A second-order linear differential equation has the form

\[ P(x) \frac{d^2y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = G(x) \]

where \( P, Q, R, \) and \( G \) are continuous functions. Equations of this type arise in the study of the motion of a spring. In Additional Topics: Applications of Second-Order Differential Equations we will further pursue this application as well as the application to electric circuits.

In this section we study the case where \( G(x) = 0 \), for all \( x \), in Equation 1. Such equations are called homogeneous linear equations. Thus, the form of a second-order linear homogeneous differential equation is

\[ P(x) \frac{d^2y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = 0 \]

If \( G(x) \neq 0 \) for some \( x \), Equation 1 is nonhomogeneous and is discussed in Additional Topics: Nonhomogeneous Linear Equations.

Two basic facts enable us to solve homogeneous linear equations. The first of these says that if we know two solutions \( y_1 \) and \( y_2 \) of such an equation, then the linear combination \( y = c_1y_1 + c_2y_2 \) is also a solution.

### Theorem
If \( y_1(x) \) and \( y_2(x) \) are both solutions of the linear homogeneous equation (2) and \( c_1 \) and \( c_2 \) are any constants, then the function

\[ y(x) = c_1y_1(x) + c_2y_2(x) \]

is also a solution of Equation 2.

**Proof** Since \( y_1 \) and \( y_2 \) are solutions of Equation 2, we have

\[ P(x)y_1'' + Q(x)y_1' + R(x)y_1 = 0 \]

and

\[ P(x)y_2'' + Q(x)y_2' + R(x)y_2 = 0 \]

Therefore, using the basic rules for differentiation, we have

\[ P(x)y'' + Q(x)y' + R(x)y = c_1[P(x)y_1'' + Q(x)y_1' + R(x)y_1] + c_2[P(x)y_2'' + Q(x)y_2' + R(x)y_2] \]

\[ = c_1(0) + c_2(0) = 0 \]

Thus, \( y = c_1y_1 + c_2y_2 \) is a solution of Equation 2.

The other fact we need is given by the following theorem, which is proved in more advanced courses. It says that the general solution is a linear combination of two linearly independent solutions \( y_1 \) and \( y_2 \). This means that neither \( y_1 \) nor \( y_2 \) is a constant multiple of the other. For instance, the functions \( f(x) = x^2 \) and \( g(x) = 5x^2 \) are linearly dependent, but \( f(x) = e^x \) and \( g(x) = xe^x \) are linearly independent.
Theorem 4 If \( y_1 \) and \( y_2 \) are linearly independent solutions of Equation 2, and \( P(x) \) is never 0, then the general solution is given by

\[
y(x) = c_1 y_1(x) + c_2 y_2(x)
\]

where \( c_1 \) and \( c_2 \) are arbitrary constants.

Theorem 4 is very useful because it says that if we know two particular linearly independent solutions, then we know every solution.

In general, it is not easy to discover particular solutions to a second-order linear equation. But it is always possible to do so if the coefficient functions \( P, Q, \) and \( R \) are constant functions, that is, if the differential equation has the form

\[
ay'' + by' + cy = 0
\]

where \( a, b, \) and \( c \) are constants and \( a \neq 0 \).

It’s not hard to think of some likely candidates for particular solutions of Equation 5 if we state the equation verbally. We are looking for a function \( y \) such that a constant times its second derivative \( y'' \) plus another constant times \( y' \) plus a third constant times \( y \) is equal to 0. We know that the exponential function \( y = e^{rx} \) (where \( r \) is a constant) has the property that its derivative is a constant multiple of itself: \( y' = re^{rx} \). Furthermore, \( y'' = r^2 e^{rx} \). If we substitute these expressions into Equation 5, we see that \( y = e^{rx} \) is a solution if

\[
ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0
\]
or

\[
(ar^2 + br + c)e^{rx} = 0
\]

But \( e^{rx} \) is never 0. Thus, \( y = e^{rx} \) is a solution of Equation 5 if \( r \) is a root of the equation

\[
ar^2 + br + c = 0
\]

Equation 6 is called the **auxiliary equation** (or characteristic equation) of the differential equation \( ay'' + by' + cy = 0 \). Notice that it is an algebraic equation that is obtained from the differential equation by replacing \( y'' \) by \( r^2 \), \( y' \) by \( r \), and \( y \) by 1.

Sometimes the roots \( r_1 \) and \( r_2 \) of the auxiliary equation can be found by factoring. In other cases they are found by using the quadratic formula:

\[
r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}
\]

We distinguish three cases according to the sign of the discriminant \( b^2 - 4ac \).

**CASE 1 \( b^2 - 4ac > 0 \)**

In this case the roots \( r_1 \) and \( r_2 \) of the auxiliary equation are real and distinct, so \( y_1 = e^{r_1 x} \) and \( y_2 = e^{r_2 x} \) are two linearly independent solutions of Equation 5. (Note that \( e^{rx} \) is not a constant multiple of \( e^{rx} \).) Therefore, by Theorem 4, we have the following fact.

If the roots \( r_1 \) and \( r_2 \) of the auxiliary equation \( ar^2 + br + c = 0 \) are real and unequal, then the general solution of \( ay'' + by' + cy = 0 \) is

\[
y = c_1 e^{r_1 x} + c_2 e^{r_2 x}
\]
EXAMPLE 1 Solve the equation \(y'' + y' - 6y = 0\).

SOLUTION The auxiliary equation is

\[r^2 + r - 6 = (r - 2)(r + 3) = 0\]

whose roots are \(r = 2, -3\). Therefore, by (8) the general solution of the given differential equation is

\[y = c_1e^{2x} + c_2e^{-3x}\]

We could verify that this is indeed a solution by differentiating and substituting into the differential equation.

EXAMPLE 2 Solve \(3y'' + y' - y = 0\).

SOLUTION To solve the auxiliary equation we use the quadratic formula:

\[r = \frac{-1 \pm \sqrt{13}}{6}\]

Since the roots are real and distinct, the general solution is

\[y = c_1e^{\left(-1+\sqrt{13}\right)x/6} + c_2e^{\left(-1-\sqrt{13}\right)x/6}\]

CASE II \(b^2 - 4ac = 0\)

In this case \(r_1 = r_2\); that is, the roots of the auxiliary equation are real and equal. Let’s denote by \(r\) the common value of \(r_1\) and \(r_2\). Then, from Equations 7, we have

\[r = \frac{-b}{2a}\quad\text{so}\quad 2ar + b = 0\]

We know that \(y_1 = e^{rx}\) is one solution of Equation 5. We now verify that \(y_2 = xe^{rx}\) is also a solution:

\[ay'' + by' + cy = a(2re^{rx} + r^2xe^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx}\]
\[= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx}\]
\[= 0(e^{rx}) + 0(xe^{rx}) = 0\]

The first term is 0 by Equations 9; the second term is 0 because \(r\) is a root of the auxiliary equation. Since \(y_1 = e^{rx}\) and \(y_2 = xe^{rx}\) are linearly independent solutions, Theorem 4 provides us with the general solution.

EXAMPLE 3 Solve the equation \(4y'' + 12y' + 9y = 0\).

SOLUTION The auxiliary equation \(4r^2 + 12r + 9 = 0\) can be factored as

\[(2r + 3)^2 = 0\]
so the only root is \( r = -\frac{3}{2} \). By (10) the general solution is
\[
y = c_1 e^{-3x/2} + c_2 xe^{-3x/2}
\]

CASE III \( b^2 - 4ac < 0 \)
In this case the roots \( r_1 \) and \( r_2 \) of the auxiliary equation are complex numbers. (See Additional Topics: Complex Numbers for information about complex numbers.) We can write
\[
r_1 = \alpha + i\beta \quad r_2 = \alpha - i\beta
\]
where \( \alpha \) and \( \beta \) are real numbers. [In fact, \( \alpha = -b/(2a) \), \( \beta = \sqrt{4ac - b^2}/(2a) \).] Then, using Euler’s equation
\[
e^{i\theta} = \cos \theta + i \sin \theta
\]
from Additional Topics: Complex Numbers, we write the solution of the differential equation as
\[
y = C_1 e^{\alpha x} + C_2 e^{\beta x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}
\]
\[
= C_1 e^{\alpha x} (\cos \beta x + i \sin \beta x) + C_2 e^{\alpha x} (\cos \beta x - i \sin \beta x)
\]
\[
= e^{\alpha x} [(C_1 + C_2) \cos \beta x + i(C_1 - C_2) \sin \beta x]
\]
\[
= e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)
\]
where \( c_1 = C_1 + C_2 \), \( c_2 = i(C_1 - C_2) \). This gives all solutions (real or complex) of the differential equation. The solutions are real when the constants \( c_1 \) and \( c_2 \) are real. We summarize the discussion as follows.

If the roots of the auxiliary equation \( ar^2 + br + c = 0 \) are the complex numbers \( r_1 = \alpha + i\beta, r_2 = \alpha - i\beta \), then the general solution of \( ay'' + by' + cy = 0 \) is
\[
y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)
\]

EXAMPLE 4 Solve the equation \( y'' - 6y' + 13y = 0 \).

SOLUTION The auxiliary equation is \( r^2 - 6r + 13 = 0 \). By the quadratic formula, the roots are
\[
r = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i
\]
By (11) the general solution of the differential equation is
\[
y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)
\]

INITIAL-VALUE AND BOUNDARY-VALUE PROBLEMS

An initial-value problem for the second-order Equation 1 or 2 consists of finding a solution \( y \) of the differential equation that also satisfies initial conditions of the form
\[
y(x_0) = y_0 \quad y'(x_0) = y_1
\]
where \( y_0 \) and \( y_1 \) are given constants. If \( P, Q, R, \) and \( G \) are continuous on an interval and \( P(x) \neq 0 \) there, then a theorem found in more advanced books guarantees the existence and uniqueness of a solution to this initial-value problem. Examples 5 and 6 illustrate the technique for solving such a problem.
EXAMPLE 5 Solve the initial-value problem

\[ y'' + y' - 6y = 0 \quad y(0) = 1 \quad y'(0) = 0 \]

SOLUTION From Example 1 we know that the general solution of the differential equation is

\[ y(x) = c_1 e^{2x} + c_2 e^{-3x} \]

Differentiating this solution, we get

\[ y'(x) = 2c_1 e^{2x} - 3c_2 e^{-3x} \]

To satisfy the initial conditions we require that

\[ y(0) = c_1 + c_2 = 1 \]
\[ y'(0) = 2c_1 - 3c_2 = 0 \]

From (13) we have \( c_2 = \frac{2}{5}c_1 \) and so (12) gives

\[ c_1 + \frac{2}{5}c_1 = 1 \quad c_1 = \frac{5}{7} \quad c_2 = \frac{2}{5} \]

Thus, the required solution of the initial-value problem is

\[ y = \frac{5}{7} e^{2x} + \frac{2}{5} e^{-3x} \]

EXAMPLE 6 Solve the initial-value problem

\[ y'' + y = 0 \quad y(0) = 2 \quad y'(0) = 3 \]

SOLUTION The auxiliary equation is \( r^2 + 1 = 0 \), or \( r^2 = -1 \), whose roots are \( \pm i \). Thus \( \alpha = 0, \beta = 1, \) and since \( e^{0x} = 1 \), the general solution is

\[ y(x) = c_1 \cos x + c_2 \sin x \]

Since

\[ y'(x) = -c_1 \sin x + c_2 \cos x \]

the initial conditions become

\[ y(0) = c_1 = 2 \quad y'(0) = c_2 = 3 \]

Therefore, the solution of the initial-value problem is

\[ y(x) = 2 \cos x + 3 \sin x \]

A boundary-value problem for Equation 1 consists of finding a solution \( y \) of the differential equation that also satisfies boundary conditions of the form

\[ y(x_0) = y_0 \quad y(x_1) = y_1 \]

In contrast with the situation for initial-value problems, a boundary-value problem does not always have a solution.

EXAMPLE 7 Solve the boundary-value problem

\[ y'' + 2y' + y = 0 \quad y(0) = 1 \quad y(1) = 3 \]

SOLUTION The auxiliary equation is

\[ r^2 + 2r + 1 = 0 \quad \text{or} \quad (r + 1)^2 = 0 \]

whose only root is \( r = -1 \). Therefore, the general solution is

\[ y(x) = c_1 e^{-x} + c_2 xe^{-x} \]
Let be a nonzero real number. Show that the boundary-value problem has only the trivial solution for

The boundary conditions are satisfied if

The first condition gives , so the second condition becomes

Solving this equation for by first multiplying through by , we get

Thus, the solution of the boundary-value problem is

\[
y = e^{-x} + (3e - 1)xe^{-x}
\]

Summary: Solutions of \( ay'' + by' + c = 0 \)

<table>
<thead>
<tr>
<th>Roots of ( ar^2 + br + c = 0 )</th>
<th>General solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_1, r_2 ) real and distinct</td>
<td>( y = c_1e^{r_1x} + c_2e^{r_2x} )</td>
</tr>
<tr>
<td>( r_1 = r_2 = r )</td>
<td>( y = c_1e^{rx} + c_2xe^{rx} )</td>
</tr>
<tr>
<td>( r_1, r_2 ) complex: ( \alpha \pm i\beta )</td>
<td>( y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) )</td>
</tr>
</tbody>
</table>

EXERCISES

1–13 Solve the differential equation.

1. \( y'' - 6y' + 8y = 0 \)
2. \( y'' - 4y' + 8y = 0 \)
3. \( y'' + 8y' + 41y = 0 \)
4. \( 2y'' - y' - y = 0 \)
5. \( y'' - 2y' + y = 0 \)
6. \( 3y'' = 5y' \)
7. \( 4y'' + y = 0 \)
8. \( 16y'' + 24y' + 9y = 0 \)
9. \( 4y'' + y' = 0 \)
10. \( 9y'' + 4y = 0 \)
11. \( \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} - y = 0 \)
12. \( \frac{d^2y}{dx^2} - 6 \frac{dy}{dx} + 4y = 0 \)
13. \( \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \)

14–16 Graph the two basic solutions of the differential equation and several other solutions. What features do the solutions have in common?

14. \( 6 \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0 \)
15. \( \frac{d^2y}{dx^2} - 8 \frac{dy}{dx} + 16y = 0 \)
16. \( \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 5y = 0 \)

17–24 Solve the initial-value problem.

17. \( 2y'' + 5y' + 3y = 0, \ y(0) = 3, \ y'(0) = -4 \)
18. \( y'' + 3y = 0, \ y(0) = 1, \ y'(0) = 3 \)
19. \( 4y'' - 4y' + y = 0, \ y(0) = 1, \ y'(0) = -1.5 \)

20. \( 2y'' + 5y' - 3y = 0, \ y(0) = 1, \ y'(0) = 4 \)
21. \( y'' + 16y = 0, \ y(\pi/4) = -3, \ y'(\pi/4) = 4 \)
22. \( y'' - 2y' + 5y = 0, \ y(\pi) = 0, \ y'(\pi) = 2 \)
23. \( y'' + 2y' + 2y = 0, \ y(0) = 2, \ y'(0) = 1 \)
24. \( y'' + 12y' + 36y = 0, \ y(1) = 0, \ y'(1) = 1 \)

25–32 Solve the boundary-value problem, if possible.

25. \( 4y'' + y = 0, \ y(0) = 3, \ y(\pi) = -4 \)
26. \( y'' + 2y' = 0, \ y(0) = 1, \ y(1) = 2 \)
27. \( y'' - 3y' + 2y = 0, \ y(0) = 1, \ y(3) = 0 \)
28. \( y'' + 100y = 0, \ y(0) = 2, \ y(\pi) = 5 \)
29. \( y'' - 6y' + 25y = 0, \ y(0) = 1, \ y(\pi) = 2 \)
30. \( y'' - 6y' + 9y = 0, \ y(0) = 1, \ y(1) = 0 \)
31. \( y'' + 4y' + 13y = 0, \ y(0) = 2, \ y(\pi/2) = 1 \)
32. \( 9y'' - 18y' + 10y = 0, \ y(0) = 0, \ y(\pi) = 1 \)

33. Let \( L \) be a nonzero real number.

(a) Show that the boundary-value problem \( y'' + \lambda y = 0, \ y(0) = 0, y(L) = 0 \) has only the trivial solution \( y = 0 \) for the cases \( \lambda = 0 \) and \( \lambda < 0 \).

(b) For the case \( \lambda > 0 \), find the values of \( \lambda \) for which this problem has a nontrivial solution and give the corresponding solution.

34. If \( a, b, \) and \( c \) are all positive constants and \( y(x) \) is a solution of the differential equation \( ay'' + by' + cy = 0 \), show that \( \lim_{x \to \infty} y(x) = 0 \).
ANSWERS

17. $y = 2e^{-3x/2} + e^{-x}$
19. $y = e^{x^2} - 2xe^{x/2}$
21. $y = 3 \cos 4x - \sin 4x$
23. $y = e^{-x}(2 \cos x + 3 \sin x)$
25. $y = 3 \cos(\frac{1}{2}x) - 4 \sin(\frac{1}{2}x)$
27. $y = \frac{e^{x^3}}{e^x - 1} + \frac{e^{2x}}{1 - e^x}$
29. No solution
31. $y = e^{-x}(2 \cos 3x - e^x \sin 3x)$
33. (b) $\lambda = n^2\pi^2/L^2$, $n$ a positive integer; $y = C \sin(n\pi x/L)$
1. The auxiliary equation is $r^2 - 6r + 8 = 0 \Rightarrow (r-4)(r-2) = 0 \Rightarrow r = 4, r = 2$. Then by (8) the general solution is $y = c_1 e^{4x} + c_2 e^{2x}$.

3. The auxiliary equation is $r^2 + 8r + 41 = 0 \Rightarrow r = -4 \pm 5i$. Then by (11) the general solution is $y = e^{-4x}(c_1 \cos 5x + c_2 \sin 5x)$.

5. The auxiliary equation is $r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow r = 1$. Then by (10), the general solution is $y = c_1 e^x + c_2 x e^x$.

7. The auxiliary equation is $4r^2 + 1 = 0 \Rightarrow r = \pm \frac{1}{2}i$, so $y = c_1 \cos \left(\frac{1}{2}x\right) + c_2 \sin \left(\frac{1}{2}x\right)$.

9. The auxiliary equation is $4r^2 + r = r(4r + 1) = 0 \Rightarrow r = 0, r = -\frac{1}{4}$, so $y = c_1 + c_2 e^{-x/4}$.

11. The auxiliary equation is $r^2 - 2r - 1 = 0 \Rightarrow r = 1 \pm \sqrt{2}$, so $y = c_1 e^{(1+\sqrt{2})t} + c_2 e^{(1-\sqrt{2})t}$.

13. The auxiliary equation is $r^2 + r + 1 = 0 \Rightarrow r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, so $y = e^{-t/2} \left[ c_1 \cos \left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin \left(\frac{\sqrt{3}}{2}t\right) \right]$.

15. $r^2 - 8r + 16 = (r - 4)^2 = 0$ so $y = c_1 e^{4x} + c_2 x e^{4x}$.

The graphs are all asymptotic to the $x$-axis as $x \to -\infty$.

and as $x \to \infty$ the solutions tend to $\pm \infty$.

17. $2r^2 + 5r + 3 = (2r+3)(r+1) = 0$, so $r = -\frac{3}{2}, r = -1$ and the general solution is $y = c_1 e^{-3x/2} + c_2 e^{-x}$. Then $y(0) = 3 \Rightarrow c_1 + c_2 = 3$ and $y'(0) = -4 \Rightarrow -\frac{3}{2} c_1 - c_2 = -4$, so $c_1 = 2$ and $c_2 = 1$. Thus the solution to the initial-value problem is $y = 2e^{-3x/2} + e^{-x}$.

19. $4r^2 - 4r + 1 = (2r-1)^2 = 0 \Rightarrow r = \frac{1}{2}$ and the general solution is $y = c_1 e^{x/2} + c_2 x e^{x/2}$. Then $y(0) = 1 \Rightarrow c_1 = 1$ and $y'(0) = -1.5 \Rightarrow \frac{1}{2} c_1 + c_2 = -1.5$, so $c_2 = -2$ and the solution to the initial-value problem is $y = e^{x/2} - 2xe^{x/2}$.

21. $r^2 + 16 = 0 \Rightarrow r = \pm 4i$ and the general solution is $y = e^{0x}(c_1 \cos 4x + c_2 \sin 4x) = c_1 \cos 4x + c_2 \sin 4x$.

Then $y\left(\frac{\pi}{4}\right) = -3 \Rightarrow -c_1 = -3 \Rightarrow c_1 = 3$ and $y'(\frac{\pi}{4}) = 4 \Rightarrow -4c_2 = 4 \Rightarrow c_2 = -1$, so the solution to the initial-value problem is $y = 3 \cos 4x - \sin 4x$.

23. $r^2 + 2r + 2 = 0 \Rightarrow r = -1 \pm i$ and the general solution is $y = e^{-x}(c_1 \cos x + c_2 \sin x)$. Then $2 = y(0) = c_1$ and $1 = y'(0) = c_2 - c_1 \Rightarrow c_2 = 3$ and the solution to the initial-value problem is $y = e^{-x}(2 \cos x + 3 \sin x)$.

25. $4r^2 + 1 = 0 \Rightarrow r = \pm \frac{1}{2}i$ and the general solution is $y = c_1 \cos \left(\frac{1}{2}x\right) + c_2 \sin \left(\frac{1}{2}x\right)$. Then $3 = y(0) = c_1$ and $-4 = y(\pi) = c_2$, so the solution of the boundary-value problem is $y = 3 \cos \left(\frac{1}{2}x\right) - 4 \sin \left(\frac{1}{2}x\right)$.

27. $r^2 - 3r + 2 = (r-2)(r-1) = 0 \Rightarrow r = 1, r = 2$ and the general solution is $y = c_1 e^x + c_2 e^{2x}$. Then $1 = y(0) = c_1 + c_2$ and $0 = y(3) = c_1 e^3 + c_2 e^6$ so $c_2 = 1/(1-e^3)$ and $c_1 = e^3/(e^3 - 1)$. The solution of the boundary-value problem is $y = \frac{e^{x+3}}{e^3-1} + \frac{e^{2x}}{1-e^3}$.
29. \( r^2 - 6r + 25 = 0 \) \( \Rightarrow \) \( r = 3 \pm 4i \) and the general solution is \( y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) \). But \( 1 = y(0) = c_1 \) and \( 2 = y(\pi) = c_1 e^{3\pi} \) \( \Rightarrow \) \( c_1 = 2/e^{3\pi} \), so there is no solution.

31. \( r^2 + 4r + 13 = 0 \) \( \Rightarrow \) \( r = -2 \pm 3i \) and the general solution is \( y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) \). But \( 2 = y(0) = c_1 \) and \( 1 = y\left(\frac{\pi}{2}\right) = e^{-\pi}(-c_2) \), so the solution to the boundary-value problem is \( y = e^{-2x}(2 \cos 3x - e^{\pi} \sin 3x) \).

33. (a) Case 1 (\( \lambda = 0 \)): \( y'' + \lambda y = 0 \) \( \Rightarrow \) \( y'' = 0 \) which has an auxiliary equation \( r^2 = 0 \) \( \Rightarrow \) \( r = 0 \) \( \Rightarrow \) \( y = c_1 + c_2x \) where \( y(0) = 0 \) and \( y(L) = 0 \). Thus, \( 0 = y(0) = c_1 \) and \( 0 = y(L) = c_2 L \) \( \Rightarrow \) \( c_1 = c_2 = 0 \). Thus, \( y = 0 \).

Case 2 (\( \lambda < 0 \)): \( y'' + \lambda y = 0 \) has auxiliary equation \( r^2 = -\lambda \) \( \Rightarrow \) \( r = \pm \sqrt{-\lambda} \) (distinct and real since \( \lambda < 0 \)) \( \Rightarrow \) \( y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \) where \( y(0) = 0 \) and \( y(L) = 0 \). Thus, \( 0 = y(0) = c_1 + c_2 \) (\( \ast \)) and \( 0 = y(L) = c_1 e^{\sqrt{-\lambda}L} + c_2 e^{-\sqrt{-\lambda}L} \) (\( \dagger \)).

Multiplying (\( \ast \)) by \( e^{\sqrt{-\lambda}L} \) and subtracting (\( \dagger \)) gives \( c_2 \left( e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L} \right) = 0 \) \( \Rightarrow \) \( c_2 = 0 \) and thus \( c_1 = 0 \) from (\( \ast \)). Thus, \( y = 0 \) for the cases \( \lambda = 0 \) and \( \lambda < 0 \).

(b) \( y'' + \lambda y = 0 \) has an auxiliary equation \( r^2 + \lambda = 0 \) \( \Rightarrow \) \( r = \pm i \sqrt{\lambda} \) \( \Rightarrow \) \( y = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x \) where \( y(0) = 0 \) and \( y(L) = 0 \). Thus, \( 0 = y(0) = c_1 \) and \( 0 = y(L) = c_2 \sin \sqrt{\lambda} L \) since \( c_1 = 0 \). Since we cannot have a trivial solution, \( c_2 \neq 0 \) and thus \( \sin \sqrt{\lambda} L = 0 \) \( \Rightarrow \) \( \sqrt{\lambda} L = n\pi \) where \( n \) is an integer \( \Rightarrow \lambda = n^2\pi^2/L^2 \) and \( y = c_2 \sin(n\pi x/L) \) where \( n \) is an integer.