f^2 \in A(v_A X), since 1/(1 + f^2) \in C^*(v_A X) \subset A(v_A X) and A(v_A X) is inverse closed. Thus f = [f/(1 + f^2)](1 + f^2) \in A(v_A X), and so A(v_A X) = C(v_A X). Since A(X) is isomorphic to A(v_A X), it follows that A(X) is a C. Specifically, A(X) \cong C(v_A X).

Two different rings of continuous functions A(X) and B(X) may give the same A-compactifications v_A X and v_B X as the following example shows.

**Example.** Let H(\mathbb{N}) denote the ring of all sequences on \mathbb{N} which are coefficients in the Taylor series of an analytic function on the disc (see [1]). It is easy to see that v_H \mathbb{N} = \mathbb{N}, that is, \mathbb{N} is H-compact. Thus, since H(\mathbb{N}) is not inverse closed, it follows by Theorem 7 that H(\mathbb{N}) is not a C. Since \mathbb{N} is realcompact, v_C \mathbb{N} = \mathbb{N}. Thus two different rings can yield the same A-compactification.

The situation in the example cannot happen for rings that are C's. Indeed, we now show that rings that are C's are the appropriate class of rings for which the converse of Theorem 6 holds. This answers a question posed at the end of [8].

**Theorem 8.** Let A(X) and B(Y) be rings of continuous functions that are C's. Then, A(X) and B(Y) are isomorphic rings if and only if v_A X and v_B Y are homeomorphic topological spaces.

**Proof.** If A(X) and B(Y) are isomorphic then v_A X and v_B Y are homeomorphic by Theorem 6. Conversely, if A(X) and B(Y) are both C's then A(X) \cong C(v_A X) and B(Y) \cong C(v_B Y). So, if v_A X and v_B Y are homeomorphic, A(X) and B(Y) are isomorphic. \qed

As an application of Theorem 8 we prove an extension theorem for A-compactifications that simultaneously generalizes those for \beta X and vX as can be seen by taking A(X) to be C^*(X) and C(X), respectively, in the statement of the theorem.

**Theorem 9.** Let A(X) be a C. Then a real-valued continuous function f on X has a continuous extension to v_A X if and only if f \in A(X).
Proof. If \( f \in A(X) \), then \( f \) has an extension \( f^A \) to \( v_A X \) as defined after Theorem 6. Conversely, since \( A(X) \) is a \( C \), it follows from Theorem 8 that \( A(v_A X) = C(v_A X) \). Thus if \( f \in C(X) \) has an extension to \( v_A X \), then \( f^A \in A(v_A X) \), and hence \( f \in A(X) \). \( \Box \)

The hypothesis that \( A(X) \) is a \( C \) in Theorem 9 is required for only one implication. Indeed, the example of \( H(\mathbb{N}) \) described above shows that if \( A(X) \) is not a \( C \), then not every continuous function \( f \) that has an extension to \( v_A X \) is necessarily in \( A(X) \).

4. PRIME IDEAL STRUCTURE

In this section we generalize the following theorem from [6]: If \( M^* \) is any maximal ideal of \( C^*(X) \) and \( M \) is the unique maximal ideal in \( C(X) \) satisfying \( M \cap C^*(X) \subseteq M^* \), then every prime ideal contained in \( M^* \) is comparable with \( M \cap C^*(X) \). Our generalization involves any two rings \( A(X) \) and \( B(X) \) and real maximal ideals in \( A(X) \). Indeed, as our proof below shows, the validity of the theorem for \( C^*(X) \) and \( C(X) \) depends crucially on the fact that every maximal ideal in \( C^*(X) \) is real.

We first need a Gelfand-Kolmogoroff-type characterization for real maximal ideals in \( A(X) \). Our characterization is in terms of the \( A \)-compactification (see [8, Theorem 5] or [3, Theorem 3.3] for a characterization valid for all maximal ideals).

For \( f \in A(X) \), we define \( S_A(f) \) to be the set of cluster points of \( Z_A(f) \) in \( v_A X \). That is \( S_A(f) = \cap \{ \text{cl}_{v_A X} E : E \in Z_A(f) \} \).

**Theorem 10.** If \( p \in v_A X \), then \( M_A^p = \{ f \in A(X) : p \in S_A(f) \} \).

**Proof.** By definition \( M_A^p = Z_A^{-1}[\cup_p] \) and so \( f \in M_A^p \) if and only if \( Z_A(f) \subseteq \cup_p \).

Since \( \cup_p \) converges to \( p \) in \( \beta X \), it follows that \( p \) is a cluster point of \( Z_A(f) \), that is \( p \in S_A(f) \). Conversely, if \( p \in S_A(f) \) then \( p \) is a cluster point of \( Z_A(f) \) in \( v_A X \).

But since \( p \) already belongs to \( v_A X \), it follows that \( p \) is a cluster point of \( Z_A(f) \) in \( \beta X \). Thus \( Z_A(f) \subseteq \cup_p \), so \( f \in M_A^p \). \( \Box \)
For $p \in v_A X$, we define the ideal $O_A^p = \{ f \in A(X) : p \in \text{int } S_A(f) \}$. For $p \in X$ this definition coincides with the usual meaning of $O_A^p$ ([3] and [7]). If $p \in v_A X$ and $P$ is a prime ideal contained in $M_A^p$ then $P$ contains $O_A^p$ (see [7, p. 48]).

**Theorem 11.** Let $A(X) \subseteq B(X)$ and let $p \in v_A X$. Then every prime ideal $P$ of $A(X)$ contained in $M_A^p$ is comparable with $M_B^p \cap A(X)$. Specifically, $P \subseteq M_B^p \cap A(X)$ if and only if $P$ contains no unit of $B(X)$, while $M_B^p \cap A(X) \subseteq P$ if and only if $P$ contains a unit of $B(X)$.

**Proof.** Let $P$ be any prime ideal of $A(X)$ with $P \subseteq M_A^p$. Choose a minimal prime ideal $Q \subseteq P$. We show that $Q \subseteq M_B^p \cap A(X)$. This will complete the proof since then both $P$ and $M_B^p \cap A(X)$ contain the prime ideal $Q$ and so are comparable (because the prime ideals in $A(X)/Q$ form a chain ([4, p. 195] and [3, Theorem 2.5])).

Let $f \in Q$. Since $Q$ is a minimal prime ideal of $A(X)$ there is $h \in A(X)\setminus Q$ such that $fh = 0$ [5, Lemma 1.1]. Now, by the remarks preceding the theorem, $O_A^p \subseteq Q$. It follows that $h \notin O_A^p$ and so $p \notin \text{int } S_A(h)$. Thus for each neighborhood $V$ of $p$ in $v_A X$, there exists $x \in V \cap X$ such that $h(x) \neq 0$ and hence $f(x) = 0$. It follows that $x \in E$ for each $E \in Z_B(f)$, and so $V \cap E \neq \emptyset$. Thus $p \in \text{cl}_{v_A X} E$ for each $E \in Z_B(f)$. Since $\cup_p$ consists of those zero-sets in $X$ whose closure in $\beta X$ contains $p$, we have $Z_B(f) \subseteq \cup_p$, and hence $f \in Z_B^{-1} \cup_p = M_B^p$. So $f \in M_B^p \cap A(X)$.

Now assume that $P$ contains no unit of $B(X)$. Let $V$ be any zero-set neighborhood of $p$ in $v_A X$. Let $I_V = \{ f \in A(X) : f(V \cap X) = 0 \}$, and let $f \in I_V$. Clearly $f \in O_A^p$ and so $Z_A(f) \subseteq Z_A(O_A^p)$. By [3, Theorem 4.1], $Z_A(O_A^p) = Z_A[P]$, so $Z_A(f) \subseteq Z_A[P]$. Now, since $P$ contains no unit of $B$, $Z_B[P]$ is a z-filter on $X$. Since $Z_A[P] \subseteq Z_B[P]$, it follows that $Z_A(f) \subseteq Z_B[P]$. Since $f$ is an arbitrary element of $I_V$, we have $Z_A[I_V] \subseteq Z_B[P]$, and so each $E \in Z_B[P]$ has nonempty intersection with every $F \in Z_A[I_V]$. By the same argument as in the proof of Lemma 4, it follows that $E \cap V \neq \emptyset$. This shows that each $E \in Z_B[P]$ meets every zero-set neighborhood $V$ of $p$ in $v_A X$. Thus $p$ is a cluster point of the z-filter $Z_B[P]$, and hence $Z_B[P] \subseteq \cup_p$, which means that $P \subseteq Z_B^{-1} \cup_p = M_B^p$; that is, $P \subseteq M_B^p \cap A(X)$. The converse is immediate, and the last statement follows from comparability. □
REFERENCES


Department of Mathematics
California State University
Long Beach, CA 90840

Department of Mathematics
The Pennsylvania State University
Abington, PA 19001

Department of Mathematics
California State University
Long Beach, CA 90840