15. Topological Divisors of Zero

Recall that in any algebra $A$ an element $x$ is a **zero divisor** if there is a $y \neq 0$ such that $xy = 0$.

**Definition:** An element $x$ in a Banach algebra $A$ is called a **topological divisor of zero** if there exists a sequence $(x_n)$ in $A$ such that

1. $\|x_n\| = 1$ for all $n$
2. $\lim_{n \to \infty} xx_n = 0$

**Example:** Let $A = C[0, 1]$. Then $x(t) = t$ is a topological divisor of zero. To show this let

$$x_n(t) = \begin{cases} 
1 - nt & 0 \leq t \leq \frac{1}{n} \\
0 & \frac{1}{n} < t \leq 1
\end{cases}$$

The sequence $x_n$ satisfies (i) and (ii). See Figure 1.

![Graphs](image.png)

**Figure 1**

**15.1 Lemma:** If $\lambda \in \mathbb{C}$ is in the boundary $\sigma(x)$ then $x - \lambda e$ is a topological divisor of zero.

**Proof:** Let $(\lambda_n)$ be a sequence of complex numbers such that $\lambda_n \notin \sigma(x)$ and $\lambda_n$ converges to $\lambda$. Then the elements $x - \lambda_n e$ are invertible and

$$\| (x - \lambda_n e)^{-1} \| \geq r((x - \lambda_n e)^{-1})$$

$$= \sup \{ f((x - \lambda_n e)^{-1}) \mid f \in M(A) \}$$

$$= \sup \{ \|f(x) - \lambda_n\|^{-1} \mid f \in M(A) \}$$

$$\geq |\lambda - \lambda_n|^{-1} \to \infty$$

Now define
\[ y_n = \frac{(x-\lambda_n e)^{-1}}{\|(x-\lambda_n e)^{-1}\|} \]

Then \( \|y_n\| = 1 \) and
\[ \|(x - \lambda e)y_n\| = \|(x - \lambda_n e + \lambda_n e - \lambda e)y_n\| \]
\[ \leq \|(x - \lambda_n e)y_n\| + |\lambda - \lambda_n||y_n| \]
\[ = \frac{1}{\|(x-\lambda_n e)^{-1}\|} + |\lambda - \lambda_n| \to 0 \]

15.2 Theorem: Let \( A \) be a complex commutative Banach algebra with identity. Then \( A \) has no nonzero topological divisors of zero if and only if \( A \) is isomorphic to \( \mathbb{C} \).

Proof: If \( A \) is not isomorphic to \( \mathbb{C} \) then there exists \( x_0 \in A \) such that \( x_0 \neq \lambda e \) for any \( \lambda \in \mathbb{C} \). Suppose \( \lambda_0 \) is in the boundary of \( \sigma(x_0) \) then \( x_0 - \lambda_0 e \) is nonzero (by the preceding sentence) and is a topological divisor of zero by Lemma 15.1.

15.3 Theorem:
(a) If \( x \) is a topological divisor of zero then \( x \) is not invertible.
(b) If \( x \in \text{Rad}(A) \) then \( x \) is a topological divisor of zero

Proof: (a) Exercise.

(b) Let \( x \in \text{Rad}(A) \).
Step 1. We show that \( x - \lambda e \) is invertible for any complex number \( \lambda \neq 0 \).
For each \( f \in M(A) \) we have \( f(x - \lambda e) = f(x) - \lambda f(e) = -\lambda \neq 0 \) so \( x - \lambda e \) is invertible by Theorem 7.4.]
Step 2. If \( y = (\lambda e - x)^{-1} \) then \( \|\lambda y\| \geq 1 \).
First note that \( xy \) is not invertible because \( x \) is not and
\[ xy = \lambda y - \lambda y + xy \]
\[ = \lambda y - (\lambda e - x)y \]
\[ = \lambda y - e \]
Now if \( \|\lambda y\| < 1 \) then \( \lambda y - e \) is invertible by Theorem 13.1. Thus we must have \( \|\lambda y\| \geq 1 \).

Step 3. We show \( x \) is a topological divisor of zero.
Let \( \lambda_k \) be a sequence of nonzero complex numbers such that \( |\lambda_k| \to 0 \).
By Step 2, \( \lambda_k e - x \) is invertible and \( y_k = (\lambda_k e - x)^{-1} \) satisfies \( \|\lambda_k y_k\| \geq 1 \).
It follows from this last inequality and the fact that \( |\lambda_k| \to 0 \) that \( \|y_k\| \to \infty \). Now,
By the calculation in Step 2
\[
\frac{\|xy_k\|}{\|y_k\|} = \frac{\|\lambda_k y_k - e\|}{\|y_k\|} \\
\leq \frac{|\lambda_k| \|y_k\| + \|e\|}{\|y_k\|} \\
= |\lambda_k| + \frac{1}{\|y_k\|} \to 0
\]
In other words \(\lim_{n \to \infty} x \frac{y_k}{\|y_k\|} = 0\) so \(x\) is a topological divisor of zero. \(\square\)

Recall that the group \(G\) of invertible elements in a Banach algebra \(A\) is open.

15.4 Theorem: Let \(A\) be a Banach algebra and \(G\) its group of invertible elements. Then every boundary point of \(G\) is a topological divisor of zero.

Proof: Suppose \(x\) belongs to the boundary of \(G\). Since \(G\) is open it follows that \(x \notin G\) and there exists a sequence \((x_n)\) in \(G\) such that \(x_n \to x\). Since \(x \notin G\), then \(xx_n^{-1} \notin G\) and so
\[
1 \leq \|xx_n^{-1} - e\| = \|xx_n^{-1} - x_n x_n^{-1}\| \\
= \|x_n^{-1}(x - x_n)\| \\
\leq \|x_n^{-1}\| \|x - x_n\|
\]
Since \(\|x - x_n\| \to 0\) it follows that \(\|x_n^{-1}\| \to \infty\). Let \(z_n = \frac{x_n^{-1}}{\|x_n^{-1}\|}\) then
\[
xz_n = xz_n - x_n z_n + x_n z_n \\
= (x - x_n)z_n + x_n \frac{x_n^{-1}}{\|x_n^{-1}\|} \\
= (x - x_n)z_n + \frac{e}{\|x_n^{-1}\|} \to 0
\]
So \(x\) is a topological divisor of zero. \(\square\)

Definition: Let \(A\) be a Banach algebra with identity.
1. A Banach algebra \(B\) is called a superalgebra of \(A\) if \(B\) contains a subalgebra isomorphic to \(A\).
2. An element \(x \in A\) is called permanently singular if it is not invertible in any superalgebra of \(A\).
In the proof of the next theorem please observe that there are three different norms all denoted by \( \| \cdot \| \): the original norm of \( A \), the norm in the power series algebra \( \widetilde{A} \), and the norm in the quotient algebra \( \widetilde{A} / I \).

15.5 Theorem: (Arens) An element \( x \in A \) is permanently singular if and only if it is a topological divisor of zero.

Proof: ( \( \iff \) ) A topological divisor of zero in \( A \) is also a topological divisor of zero in any superalgebra of \( A \), so is not invertible in any superalgebra (by Exercise 1).
( \( \Rightarrow \) ) Assume \( w \neq 0 \) is not a topological divisor of zero.

Step 1. We exhibit a superalgebra \( \widetilde{A} \) of \( A \).

Let \( \widetilde{A} \) be the algebra whose elements are the formal power series
\[
\widehat{x}(t) = \sum_{n=0}^{\infty} x_n t^n = x_0 + x_1 t + x_2 t^2 + \cdots
\]
where \( x_n \in A \) with
\[
\| \widehat{x}(t) \| = \sum_{n=0}^{\infty} \| x_n \| < \infty
\]
It's easy to check that \( \widetilde{A} \) is a Banach algebra under Cauchy multiplication of power series. Also \( \widetilde{A} \) is a superalgebra of \( A \) because the map \( x \to \widehat{x} \), where \( \widehat{x} = x + 0 \cdot t + 0 \cdot t^2 + \cdots \) (the constant series) is an isomorphism onto its image.

Step 2. We find an ideal \( I \) such that \( \widetilde{A} / I \) is a superalgebra of \( A \).

We now consider the ideal \( I = (e - wt)\widetilde{A} \) and the quotient algebra \( \widetilde{A} / I \). Let \([\widehat{x}(t)]\) denote the coset of \( x(t) \) in \( \widetilde{A} / I \). We show that the map
\[
A \to \widetilde{A} / I
\]
\[
x \mapsto [\widehat{x}]
\]
is an isometry; that is for every \( x \in A \), \( \| x \| = \|[\widehat{x}]\| \). Note that again \( \widehat{x} \) denotes the constant series in \( \widetilde{A} \) associated with \( x \).

\[
\|[\widehat{x}]\| = \inf_{\tilde{y}(t) \in \widetilde{A}} \| \widehat{x} + (e - wt) \tilde{y}(t) \| \leq \| x \| \quad \text{norm in } \widetilde{A} / I
\]

For the other inequality we first, observe that since \( w \) is not a topological divisor of zero we have \( \delta = \inf \{ \| wy \| \mid \| y \| = 1 \} > 0 \). So without loss of generality assume \( \delta \geq 1 \).

Then for all \( y \in A \) we have \( \| wy \| \geq \| y \| \). Now,
\[ \| \tilde{x} - (e - wt) \tilde{y}(t) \| = \left\| \tilde{x} + (e - wt) \sum_{n=0}^{\infty} y_n t^n \right\| \\
= \left\| x - y_0 + \sum_{n=1}^{\infty} (w y_{n-1} - y_n) t^n \right\| \\
= \left\| x - y_0 \right\| + \sum_{n=1}^{\infty} \| w y_{n-1} - y_n \| \quad \text{norm in } \tilde{A} \\
\geq \left\| x - y_0 \right\| + \sum_{n=1}^{\infty} (\| w y_{n-1} \| - \| y_n \|) \quad \text{norm in } A \\
\geq \left\| x - y_0 \right\| + \sum_{n=1}^{\infty} (\| y_{n-1} \| - \| y_n \|) \quad \text{Inequality above} \\
= \left\| x - y_0 \right\| + \| y_0 \| \quad \text{Telescoping} \\
\geq \| x \| \\
\]

**Step 3.** \([\tilde{w}]\) is invertible in \(\tilde{A} / I\).

Since \(e - wt \in I\) it follows that \([e - wt] = [0]\) in \(\tilde{A} / I\). We have

\[ [e - wt] = [e] - [w][et] = [0] \]

So \([w][et] = [e]\)

Thus \([w]\) is invertible in \(\tilde{A} / I\). \(\square\)

**Example:** Let \(A = C[0,1]\) and let \(B = B[0,1]\) be the Banach algebra of all bounded functions on \([0,1]\) with the sup norm. Clearly \(B\) is a superalgebra of \(A\). Moreover, every topological divisor of zero in \(A\) is actually a zero divisor in \(B\). For instance, the function \(x(t) = t\) is a topological divisor of zero in \(A\) (see the preceding example). The function \(y \in B\) defined by

\[
y(t) = \begin{cases} 
1 & t = 0 \\
0 & 0 < t \leq 1 
\end{cases}
\]

is nonzero but clearly \(xy \equiv 0\). This shows that \(x\) is a zero divisor in \(B\).

**15.6 Theorem:** Let \(A\) be a Banach algebra. There exists a superalgebra \(B\) of \(A\) in which every topological divisor of zero in \(A\) is a zero divisor in \(B\).

**Proof:** Let \(B\) be the algebra consisting of all sequences \(x = (x_n)\) in \(A\) satisfying
\[ ||x|| = \limsup \|x_n\| < \infty \]

with pointwise operations and let \( I \) be the ideal \( I = \{ x \in B \mid ||x|| = 0 \} \). It can be shown that \( B/I \) is a Banach algebra in the norm \( ||\cdot|| \). The subalgebra of \( A \) consisting of constant sequences is isomorphic to \( A \) by the map

\[ x \mapsto [(x, x, x, \ldots)] \]

Now if \( x \) is a topological divisor of zero in \( A \) then there exists a sequence \( (y_k) \) such that \( \|y_k\| = 1 \) and \( xy_k \to 0 \). Let's consider the following elements in \( B/I \).

\[ [(x, x, x, \ldots)] \]
\[ [(y_1, y_2, y_3, \ldots)] \]
\[ [(xy_1, xy_2, xy_3, \ldots)] \]

The first two elements are not zero, but the third is zero in \( B/I \) (because the limit is zero). But the product of the first two elements is the third. Thus \( x \) (or rather its image in \( B/I \)) is a zero divisor in \( B/I \).

\[ \square \]

**Exercises**

1. If \( x \) is a topological divisor of zero then \( x \) is not invertible.
2. In \( C(X) \), where \( X \) is a compact Hausdorff space, every element is either invertible or is a topological divisor of zero. [Hint: Use Uryshon's Lemma]
3. The set of topological divisors of zero in \( A \) is closed.
4. If \( A \) and \( B \) are Banach algebras and \( B \subset A \) and \( e \in B \), then
   
   - (a) \( \sigma_A(x) \subset \sigma_B(x) \)
   - (b) \( \partial \sigma_B(x) \subset \partial \sigma_A(x) \)

where \( \sigma_A(x) \) and \( \sigma_B(x) \) denote the spectrum of \( x \) in the algebras \( A \) and \( B \), respectively.
16. The Shilov Boundary

Consider the Banach algebra $A(D)$ of functions which are analytic on the disc and continuous on its boundary. Every function $x \in A(D)$ attains maximum on the boundary of the disc. In the Gelfand representation of $A(D)$ the maximal ideal space is homeomorphic with $D$ so each $\hat{x}$ on $M(A)$ attains its maximum on the points of $M(A)$ that correspond to the boundary of $D$. This leads to the following question: Does every Banach algebra have a boundary? That is, is there a subset of the maximal ideal space on which every Gelfand transform attains its maximum? These questions are the reason behind the following definition.

**Definition:** A maximizing set for a Banach algebra $A$ is any closed subset $\mathcal{F} \subset M(A)$ such that

$$\sup_{M \in M(A)} |\hat{x}(M)| = \sup_{M \in \mathcal{F}} |\hat{x}(M)|$$

for every $x \in A$. A Shilov boundary for $A$, denoted by $\Gamma(A)$ is any minimal maximizing set.

**16.1 Lemma:** $M_0 \in \Gamma(A)$ if and only if for every neighborhood $U$ of $M_0$ and $0 < \epsilon < 1$ there exists $y \in A$ such that

$$\sup_{M \in U} |\hat{y}(M)| = 1 \quad \text{and} \quad \sup_{M \in M(A) \setminus U} |\hat{y}(M)| < \epsilon$$

Proof: Let $M_0 \in \Gamma$ and let $U$ be a neighborhood of $M_0$. Since $\Gamma$ is minimal there exists $y \in A$ such that $\hat{y}$ attains its maximum modulus $r(y)$ on $U \cap \Gamma$ and

$$\max_{M \in \Gamma \setminus U} |\hat{y}(M)| < r(y)$$

(otherwise $\Gamma \setminus U$ would be a maximizing set smaller that $\Gamma$ contradicting the minimality of $\Gamma$). Thus

$$r(y) = \sup_{M \in M(A)} |\hat{y}(M)| = \sup_{M \in U} |\hat{y}(M)|$$

Without loss of generality we may assume this maximum value is 1 (replace $y$ by $\alpha y$ for an appropriate $\alpha$). Now, since $\max_{M \in \Gamma \setminus U} |\hat{y}(M)|$ is strictly less than 1 we can replacing $y$ by a large enough power $y^n$ so that $\max_{M \in \Gamma \setminus U} |\hat{y}(M)| < \epsilon$.

The converse is trivial. \qed

**16.2 Theorem:** Every Banach algebra has a unique Shilov boundary.
Proof: \textbf{Step 1.} (Existence of a minimal maximizing set)
Let $\mathfrak{H}$ be the family of maximizing sets for $A$. $\mathfrak{H}$ is nonempty because $M(A) \in \mathfrak{H}$ and is a partially ordered set under inclusion. We apply Zorn's Lemma to find a minimum element in $\mathfrak{H}$, which would be the Shilov boundary.

If $\{H_{\alpha} \mid \alpha \in \Lambda\}$ is a chain in $\mathfrak{H}$ then $H = \bigcap H_{\alpha}$ is nonempty (because $\mathfrak{H}$ is compact). We claim $H \in \mathfrak{H}$ (in other words, we claim $H$ is a maximizing set). To see this, note that for any $x \in A$ the set $S_x = \{M \in M(A) \mid \widehat{x}(M) = r(x)\}$ is compact and

$$H_{\alpha} \cap S_x \neq \emptyset \quad \text{for all } \alpha$$

So $H \cap S_x \neq \emptyset$. Since this holds for all $x \in A$, it follows that each $\widehat{x}$ attains its maximum value on $H$, so $H$ is a maximizing set. [We have used the fact that a space is compact iff every collection of closed sets with the finite-intersection property has nonempty intersection.]

We've shown that $\mathfrak{H}$ is a partially ordered set in which every chain has a lower bound, so Zorn's Lemma implies that $\mathfrak{H}$ has a minimal element, that is, a minimal maximizing set.

\textbf{Step 2.} (Uniqueness of the minimal maximizing set)
Suppose there are two Shilov boundaries $\Gamma_1$ and $\Gamma_2$. Let $M_0 \in \Gamma_1$ and let $U$ be a basic neighborhood of $M_0$

$$U = \{M \in M(A) \mid |\widehat{x}_1(M)| < \epsilon, \ldots, |\widehat{x}_n(M)| < \epsilon\}$$

By Lemma 16.1 there exists $y \in A$ such that $\widehat{y}$ attains its maximum modulus $r(y)$ on $U \cap \Gamma_1$ and

$$\max_{M \in U} |\widehat{y}(M)| = 1 \quad \text{and} \quad \max_{M \in M(A) \setminus U} |\widehat{y}(M)| < \frac{\epsilon}{\max_{1 \leq k \leq n} r(x_k)}$$

With this choice of $y$ we have

$$\max_{M \in \Gamma_1} |\widehat{y}(M)\widehat{x}_k(M)| = \max_{M \in M(A)} |\widehat{y}(M)\widehat{x}_k(M)| < \epsilon$$

Now there exists $M_1 \in \Gamma_2$ such that $\widehat{y}(M_1) = 1$. Since $|\widehat{y}(M_1)\widehat{x}_k(M_1)| < \epsilon$ for every $M \in M(A)$ it follows that

$$|\widehat{x}_k(M_1)| < \epsilon \quad \text{for } k = 1, 2, \ldots n$$

Thus $M_1 \in U$, so $U$ meets $\Gamma_2$. Since $U$ was any basic neighborhood it follows that every basic neighborhood of $M_0$ meets $\Gamma_2$. Since $\Gamma_2$ is closed, it follows that $M_0 \in \Gamma_2$.

Thus $\Gamma_1 \subset \Gamma_2$. But since $\Gamma_1$ is minimal, it follows that $\Gamma_1 = \Gamma_2$. \hfill \qed

\textbf{16.3 Theorem:} For every $x \in A$ we have

$$\partial[\sigma(x)] \subset \widehat{x}(\Gamma)$$
Proof: Since $\Gamma$ is compact and $\hat{x}$ is continuous it follows that $\hat{x}(\Gamma)$ is a compact subset of $\mathbb{C}$. We show that if $\lambda_0 \notin \hat{x}(\Gamma)$ then $\lambda_0 \notin \partial[\sigma(x)]$.

If $\lambda_0 \notin \hat{x}(\Gamma)$ there exists $\delta > 0$ such that

$$\min_{M \in \Gamma} |\hat{x}(M) - \lambda_0| > \delta$$

Suppose to the contrary that $\lambda_0 \in \partial[\sigma(x)]$ then there exists a $\lambda_1 \notin \sigma(x)$ such that $|\lambda_0 - \lambda_1| < \frac{1}{2}\delta$. Then for any $M \in \Gamma$

$$|\hat{x}(M) - \lambda_1| \geq |\hat{x}(M) - \lambda_0| - |\lambda_0 - \lambda_1| > \delta - \frac{1}{2}\delta = \frac{\delta}{2}$$

so that for $u = (x - \lambda_1e)^{-1}$ we have

$$r(u) = \max_{M \in \Gamma} \frac{1}{|\hat{x}(M) - \lambda_1|} = \left( \min_{M \in \Gamma} |\hat{x}(M) - \lambda_1| \right)^{-1} \leq \frac{2}{\delta}$$

However, since $\hat{x}(M_0) = \lambda_0$ for some $M_0$, then

$$r(u) \geq |\hat{u}(M)| = |\lambda_0 - \lambda_1|^{-1} > \frac{2}{\delta}$$

This contradicts the preceding displayed inequality. $\square$

The next theorem concerns the possibility of extending a multiplicative linear functional from a subalgebra $B$ of $A$ to a multiplicative linear functional on all of $A$.

16.4 Theorem: Let $B$ be a subalgebra of a Banach algebra $A$. Then every ideal $N \in \Gamma(B)$ is contained in a maximal ideal $M \in M(A)$.

Proof: Let $x \in B \subset A$.

Step 1. Notice that the spectral radius of $x$ does not depend on which algebra $x$ is in: $A$ or $B$. We can see this from the spectral radius formula $r(x) = \lim_{n \to \infty} \|x^n\|^{1/n}$, which depends only on the norm of $x$ itself. Thus

$$r(x) = \max_{M \in \Gamma(B)} |\hat{x}(M)| = \max_{M \in M(A)} |\hat{x}(M)|$$

Step 2. Let $N \in \Gamma(B)$ and suppose that $N$ is not contained in a maximal ideal of $A$. We show that this assumption leads to a contradiction.

If $N$ is not contained in a maximal ideal of $A$ then the ideal generated by $N$ in $A$ is all of $A$. That is, each $x \in A$ can be expressed as $x = x_1z_1 + x_2z_2 + \cdots + x_mz_m$ for some $x_k \in N$ and $z_k \in A$. In particular,

$$e = x_1z_1 + x_2z_2 + \cdots + x_nz_n$$
for some \( x_k \in N \) and \( z_k \in A, k = 1, 2, \ldots n \). Without loss of generality we may assume that \( r(x_k) \leq 1 \) for \( k = 1, 2, \ldots n \) (multiply \( x_k \) by an appropriate \( \alpha \) and \( z_k \) by \( 1/\alpha \)). Choose \( \delta \) so that

\[
\delta > \max \{r(z_1), r(z_2), \ldots, r(z_n)\}
\]

and set \( \epsilon = 1/2n\delta \). Let \( U \) be the following neighborhood of \( N \) in \( M(B) \):

\[
U = \{M \in M(B) | |\widehat{x}_1(M)| < \epsilon, \ldots, |\widehat{x}_n(M)| < \epsilon\}
\]

By Lemma 16.1 we can find \( y \in A \) such that

\[
\sup_{M \in U} |\widehat{y}(M)| = 1 \quad \text{and} \quad \sup_{M \in M(B) \setminus U} |\widehat{y}(M)| < \epsilon
\]

This together with the definition of \( U \) shows that

\[
\sup_{M \in U} |\widehat{x}_k(M)\widehat{y}(M)| \leq \sup_{M \in U} |\widehat{x}_k(M)| \sup_{M \in U} |\widehat{y}(M)| \leq \epsilon
\]

and

\[
\max_{M \in \Gamma(B) \setminus U} |\widehat{x}_k(M)\widehat{y}(M)| \leq \max_{M \in \Gamma(B) \setminus U} |\widehat{x}_k(M)| \max_{M \in \Gamma(B) \setminus U} |\widehat{y}(M)| \leq \epsilon
\]

Now we have

\[
1 = r(y) = r\left[ y(x_1z_1 + x_2z_2 + \cdots + x_nz_n) \right]
\]

\[
\leq r(yx_1) r(z_1) + \cdots + r(yx_n) r(z_n)
\]

\[
\leq \delta \left[r(yx_1) + \cdots + r(yx_n)\right]
\]

\[
= \delta \max_{M \in M(A)} (|\widehat{x}_1(M)\widehat{y}(M)| + \cdots + \max_{M \in M(A)} |\widehat{x}_n(M)\widehat{y}(M)|)
\]

\[
\leq \delta n \epsilon = \delta n\left(\frac{1}{2n\delta}\right) = \frac{1}{2}
\]

This contradiction shows that \( N \) can be extended to a maximal ideal in \( A \). \( \square \)

**Definition:** The **cortex** for a Banach algebra \( A \), denoted by \( \text{cor}(A) \), is the set of \( f \in M(A) \) which extend to multiplicative linear functionals on any superalgebra of \( A \).

**16.5 Theorem:** For any Banach algebra \( A \) we have

1. \( \Gamma(A) \subset \text{cor}(A) \) and
2. \( \text{cor}(A) \) is a compact subset of \( M(A) \).
Proof: (1) Follows immediately from Theorem 16.4
(2) Let $B$ be any superalgebra of $A$. The restriction map

$$\varphi_B : M(B) \rightarrow M(A)
\quad f \mapsto f|_A$$

is clearly continuous. So $\varphi_B(M(B))$ is a compact subset of $M(A)$. Since

$$\text{cor}(A) = \cap \{\varphi_B(M(B)) | B \text{ is a superalgebra of } A\}$$

it follows that $\text{cor}(A)$ is compact.

**Definition** A subset $E$ of a Banach algebra $A$ consists of **joint topological divisors of zero** if for every finite subset $\{x_1, x_2, \ldots, x_n\}$ of $E$ we have

$$\inf_{\|z\|=1} \sum_{k=1}^n \|zx_k\| = 0$$

It has been shown (Zelazko) that if $M \in \Gamma(A)$ then $M$ consists of joint topological divisors of zero. Also, if $M$ consists of joint topological divisors of zero then $M \in \text{cor}(A)$. Zelazko has made the following conjecture.

**Conjecture:** $M \in \text{cor}(A)$ if and only if $M$ consists of topological divisors of zero.

**Exercises**

1. Find the Shilov boundary for each of the following Banach algebras.
   (a) $C(X)$
   (b) $A(D)$

2. Let $A = < x_0 >$ be a singly-generated Banach algebra. Then
   (a) $M(A)$ is homeomorphic to $\sigma(x_0)$.
   (b) $\Gamma(A) = \partial \sigma(x_0)$

3. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.
   (a) $A(D) \subset C(\mathbb{T})$.
   (b) The only multiplicative linear functionals which extend from $A(D)$ to $C(\mathbb{T})$ are the ones in the Shilov boundary $\Gamma(A(D))$.

4. $M$ consists of joint topological divisors of zero then $M \in \text{cor}(A)$.

5. Consider $H(\Omega)$ where $\Omega$ is the polydisk $\Omega = \{(z_1, z_2) : |z_1| < 1, |z_2| < 1\} \subset \mathbb{C}^2$. The Shilov boundary of $H(\Omega)$ is the distinguished boundary of $\Omega$ (not the whole topological boundary).
17. Derivations on Banach Algebras

Recall that the derivative (as in calculus) can be thought of as a linear operator. Of course the derivative also satisfies the product rule. In a Banach algebra we can define a linear operator that satisfies this additional condition.

**Definition:** A derivation on an algebra $A$ is a linear mapping $D$ of $A$ into $A$ such that

$$D(ab) = aDb + (Da)b \quad (a, b \in A).$$

This relation is called the **product rule**.

We will be interested in the following questions:

1. When is every derivation continuous?
2. When is every derivation identically zero?

First, some examples of derivations.

**Example**

1. Let $c \in A$. Define $D_c$ to be the mapping of $A$ into $A$ such that:

$$D_c(a) = ac - ca \quad (a \in A).$$

This is a derivation since:

$$D_c(ab) = (ab)c - c(ab)$$

$$= (abc - acb) + (acb - cab)$$

$$= a(bc - cb) + (ac - ca)b$$

$$= aD_c b + (D_c a)b$$

Each derivation of this form is called an **inner derivation**. And its clear that $A$ is commutative if and only if $0$ is the only derivation on $A$.

2. Let $A = C^\infty[0, 1]$ be the algebra of infinitely differentiable functions on $[0, 1]$ with pointwise operations. Define $D : C^\infty[0, 1] \rightarrow C^\infty[0, 1]$ by $Df = f'$ for $f \in C^\infty[0, 1]$. The product rule from calculus tells us that $D$ is a derivation on $A$. It will be later shown there is no norm which makes $C^\infty$ a Banach algebra. In this example the identity is the function $e(t) \equiv 1$.  

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Note: If $A$ is any commutative Banach algebra with identity $e$ and $D$ is a derivation on $A$ then $D(e) = 0$. 

Proof: We have 

$$D(e) = D(ee) = eD(e) + eD(e) = D(e) + D(e) = 2D(e)$$

Thus $D(e) = 2D(e)$, so $D(e) = 0$.

Definition: An element $j$ in an algebra $A$ is called idempotent if $j^2 = j$.

17.1 Proposition: Let $D$ be a derivation on a commutative algebra $A$, and let $j$ be an idempotent in $A$. Then $D(j) = 0$.

Proof - We have 

$$D(j) = D(jj) = jD(j) + D(j)j = 2jD(j)$$

Multiplying the above equation by $j$ we get 

$$jD(j) = 2j^2D(j) = 2jD(j)$$

So $jD(j) = 0$. From the first equation $D(j) = 2jD(j) = 2 \cdot 0 = 0$. \qed

17.2 Proposition: Let $D$ be a derivation on an algebra $A$. Then the following statements hold (with the convention that $D^0 = I$):

1) (Leibnitz rule) $D^n(ab) = \sum_{r=0}^{n} \binom{n}{r} (D^{n-r}a)(D^rb) \quad (n \in \mathbb{N}; a, b \in A)$

2) $D(a^n) = na^{n-1}Da \quad (n - 1 \in \mathbb{N})$ if and only if $aDa = (Da)a$

Proof - (1) This can be proved by induction.

If $n = 1$, then $D^1(ab) = aDb + (Da)b = \sum_{r=0}^{1} \binom{1}{r} (D^{1-r}a)(D^rb)$

Next, assume the result holds for $n \leq k$.

Then, $D^{k+1}(ab) = D(D^k(ab)) = D \left( \sum_{r=0}^{k} \binom{k}{r} (D^{k-r}a)(D^rb) \right)$

$$= \left( \sum_{r=0}^{k} \binom{k}{r} D((D^{k-r}a)(D^rb)) \right)$$

$$= \left( \sum_{r=0}^{k} \binom{k}{r} (D^{k-r}a D^{r+1}b + D^{k-r-1}a D^rb) \right)$$

$$= \sum_{r=0}^{k} \binom{k}{r} (D^{k-r}a D^{r+1}b) + \sum_{r=0}^{k} \binom{k}{r} (D^{k-r-1}a D^rb)$$
\[
= \sum_{r=1}^{k+1} \binom{k+1}{r-1} (D^{k-r+1} a D^r b) + \sum_{r=0}^{k} \binom{k}{r} (D^{k-r+1} a D^r b)
\]

\[
= \sum_{r=1}^{k+1} \binom{k+1}{r-1} (D^{k-r+1} a D^r b) + a D^{k+1} b + (D^{k+1} a) b
\]

\[
= \sum_{r=1}^{k} \binom{k+1}{r} (D^{k-r+1} a D^r b) + D^{k+1} (ab)
\]

\[
= \sum_{r=1}^{k+1} \binom{k+1}{r} (D^{k-r+1} a D^r b)
\]

(2) \((\Rightarrow)\) 2aDa = D(a^2) = D(aa) = aDa + (Da)a

\[
aDa = (Da)a
\]

\((\Leftarrow)\) If \(n = 2\), then \(D(a^2) = aDa + (Da)a = 2aDa\)

Next, assume the result holds for \(n \leq k\).

Then, \(D(a^{k+1}) = D(aa^k) = aDa^k + (Da)a^k\)

\[
= a(ka^{k-1} Da) + (Da)a^k
\]

\[
= ka^k Da + (Da)a^k
\]

\[
= ka^k Da + a^k Da
\]

\[
= (k + 1)a^k Da
\]

\[
\square
\]

17.3 Proposition: Let \(a \in A\) and \(p(x) = \sum_{k=0}^{n} \alpha_k x^k\) be a polynomial with complex coefficients. If \(D : A \rightarrow A\) is a derivation then

\[
D(p(a)) = p'(a)D(a)
\]

where \(p'\) is the usual derivative of the polynomial \(p\).

Proof - Since \(D\) is linear we have

\[
D(p(a)) = D(\sum_{k=0}^{n} \alpha_k a^k)
\]
This can be proved by induction. □

Recall that the differential operator on $C^\infty[0, 1]$ is not bounded (or continuous). It is natural to ask for conditions on a derivation that would guarantee its continuity.

**Example: Derivations on $C[a, b]$**
We show that every derivation (continuous or not) on $C[a, b]$ is zero.

Proof: Fix $s \in [a, b]$. We'll show that for any $u \in C[a, b]$, $D(u(s)) = 0$.

Step 1. We show that if $f \in C[a, b]$ and $f(s) = 0$, then $D(f)(s) = 0$.
First, if $f \geq 0$ then there is a $g$ such that $f = g^2$. It follows that

$$Df = Dg^2 = gDg + gDg = 2gDg$$

so

$$(Df)(s) = 2g(s)(Dg)(s) = 2 \cdot 0 \cdot (Dg)(s) = 0$$

In general we can express $f = g^2 - h^2$ where $g^2 = f^+$ and $h^2 = f^-$. (Recall that

$f^+ = f \vee 0$ and $f^- = f \wedge 0$). Again we have

$$(Df)(s) = 2g(s)(Dg)(s) - 2g(s)(Dg)(s) = 0$$

Step 2. Next we show that for any $u \in C[a, b]$, we have $(Du)(s) = 0$

Let $f = u - u(s)e$ and note that $f(s) = 0$. Now $u = f + u(s)e$ and

$$Du = Df + u(s)D(e) = Df + 0 = Df$$

Since $f(s) = 0$ it follows from Step 1 that $(Du)(s) = 0$.

Step 3. Since $s$ was arbitrary it follows that $D(u) \equiv 0$ and since $u$ was arbitrary if follows that $D$ is the zero operator.

The next result is the catalyst for the entire subject of derivations.

**17.4 Theorem: (Wermer-Singer)** Let $D$ be a continuous derivation on $A$. Then $Da \in \text{Rad}(A) \ (a \in A)$. In particular, if $A$ is semisimple then the only derivation on $A$ is the zero derivation.

Proof - Let $D$ be a continuous derivation on $A$. 

Define $e^{\lambda D} = \sum_{n=0}^{\infty} \frac{(\lambda D)^n}{n!}$ which is bounded, since $D$ is bounded.

Note that for $a, b \in A$,

$$e^{\lambda D}(ab) = \sum_{n=0}^{\infty} \frac{(\lambda D)^n}{n!} (ab) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} ((\lambda D)^{n-r} a)((\lambda D)^r b)$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \left( \frac{1}{(n-r)!} (\lambda D)^{n-r} a \right) \left( \frac{1}{r!} (\lambda D)^r b \right)$$

$$= \left( \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda D)^n a \right) \left( \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda D)^n b \right)$$

$$= (e^{\lambda D} a) (e^{\lambda D} b).$$

Let $f \in M(A)$. Define a mapping $g_\lambda : A \to \mathbb{C}$ by

$$a \mapsto f(e^{\lambda D} a).$$

The map $g_\lambda$ is clearly linear and is also multiplicative because

$$g_\lambda(ab) = f(e^{\lambda D}(ab)) = f((e^{\lambda D} a)(e^{\lambda D} b))$$

$$= f(e^{\lambda D} a) f(e^{\lambda D} b)$$

$$= g_\lambda(a) g_\lambda(b)$$

Since $g_\lambda$ is a multiplicative linear functional on $A$, it is bounded with norm $1$.

Thus, given $a \in A$,

$$|f(e^{\lambda D} a)| \leq \|a\| \quad (\lambda \in \mathbb{C})$$

Since this holds for any $\lambda \in \mathbb{C}$, the mapping $\lambda \mapsto f(e^{\lambda D} a)$ is a bounded entire function. By the ordinary Liouville theorem for complex-valued analytic functions, the function is thus constant.

Looking at the power series expansion of $f(e^{\lambda D} a)$:

$$f \left( \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda D)^n a \right) = f(a) + \lambda f(Da) + \frac{1}{2!} \lambda^2 f(Da)^2 + \frac{1}{3!} \lambda^3 f(Da)^3 \cdots,$$

we see that $f(Da) = 0.$
Since $f$ is an arbitrary element of $M(A)$, then $Da \in \text{Rad}(A)$.

\[\square\]

**17.5 Corollary:** There exists no norm under which the algebra $C^\infty[a, b]$ is a Banach algebra.

Proof - Let $Df = f'$ (obviously a derivation). We will show that $C^\infty[a, b]$ is semisimple, and that $D$ is continuous in any Banach algebra norm on $C^\infty[a, b]$. It follows from Theorem 17.4 that $D$ must be zero, contradicting the obvious fact that $D$ is not zero.

**Step 1.** $C^\infty[a, b]$ is semisimple. This follows from the fact that the evaluation functionals $f_{t_0}(x) = x(t_0)$ are multiplicative. So if $f_t(x) = 0$ for all $f_t \in M(A)$, or equivalently for all $t \in [a, b]$, then clearly $x \equiv 0$.

**Step 2.** $D$ is continuous.

First we show that the functionals defined by $f_{t_0}(x) = x'(t_0)$, for $t_0 \in [a, b]$, are bounded. To show this note that the functionals $f_{t_0+1/n}$ and $f_{t_0}$ are multiplicative and hence bounded so the map

$$L_n(x) = \frac{f_{t_0+1/n}(x) - f_{t_0}(x)}{1/n} = \frac{x(t_0+1/n)-x(t_0)}{1/n}$$

is bounded and linear. Observe that

$$\lim_{n \to \infty} L_n(x) = x'(t_0) = f_{t_0}(x)$$

So by the Uniform Boundedness Theorem $f_{t_0}$ is a bounded linear functional.

Now we show that $D$ has closed graph: Let $x_n$ converge to $x$ and suppose that $D(x_n)$ converges to $y$. For each $t_0 \in [a, b]$ we have

$$f_{t_0}(x_n) \to f_{t_0}(x) \quad \text{for} \ t_0 \in [a, b]$$

or equivalently

$$x_n'(t_0) \to x'(t_0) \quad \text{for} \ t_0 \in [a, b]$$

So

$$D(x_n)(t_0) \to D(x)(t_0) \quad \text{for} \ t_0 \in [a, b]$$

Hence $D(x) = y$. Thus $D$ has closed graph so $D$ is bounded by the Closed Graph Theorem. 

\[\square\]

Note: Although we won't prove it here, it can be shown that any derivation (continuous or not) on a semisimple Banach algebra is zero.