MATH 410
Assignment 7

1. (a) State Fermat’s little theorem.
   (b) Prove that if \( x \) is relatively prime to 7, then \( x^6 \equiv 1 \pmod{7} \).
   (c) Prove that if \( x \) is relatively prime to 7, then \( x^3 \equiv \pm 1 \pmod{7} \).
   (d) Prove that the Diophantine equation \( x^3 + y^3 = z^3 \) has no solutions where \( 7 \nmid xyz \).

2. Let \( n \in \mathbb{N} \) and let \( G = \{ k \in \mathbb{N} \mid k < n \text{ and } k \text{ is relatively prime to } n \} \), and let \( \phi(n) \) denote the number of elements in \( G \).
   (a) Prove that \( G \) is a group under multiplication mod \( n \).
   (b) Prove that for any \( a \in G \), we have \( a^{\phi(n)} = 1 \).
   (c) State and prove Euler’s generalization of Fermat’s little theorem.

3. Consider the Diophantine equation \( y^3 = x^2 + 2 \).
   (a) Observe that \( x = \pm 5 \) and \( y = 3 \) are solutions.
   (b) Factor the equation as \( y^3 = (x + \sqrt{-2})(x - \sqrt{-2}) \). We’ve shown that this equation has only the solutions in part (a) in the ring \( R = \{ a + b\sqrt{-2} \mid a, b \in \mathbb{N} \} \). Explain why this proves that the original equation has only these same solutions in the ring of integers.

4. The Riemann zeta function is defined by \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \).
   (a) Explain how the Euler product formula is derived by writing each factor in the product as a geometric series:
   \[
   \prod_{p \text{ prime}} \left(1 - p^{-s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s}
   \]
   (b) Define the Möbius function \( \mu(n) \) and prove that
   \[
   \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}
   \]