In these notes we give the main theory of Banach algebras, restricting the discussion to the clearest case—commutative Banach algebras with identity. This allows for an uncluttered presentation of the main ideas. Generalizations and refinements can then be studied with the main principles already at hand.

Let's briefly discuss the purpose of this study. It's easy to prove that the reciprocal of a nonvanishing continuous function on the unit interval is a continuous function on the unit interval. This can be proved directly by showing that the reciprocal satisfies the definition of continuity. But here's an immeasureably harder problem. Norbert Wiener's theorem on absolutely convergent trigonometric series: The reciprocal of a nonvanishing absolutely convergent trigonometric series is an absolutely convergent trigonometric series. Precisely, let \( f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int} \) be an absolutely convergent trigonometric series. If \( f(t) \neq 0 \) \( \forall t \in [0, 2\pi) \), then \( \frac{1}{f(t)} \) is an absolutely convergent trigonometric series.

Wiener proved this theorem analytically, by basically inverting the series. His proof is long and difficult. But we will see that we can prove Wiener's theorem as a simple consequence of Banach Algebra theorems (a result first proved by Gelfond). Gelfond shows that the reciprocal of \( f \) is an absolutely convergent trigonometric series by showing that \( f \) belongs to a collection of functions that have this property. Gelfond's proof using Banach Algebras is so simple and elegant that it inspired mathematicians to find other uses for this "functional analysis" approach (that is, looking at algebraic collections of functions instead of working with single functions).
0. Definitions

**Definition:** An algebra is a vector space $A$ with addition ($+$) and scalar multiplication (juxtaposition) with an additional operation called multiplication (also juxtaposition) with the following properties:

\begin{align*}
(1) \quad (xy)z &= x(yz) & \forall x, y, z \in A \\
(2) \quad x(y + z) &= xy + xz & \forall x, y, z \in A \\
(y + z)x &= yx + zx & \forall x, y, z \in A \\
(3) \quad \alpha(xy) &= (\alpha x)y = x(\alpha y) & \forall \alpha \in \mathbb{C}; x, y \in A
\end{align*}

An algebra $A$ is called **commutative** if

$$\alpha \beta = \beta \alpha$$

An algebra $A$ has an **identity** if $\exists e \in A$ such that $ex = xe = x$ for all $x \in A$.

A subalgebra $B$ of an algebra $A$ is a subset of $A$ which is closed under addition and multiplication.

**Definition:** A normed algebra is an algebra $A$ with a norm $\| \cdot \|$ such that $(A, \| \cdot \|)$ is a normed vector space (i.e. $\| \cdot \|: A \to \mathbb{R}^+$) with:

\begin{align*}
(1) \quad \| x \| = 0 & \text{ iff } x = 0 \\
(2) \quad \| \alpha x \| &= \| \alpha \| \| x \| \\
(3) \quad \| x + y \| &\leq \| x \| + \| y \| & \text{ (which means } + \text{ is continuous)}
\end{align*}

with the additional property that

$$\| xy \| \leq \| x \| \| y \|$$

(which means $\cdot$ is continuous)

**1.1 Theorem:** Let $A$ be a normed algebra. Then multiplication is continuous from $A \times A \to A$ (i.e. jointly continuous).

**Proof:** Suppose $x_n \to x$ and $y_n \to y$

We need to show that $x_n y_n \to xy$.

\[\| x_n y_n - xy \| = \| x_n y_n - x_n y + x_n y - xy \| \]
\[\leq \| x_n y_n - x_n y \| + \| x_n y - xy \| \]
\[= \| x_n (y_n - y) \| + \| (x_n - x) y \| \]
\[\leq \| x_n \| \| y_n - y \| + \| x_n - x \| \| y \| \]
\[\leq M \| y_n - y \| + \| x_n - x \| \| y \| \quad \text{ (since } (x_n) \text{ converges)} \]
\[\to 0 \]

**Definition:** A Banach algebra is a complete normed algebra.
**Definition:** A **homomorphism** from an algebra $A$ to an algebra $B$ is a linear map $\phi$ satisfying

\[
\phi(x + y) = \phi(x) + \phi(y) \\
\phi(xy) = \phi(x)\phi(y)
\]

A bijective homomorphism $\phi$ from $A$ onto $B$ is called an **algebra isomorphism**. The normed algebras $A$ and $B$ are **isomorphic (as normed algebras)** if there is an algebra isomorphism $\phi : A \to B$ which preserves the norm

\[
\|\phi(x)\| = \|x\|
\]

In this case $A$ is identical to $B$ as a normed algebra (the isomorphism merely renames the elements of the algebra). If $A$ is isomorphic with a subalgebra of $B$ by a $\phi$ then we say that $A$ is **embedded** in $B$ and $\phi$ is called an embedding.

**Exercises**

1. The center of an algebra $A$ is the subset $C$ consisting of those elements that commute with every element in $A$. The center is a commutative subalgebra of $A$.

2. If $A$ is a Banach algebra with norm $\| \cdot \|$ and identity $e$, then the norm $| \cdot |$ defined by

\[
|x| = \frac{1}{\|e\|}\|x\|
\]

is a vector space norm on $A$ which is equivalent to the original norm. The norm $|x|$ is not necessarily a Banach Algebra norm (i.e, not necessarily submultiplicative.).

3. Show that the algebras $A = C[1, 2]$ and $B = C[3, 4]$ are isomorphic Banach algebras. In each case the algebras are equipped with pointwise operations and the sup norm.
1. Examples and Constructions

In this section we give some fundamental examples of Banach algebras. We also define some basic constructions including products, quotients, and direct sums of Banach algebras. Other are presented as needed.

Example of Banach Algebras

There are three general categories of Banach algebras: "algebras of continuous functions," "algebras of operators," and "group algebras."

Example 1. Algebras of Continuous Functions

Let \( C[0,1] \) = the set of continuous complex-valued functions on \([0, 1]\), where operations are defined pointwise. The norm is

\[
\|x\| = \sup_{t \in [0,1]} |x(t)|
\]

It's easy to show that \( \|xy\| \leq \|x\|\|y\| \) and \( \|x + y\| \leq \|x\| + \|y\| \). So, with this norm \( C[0,1] \) is a commutative Banach algebra with identity \( e(t) \equiv 1 \).

A Note on \( C(X) \)

The Banach algebra \( C[0,1] \) is a special case of the following more general construction. Let \( X \) be a compact topological space and \( C(X) \) be the collection of real (or complex) valued continuous functions on \( X \). \( C(X) \) is a Banach algebra with pointwise operations and the norm

\[
\|x\| = \sup_{t \in X} |x(t)|
\]

If \( X = [0, 1] \) then \( C(X) \) is the algebra in Example 1.

Example 2. Algebras of Analytic Functions

Let \( A(D) \) = the analytic functions on the interior of \( D = \{ z \|z\| \leq 1 \} \) and continuous on the boundary \( T = \{ z \|z\| = 1 \} \). Operations are +, \( \cdot \) and scalar multiplication. The norm is

\[
\|f\| = \sup_{t \in D} |f(t)| = \sup_{t \in T} |f(t)| \quad \text{(by the Maximum Modulus Principle)}
\]

Example 3. Algebras of Operators

Let \( X \) be a Banach space. \( A=B(X) \), i.e. \( A \) is the collection of all bounded linear operators on \( X \). We know \( B(X) \) is a Banach space (since \( X \) is Banach). Multiplication is given by functional composition:
\[ TSx = T(Sx) \quad \forall x \in X \]

Also, \[ \|TS\| \leq \|T\| \|S\| \]

Thus, \( B(X) \) is a Banach algebra with identity \( I \), where \( I(x) = x \) for all \( x \in X \)

**Example 4. Algebras of Integrable Functions**

Let \( L^1(\mathbb{R}) = \) the set of all equivalent classes of integrable functions on \( \mathbb{R} \).

i.e. \( \|f\| = \int_{\mathbb{R}} |f| \, d\mu < \infty \), where \( f \sim g \) iff \( f = g \) except possible on a set of measure zero, and the integral is the Lebesgue integral.

\( L^1(\mathbb{R}) \) is a vector space under addition and scalar multiplication with the norm

\[ \|f\| = \int_{\mathbb{R}} |f| \, d\mu. \]

We have:

a) \( \|f\| = 0 \) iff \( f = 0 \) a.e. (almost everywhere)

b) \( \int_{\mathbb{R}} |\alpha f| \, d\mu = |\alpha| \int_{\mathbb{R}} |f| \, d\mu \) means \( \|\alpha f\| = |\alpha| \|f\| \)

c) \( \int_{\mathbb{R}} |f + g| \, d\mu \leq \int_{\mathbb{R}} |f| \, d\mu + \int_{\mathbb{R}} |g| \, d\mu \) means \( \|f + g\| \leq \|f\| + \|g\| \)

\[ \therefore L^1(\mathbb{R}) \) is a Banach space

Multiplication is given by convolution:

\( (f * g)(x) = \int_{\mathbb{R}} f(x - y)g(y) \, dy \)

Need to show \( (\ast) \) is associative, distributive, compatible with multiplication, and

\[ \|f * g\| \leq \|f\| \|g\|. \]

Proof - \( \|f * g\| = \int_{-\infty}^{\infty} |(f * g)(x)| \, dx \)

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - y)g(y) \, dy \, dx \]

\[ \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x - y)|g(y) \, dy \, dx \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x - y)g(y)| \, dx \, dy \]

\[ = \int_{-\infty}^{\infty} g(y) \int_{-\infty}^{\infty} |f(x - y)| \, dx \, dy \]

\[ = \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} g(y) \, dy \]

\[ = \|f\| \|g\| \]

\[ \therefore \|f * g\| \leq \|f\| \|g\| \]
Now, in the expression \((f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y) \, dy\)
let \(u = x - y \Rightarrow du = -dy\) and \(y = x - u\). Then,

\[
(f * g)(x) = -\int_{-\infty}^{\infty} f(u)g(x-u) \, du \\
= \int_{-\infty}^{\infty} g(x-u)f(y) \, du \\
= (g * f)(x)
\]

So multiplication is commutative.
There is no identity.
Thus \(L^1(\mathbb{R})\) is a commutative Banach algebra without identity.

### A Note on Topological Groups

\(L^1(\mathbb{R})\) is a special case of \(L^1(G)\) where \(G\) is a topological group with Haar measure. A **topological group** is a group \(G\) with a topology \(\tau\) such that addition and inversion are \(\tau\)-continuous. Every topological group has a unique translation invariant measure \(\mu\) called **Haar measure**. (Translation invariant means \(\mu(A + x) = \mu(A), \text{ for } A \subset G\). Examples of topological groups are

1. \((\mathbb{R}, +)\) with usual topology.
2. \((\mathbb{T}, \cdot) = \text{the unit circle, with complex multiplication. } \mathbb{T} = \{e^{it} | t \in [0, 2\pi]\}\)
3. \(\mathbb{Z}/(n)\) with discrete topology.

The algebra \(L^1(G)\) consisting of the measurable functions on \(G\) satisfying

\[
\|f\| = \int_G |f| \, d\mu < \infty
\]

is a Banach algebra under convolution (*)

\[
f * g(s) = \int_G f(s-t)g(t) \, d\mu(t)
\]

### Example 5. The Group Algebra \(l^1(\mathbb{Z})\)

Let \(l^1(\mathbb{Z}) = \left\{ (a_n)_{n=-\infty}^{\infty} | a_n \in \mathbb{C} \text{ and } \sum_{n=-\infty}^{\infty} a_n < \infty \right\}\) where

\[
\|a\| = \sum_{n=-\infty}^{\infty} a_n
\]

and multiplication is given by

\[
a * b = \sum_{n=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} a_{n-k}b_k \right).
\]
Example 6. Wiener's Algebra
The algebra $W$ (for Weiner) consists of the absolutely convergent trigonometric series

$$W = \left\{ f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{int} \left\| f \right\| = \sum_{n=-\infty}^{\infty} |\alpha_n| < \infty \right\}$$

with convolution as the multiplication: for $f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{int}$ and $g(t) = \sum_{n=-\infty}^{\infty} \beta_n e^{int}$

$$fg(t) = \left( \sum_{n=-\infty}^{\infty} \alpha_n e^{int} \right) \left( \sum_{n=-\infty}^{\infty} \beta_n e^{int} \right)$$

$$= \sum_{n=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} \alpha_{n-k} \beta_k \right) e^{int}$$

Constructions with Banach Algebras

Here we show how new Banach algebras can be constructed from other Banach algebras.

**Definition:** If $A_1$ and $A_2$ are Banach algebras then their *direct sum* is the algebra

$$A = A_1 \oplus A_2$$

consisting of ordered pairs $(x_1, x_2)$ with coordinate wise operations and with the norm

$$\|x\| = \max (\|x_1\|, \|x_2\|).$$

**Example 7. Direct Sums**
1. The direct sum $C[0, 1] \oplus C[0, 1]$ is isomorphic to $C([0, 1] \cup [2, 3])$.
2. The direct sum $c_0 \times c_0$ is isomorphic to $c_0$.

**Definition:** If $A$ is a Banach algebras and $I$ is a proper ideal in $A$, then the quotient algebra $A/I$ is a normed algebra with the norm

$$\|x + I\| = \inf_{y \in I} \|x + y\|$$

Quotient algebras are studied in more detail in Section 4. This idea is key to the development of the Gelfand theory for Banach algebras.

**Example 7. Quotient Algebras**
1. Let $A = L^1(G) \oplus C[0, 1]$. What is $A/C[0, 1]$.
2. In general, if $A = A_1 \oplus A_2$, it's easy to see that $A/A_1 = A_2$. 
Exercises

1. The collection $B(0, 1)$ of bounded continuous functions on $(0, 1)$ with pointwise operations and sup norm is a Banach algebra.

2. The Banach space $l^\infty(\mathbb{N})$ is a Banach algebra if multiplication is defined pointwise.

3. The Banach space $L^\infty(\mathbb{R})$ is a Banach algebra if multiplication is defined pointwise.
   [Note on 2 and 3: In general, if $(X, \Omega, \mu)$ is a $\sigma$-finite measure space then $L^\infty(X, \Omega, \mu)$ is a Banach algebra under pointwise multiplication.]

4. $L^1(0, \infty) = \{ x \in L^1(\mathbb{R}) : x(t) = 0, t \leq 0 \}$ is a closed subalgebra of $L^1(\mathbb{R})$.

5. If $E \subset [0, 1]$ then $B = \{ x \in C[0, 1] : x(E) = 0 \}$ is a closed subalgebra of $C[0, 1]$.

6. $L^1(0, 1) = \{ x : \| x \| = \int_0^1 |x(t)| \, dt < \infty \}$ is a Banach algebra with convolution
   \[ x*y = \int_0^1 x(t - \tau)y(\tau) \, d\tau \quad (0 < t < 1) \]
   and the obvious norm. (This algebra has no identity).

7. $L^p(\mathbb{T}) = \{ x : \| x \| = \int_{\mathbb{T}} |x(t)|^p \, dt < \infty \}$ is a Banach algebra with convolution
   \[ x*y = \int_{\mathbb{T}} x(t \tau^{-1})y(\tau) \, d\tau \]
   and the obvious norm.
   [Note on 6: If $G$ is a compact topological group, and $p \geq 1$, then $L^p(G)$ is a Banach algebra under convolution.]

8. Let $C^n[0, 1]$ denote the algebra of all $n$-times differentiable functions on $[0, 1]$ with pointwise operations. Then $C^n[0, 1]$ is a Banach algebra with the norm
   \[ \| x \|_n = \sup_{0 \leq t \leq 1} \sum_{k=0}^n \frac{|x^{(k)}(t)|}{k!} \]
   (We use the convention that the derivatives at the endpoints are one-sided)

9. An approximate identity in $A$ is a net $(e_\alpha)$ of bounded norm such that
   \[ \lim \alpha e_\alpha x = \lim \alpha xe_\alpha = x \]
   for every $x \in A$. In $L^1(\mathbb{R})$ the sequence of functions
   \[ e_n(t) = \begin{cases} 
   2n & \text{for } -\frac{1}{n} \leq t \leq \frac{1}{n} \\
   0 & \text{for } |t| > \frac{1}{n}
   \end{cases} \]
is an approximate identity.

10. (a) If $A_1$ and $A_2$ are Banach algebras then $A_1 \oplus A_2$ is a Banach algebra. 
(b) $A_1 \oplus A_2$ has an identity if and only if $A_1$ and $A_2$ have identities.

11. $C[0, 1] \oplus C[0, 1]$ is isomorphic to $C([0, 1] \cup [2, 3])$.

12. If $A_1, A_2, A_3, \ldots$ is a sequence of Banach algebras then their bounded direct sum is 
the subset $\sum_n A_n$ of the product consisting of those $x = (x_n)$ for which 
$\sup_n \|x_n\| < \infty$. The bounded direct sum is a Banach algebra.
2. Basic Properties of Commutative Banach Algebras

Let $A$ be a commutative Banach algebra over $\mathbb{C}$. In this section we prove that a Banach algebra without identity can be imbedded in a Banach algebra with identity. Also, in any Banach algebra with identity we can always find an equivalent norm in which the norm of the identity is 1.

2.1 Theorem: (Adjoining an Identity) A Banach algebra $A$ without identity can always be embedded as a subalgebra of a Banach algebra with identity. (This procedure is called adjoining an identity and denoted $\tilde{A} = A \oplus \{1\}$).

Proof - Let $B = A \oplus \mathbb{C}$ with the following operations:

1) $(x, \alpha) + (y, \beta) = (x + y, \alpha + \beta)$
2) $c(x, \alpha) = (cx, c\alpha)$
3) $(x, \alpha)(y, \beta) = (xy + \alpha y + \beta x, \alpha \beta)$
4) $\| (x, \alpha) \| = \| x \| + |\alpha|$

Note that $(0, 1)$ is an identity for $B$.

The embedding is given by:

$$T : A \rightarrow B \text{ where } x \mapsto (x, 0)$$

Note that $T$ is a homomorphism of algebras and preserves the norm so that $T$ is injective and an isometry.

i.e. $\| Tx \| = \| (x, 0) \| = \| x \| + \| 0 \| = \| x \|$  \[\square\]

The next theorem shows that any Banach algebra can be represented as a subalgebra of $B(X)$ where $X$ is a suitable Banach space.

2.2 Theorem: (Embedding an Banach Algebra in $B(X, X)$)

Let $A$ be a Banach algebra. Then there exists a Banach space $X$ such that $A$ is somorphic as a closed subalgebra of $B(X, X)$.

Proof - Let $X = A$ (or let $X = \tilde{A}$ if $A$ does not have an identity. So assume W.L.O.G. that $A$ has an identity). Consider the left-multiplication operator

$$T_x : A \rightarrow A$$

$$y \mapsto xy.$$ 

Clearly

$$\| T_x y \| = \| xy \| \leq \| x \| \| y \|$$

Therefore $T_x$ is bounded and $\| T_x \| \leq \| x \|$. 


Define $\phi : A \to B(X, X)$ by $x \mapsto T_x$

0) $\phi$ is a homomorphism. This follows from the following observations:

$$T_{x+y} = T_x + T_y, \quad T_{\alpha x} = \alpha T_x, \quad \text{and} \quad T_{xy} = T_x T_y$$

1) $\phi$ is injective.

$$T_x = T_y \Rightarrow xz = yz \forall z \in X$$

$$\Rightarrow xe = ye$$

$$\Rightarrow x = y$$

2) $\phi^{-1}$ is continuous.

$$\|\phi(x)\| = \|T_x\| = \sup_{\|x\| \leq 1} \|T_x y\| = \sup_{\|x\| \leq 1} \|xy\|$$

$$\geq \left\| x \cdot \frac{y}{\|e\|} \right\| = \frac{1}{\|e\|} \|x\| e = \frac{\|x\|}{\|e\|}$$

$$\therefore \|\phi(x)\| \geq \frac{1}{\|e\|} \|x\|$$

Therefore $\phi^{-1}$ is bounded.

3) $\phi(A)$ is complete.

Since $B(X, X)$ is complete, we need only show that a convergent sequence in $\phi(A)$ has its limit in $\phi(A)$, (i.e. show $\phi(A)$ is closed).

Suppose $T_{x_n} \to T$, then $T_{x_n}(yz) = (T_{x_n}y)z \forall y, z \in A$.

Taking the limit, set $T(yz) = T(y)z$.

Set $y = e$, then $T(z) = T(e)z \forall z$

Let $x_o = T(e)$, then $T = T_{x_o}$.

4) $\phi$ is bounded and surjective.

Since $\phi^{-1} : \phi(A) \to A$ is bounded and surjective, and $\phi(A)$ and $A$ are Banach space, then $(\phi^{-1})^{-1} = \phi$ is continuous by the Open Mapping Theorem. $\square$

2.3 Corollary: In every Banach algebra with identity, there exists an equivalent norm for which $\|e\| = 1$.

Proof - Define $\|x\|_N = \|T_x\|$.

Now, $\|e\|_N = \|T_e\|.$

But, $\|T_e x\| = \|e x\| = \|x\| \forall x$.

Hence, $\|T_e\| = 1$. $\square$
**Definition:** $x \in A$ is **invertible** if there exists $y \in A$ such that $xy = e$ and $yx = e$. In this case $y$ is denoted by $x^{-1}$.

For the next theorem we need to recall that a series $\sum_{n=1}^{\infty} x_n$ in a Banach spaces is said to **converge absolutely** if $\sum_{n=1}^{\infty} \|x_n\| < \infty$. A series that converges absolutely converges (in the norm of the Banach space).

**2.4 Theorem:** Let $A$ be a Banach algebra and $x \in A$. If $\|x - e\| < 1$, then $x$ is invertible. And $x^{-1} = \sum_{n=0}^{\infty} (e - x)^n$.

**Proof** - Let $y = e - x$, then $\|y\| = \alpha < 1$. for some $\alpha$.

So, $\|y^n\| \leq \|y\|^n = \alpha^n$.

Thus $z = \sum_{n=0}^{\infty} y^n$ converges absolutely in $A$. (Note that $y^0 = e$.)

Now, $xz = (e - y)z = ez - yz$

$$= e \sum_{n=0}^{\infty} y^n - y \sum_{n=0}^{\infty} y^n$$

$$= \sum_{n=0}^{\infty} y^n - \sum_{n=1}^{\infty} y^n \quad \text{(continuity of multiplication)}$$

$$= e$$

Similarly, $zx = e$. So $z$ is the inverse of $x$. \qed

**Examples**

1. $A = B(X)$
   $$\|T - I\| < 1 \Rightarrow T \text{ is invertible.}$$

2. If $A = C[0, 1]$, then
   $$\|f - 1\| < 1 \Rightarrow f \text{ is invertible.}$$

**Definition:** The set $G$ of invertible elements is called the **group of invertible elements**.

**2.5 Corollary:** The set $G$ of invertible elements of $A$ is open.

**Proof** - Let $x \in G$. Consider the map $T_x : A \to A$ where $y \mapsto xy$.

a) $T_x$ is continuous. Moreover,

$$\|T_xy\| = \|xy\| \leq \|x\|\|y\|$$

Therefore $\|T_x\| \leq \|x\|$ and taking $y = e$. we see that $\|T_x\| = \|x\|$.

b) $T_x$ is onto $A$.

Proof - Let $z \in A$, then $T_x(x^{-1}z) = z$

$\therefore T_x$ is onto $A$.

c) $T_x$ is 1-1 (it is in fact an isometry)
by the Open Mapping Theorem, \( T_x \) is open.

(i.e. \( T \) maps open sets to open sets)

Now \( U = \{ x \mid \|x - e\| < 1 \} \) is an open neighborhood of \( E \) contained in \( G \).
Thus \( T_x(U) \) is an open neighborhood of \( x \).
Moreover \( U \subset G \Rightarrow xU \subset xG = G \) (because \( x \in G \)) \( \square \)

2.6 Theorem: The map \( x \mapsto x^{-1} \) defined on \( G \) is continuous.

Proof - Let \( x_n \in G \), where \( x_n \to y \).
We need to show \( x^{-1}_n \to y^{-1} \)
First, suppose \( x_n \to e \).

Then \( \exists N > 0 \) such that \( n > N \Rightarrow \|e - x_n\| < \frac{1}{2} \)

Then \( \|x^{-1}_n\| = \left\| \sum_{k=0}^{\infty} (e - x_n)^k \right\| \leq \sum_{k=0}^{\infty} \|e - x_n\|^k \leq \sum_{k=0}^{\infty} (\frac{1}{2})^k = 2 \)

Therefore \( \|x^{-1}_n - e\| = \|x^{-1}_n(e - x_n)\| \leq \|x^{-1}_n\|\|e - x_n\| \leq 2\|e - x_n\| \)
\( \to 0 \) as \( n \to \infty \).

Thus, we have shown if \( x_n \to e \), then \( x^{-1}_n \to e \).

Now suppose \( x_n \to y \in G \). Then \( y^{-1}x_n \in G \) and \( y^{-1}x_n \to e \).
So, by the first part of the proof, \( (y^{-1}x_n)^{-1} \to e \) or \( x^{-1}_ny \to e \).
Multiplying by \( y^{-1} \) we get \( x^{-1}_ny^{-1} \to y^{-1} \) \( \square \)

Definition: A division algebra is an algebra where every nonzero element is invertible.

Example.
1. \( C[0, 1] \) is not a division algebra
2. \( \mathbb{C} \) is a division algebra.

2.7 Theorem (Gelfand-Mazur): Let \( A \) be a complex Banach division algebra, then \( A \) is isomorphic to \( \mathbb{C} \).

Proof - We show that every \( x \in A \) is of the form \( x = \lambda e \) for some \( \lambda \in \mathbb{C} \).
Suppose not, then \( \exists x \in A \) such that \( x \neq \lambda e \ \forall \lambda \in \mathbb{C} \).
Hence, \( x - \lambda e \neq 0 \ \forall \lambda \in \mathbb{C} \).
Let \( f \in A' \) (the dual of \( A \)) such that \( f(x^{-1}) \neq 0 \).
Define \( \phi: \mathbb{C} \to \mathbb{C} \) by \( \phi(\lambda) = f([x - \lambda e]^{-1}) \)
1) Show \( \phi \) is entire (analytic on all of \( \mathbb{C} \)).
\[
\phi(\lambda + h) - \phi(\lambda) = \frac{f[(x-(\lambda + h)e)^{-1}] - f[(x-\lambda e)^{-1}]}{h} = f\left[\frac{(x-(\lambda + h)e)^{-1} - (x-\lambda e)^{-1}}{h}\right] = f\left[\frac{h(x-\lambda e)^{-1}(x-(\lambda + h)e)^{-1}}{h}\right] = f\left[(x-\lambda e)^{-1}(x-(\lambda + h)e)^{-1}\right]
\]

By Theorem 2.6, the limit exists as \( h \to 0 \).

\[ \text{namely as } h \to 0, (x - (\lambda + h)e)^{-1} \to (x - \lambda e)^{-1} \]

\[ \therefore \text{ } \phi \text{ is entire because this limit exists for all } \lambda. \]

2) Show that \( \phi \) is bounded.

\[ \phi(\lambda) = f\left((x-\lambda e)^{-1}\right) = \frac{1}{\lambda} f\left[\left(\frac{x}{\lambda} - e\right)^{-1}\right] \]

So, \( \lim_{|\lambda| \to \infty} \phi(\lambda) = \lim_{|\lambda| \to \infty} \frac{1}{\lambda} f\left[\left(\frac{x}{\lambda} - e\right)^{-1}\right] = \lim_{|\lambda| \to \infty} \frac{1}{\lambda} \cdot \lim_{|\lambda| \to \infty} f\left[\left(\frac{x}{\lambda} - e\right)^{-1}\right] \]

\[ = \lim_{|\lambda| \to \infty} \frac{1}{\lambda} \cdot f\left[\lim_{|\lambda| \to \infty} \left(\frac{x}{\lambda} - e\right)^{-1}\right] \text{ since } f \in A' \]

\[ = \lim_{|\lambda| \to \infty} \frac{1}{\lambda} \cdot f\left[\left(\lim_{|\lambda| \to \infty} \frac{x}{\lambda} - e\right)^{-1}\right] \text{ by Theorem 2.6} \]

\[ = 0 \cdot f(-e) = 0. \]

3) By Louville's Theorem, \( \phi \) is constant and so, by the above, \( \phi \equiv 0 \).

This contradicts \( f(x^{-1}) \neq 0 \).

[\text{note, by the definition of } \phi, \phi(0) = f(x - 0 \cdot e)^{-1} = f(x^{-1}) = 0] \square

**Exercises**

1. The operator \( T_x \) defined in Theorem 2.2 is linear.

2. The map \( \phi \) defined in Theorem 2.2 is a homomorphism.

3. \( C[0, 1] \) is not a division algebra. Which elements are invertible?

4. If \( |\lambda| > \|x\| \) then \( \lambda e - x \) in invertible and

\[ (\lambda e - x)^{-1} = \sum_{n=1}^{\infty} \lambda^{-n} x^{n-1} \]

5. If \( T \in B(X) \) where \( X \) is a Banach space and if \( |\lambda| > \|T\| \) then \( \lambda I - T \) is invertible.

6. If \( x \) is invertible and \( \|y - x\| > \frac{1}{\|x^{-1}\|} \) then \( y \) is invertible.
7. If $A$ is a complex Banach algebra and if there exists $c > 0$ such that $\|xy\| \geq c\|x\|\|y\|$ for all $x, y \in A$, then $A$ is isomorphic to $\mathbb{C}$.

8. An element $x \in A$ is a **topological divisor of zero** if there exists a sequence $(x_n)$ with $\|x_n\| = 1$ and $xx_n \to 0$ (or $x_n x \to 0$). (a) A topological divisor of zero is not invertible. (b) The boundary of the set of noninvertible elements is a subset of the set of topological divisors of zero.
3. Ideals

In this section we study special subalgebras called ideals. The concept of an ideal is very natural because, as we will see, the kernel of a homomorphism is always an ideal.

**Definition:** An **ideal** is an algebra $A$ is a subset $I$ of $A$ such that:

1) $I$ is a subalgebra of $A$
2) $\forall x \in A$, $xI \subset I$ and $Ix \subset I$

An ideal is **proper** if $I \neq \{0\}$ and $I \neq A$. The term ideal (without qualification) will mean proper ideal. A proper ideal consists of noninvertible elements only, so a division algebra has no proper ideals.

Note that the concept of ideal is purely algebraic (i.e., does not involve the norm).

**Definition:** An ideal is **maximal** if it is not properly contained in any proper ideal.

**Example. Ideals in $C[0,1]$**

Let $A = C[0, 1]$ and let $E \subset [0, 1]$

Let $I_E = \{f \mid f(E) = 0\}$

Let $M_p = \{f \mid f(p) = 0\}$ where $p \in [0, 1]$

$I_E$ is an ideal because: If $g \in A$, $f \in I_p$, then $fg(p) = f(p)g(p) = 0$. $\therefore fg \in I_p$.

Note that if $F \subset E$, then $I_F \supset I_E$.

$M_p$ is a maximal ideal.

Proof - Suppose $M_p$ is not maximal (i.e. $M_p \subset M$ for some ideal $M$).

Let $f \in M \setminus M_p$. Then $f(p) \neq 0$.

Choose $g \in M_p$ such that $g(x) = 0$, but $g(t) \neq 0 \forall t \neq p$.

Clearly $f, g \in M$. Thus, $f^2 + g^2 \in M$ since is an algebra.

But then $(f^2 + g^2)(t) \neq 0 \forall t \in [0, 1] \Rightarrow f^2 + g^2$ is invertible.

Thus, $M = A$.

If $M$ is maximal in $A = C[0, 1]$, then $M = M_p$ for some $p$.

Proof - For each $f \in M$, consider the zero set of $f$: $Z(f) = \{t \mid f(t) = 0\} = f^{-1}(\{0\})$.

Since $f$ is continuous and $\{0\}$ is a closed set, then $Z(f)$ is a closed set.

If $f, g \in M$, then $Z(f) \cap Z(g) \neq \emptyset$.

If not, $(f^2 + g^2)(t) \neq 0 \forall t$. But then $f^2 + g^2$ is invertible, which is a contradiction.

Thus the collection $\{Z(f) \mid f \in M\}$ of closed sets has the finite intersection property.

Since $[0, 1]$ is compact, then $\bigcap_{f \in M} Z(f) \neq \emptyset$.

If $p \in K$, then $M_p \subseteq M$.  

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Since $M$ is maximal, then $K = \{p\}$. So, $M = M_p$.

3.1 Theorem: The intersection of any collection of ideals is an ideal.

Proof - Let $\{I_\alpha : \alpha \in \Lambda\}$ be a collection of ideals.

Let $I = \bigcap_{\alpha \in \Lambda} I_\alpha$.

Suppose $x, y \in I$, then $x, y \in I_\alpha \forall \alpha$.

But $I_\alpha$ is an ideal, so $x + y, xy \in I_\alpha \forall \alpha \Rightarrow x + y, xy \in I$.

Let $z \in A$. Since $x \in I \Rightarrow x \in I_\alpha \forall \alpha \Rightarrow xz \in I_\alpha \forall \alpha \Rightarrow xz \in I$. \hfill \square

3.2 Theorem: The closure of an ideal is an ideal.

Proof - Let $I$ be an ideal and $x, y \in \overline{I}$.

Then $\exists (x_n), (y_n) \in I$ such that $x_n \to x, y_n \to y$.

By continuity of addition and multiplication,

$$x_n + y_n \to x + y \quad \text{and} \quad x_n y_n \to xy.$$  

Thus, $x + y \in \overline{I}$ and $xy \in \overline{I}$.

Suppose $z \in A$, then $xz_n \in I$ and $zx_n \to zx$.

Thus $zx \in \overline{I}$. \hfill \square

3.3 Theorem: The closure of a proper ideal is a proper ideal.

Proof - Let $U = \{x|\|x - e\| < 1\}$ consist of invertible elements and is open (Theorem 2.4).

So $A \setminus U$ is closed.

So, for any proper ideal $I$, $I \subset A \setminus U$

$$\Rightarrow \overline{I} \subset \overline{A \setminus U} = A \setminus U$$

$$\Rightarrow \overline{I} \text{ is proper}.$$ \hfill \square

3.4 Theorem: Every maximal ideal is closed.

Proof - Let $M$ be a maximal ideal.

By Theorem 3.3, $\overline{M}$ is a proper ideal.

But $\overline{M} \supset M$ and $M$ is maximal.

So, $\overline{M} = M$.

Thus, $M$ is closed. \hfill \square

3.5 Theorem: Every proper ideal is contained in a maximal ideal.

Proof - Let $I$ be an ideal.

Consider the collection of all ideals containing $I$.

This is a partially ordered set under inclusion. Apply Zorn's Lemma.
3.6 Theorem: Every noninvertible element $x$ is contained in some maximal ideal.

Proof - Let $I = xA = \{xa | a \in A\}$.
Then $I$ is a proper ideal.
By Theorem 3.5, $I$ is contained in a maximal ideal $M$ and $x \in I \subseteq M$. □

Exercises
1. If $x$ is a noninvertible element in $A$ then $I = xA = \{xa : a \in A\}$ is a proper ideal.
2. Let $A = C[0, 1]$ and let $x_0$ be the element in $A$ defined by $x_0(t) = t$. Find the closure of the \{ $x_0$ \} generated by $A$.
3. If $E \subset F$ are subsets of $[0, 1]$ then $I_E \supset I_F$ in $C[0, 1]$.
4. If $A = A_1 \oplus A_2$ then $\widehat{A}_1 = \{(x, 0) : x \in A_1\}$ is an ideal in $A$. Also $\widehat{A}_1$ is isometrically isomorphic to $A_1$.
5. Prove Theorem 3.5.
4. Quotient Algebras

First we review some basic facts about rings and then apply those results to Banach algebras. Note that an algebra is a ring with some additional properties.

Results from Ring Theory

**Definition:** Let \( A \) be a commutative ring and \( I \) an ideal in \( A \). The **quotient ring** \( A/I \) is defined as follows:
\[
A/I = \{ x + I \mid x \in A \}
\]
With the following operations:
\[
(x + I) + (y + I) = (x + y) + I
\]
\[
(x + I)(y + I) = xy + I
\]
\[
\alpha(x + I) = \alpha x + I
\]
With these operations, \( A/I \) is a ring.

**Note:** \( x \) and \( y \) are in the same coset of \( I \) iff \( x - y \in I \).

**4.1 Theorem:** If \( A \) and \( B \) are rings with identity and \( \phi : A \to B \) is a ring homomorphism then

1. the kernel of \( \phi \) is an ideal in \( A \)
2. If \( \phi \) is an epimorphism then \( A/\ker(\phi) \) is isomorphic to \( B \).

**Proof** - (a) Let \( x, y \in \ker(\phi) = K \), and \( z \in A \), then

a. \( \phi(x + y) = \phi(x) + \phi(y) = 0 + 0 = 0. \therefore x + y \in K. \)

b. \( \phi(xy) = \phi(x)\phi(y) = 0 \cdot 0 = 0. \therefore xy \in K \)

c. \( \phi(zx) = \phi(z)\phi(x) = \phi(z) \cdot 0 = 0. \therefore zx \in K \)

\( \therefore K \) is an ideal.

(b) \( \phi \) can be factored as follows:

\[
A \xrightarrow{g} A/\ker(f) \xrightarrow{h} B
\]
\[
x \mapsto x + K \mapsto f(x) \quad \blacksquare
\]

**4.2 Theorem:** \( M \) is a maximal ideal in the ring \( A \) if and only if \( A/M \) is a field.
Proof - (⇒) Suppose $x \notin M$.
  Let $W$ be the ideal generated by $x$ and $M$. i.e. $W = \{ y + xz | y \in M, z \in A \}$

Now, setting $y = 0$ and $z = e$ we see that $0 + xe = x \in W$, so $W$ properly contains $M$. Since $M$ is maximal, $W = A$.
So, $\exists y_o \in M$ such that $e = y_o + z_o x$.

$\Rightarrow e + M = y_o + z_o x + M$

$= y_o + M + z_o x + M$

$= M + z_o x + M$

$= z_o x + M$

$= (z_o + M)(x + M)$

Note that $e + M$ is the multiplicative identity of $A/M$. So, $z_o + M$ is the inverse of $x + M$.

(⇐) If $K$ is an ideal properly containing $M$, then $K/M$ is an ideal in $A/M$.
But $A/M$ is a field so it has no proper ideals. This contradiction proves that $K$ doesn't exist, so $M$ is maximal.

Qotients of Banach Algebras

Recall from Section 1 that if $A$ is a Banach algebra and $I$ an ideal in $A$, then the quotient algebra $A/I$ is a normed algebra with the norm

$\|x + I\| = \inf_{y \in I} \|x + y\|$

4.3 Theorem: If $I$ is a closed ideal, then $A/I$ is a Banach algebra.
Proof - Exercise.

4.4 Theorem: If $M$ is a maximal ideal, then $A/M \approx \mathbb{C}$.
Proof - Note that $A/M$ is a Banach algebra and $A/M$ is a field.
  By the Gelfand-Mazur Theorem, $A/M \approx \mathbb{C}$.

Exercises

1. If $I$ is an ideal in the ring $A$ then $x$ and $y$ are in the same coset of $I$ if and only if $x - y \in I$

2. What is $C[0, 1]/I_0$ where $I_0 = \{ x \in C[0, 1] : x(0) = 0 \}$.
3. Let $E = [0, \frac{1}{2}]$. What is $C[0, 1]/I_E$ where $I_E = \{x \in C[0, 1] : x(E) = 0\}$, and 

4. Prove Theorem 4.3
5. Multiplicative Linear Functionals

Linear functionals on a Banach algebra which preserve the multiplicative structure of the algebra play a special role. We will see that they are in one-to-one correspondence with the maximal ideals of the algebra.

**Definition:** Let $A$ be a complex commutative Banach algebra with identity. A nonzero linear functional $f : A \rightarrow \mathbb{C}$ which satisfies $f(xy) = f(x)f(y)$ $\forall x, y \in A$ is called a **multiplicative linear functional** (i.e. $f$ is a homomorphism from $A$ into $\mathbb{C}$).

**5.1 Theorem:** The kernel of a nonzero m.l.f. is a maximal ideal.

**Proof -** Let $f : A \rightarrow \mathbb{C}$ be a m.l.f. By Theorem 4.1(1), $\ker(f)$ is an ideal.

By Theorem 4.1(2),

$$A/\ker(f) \simeq \mathbb{C}.$$ 

Since $\mathbb{C}$ is a field, $\ker(f)$ is a maximal ideal (by Theorem 4.2). 

**Note:** Recall from Functional Analysis that if $f$ is a linear functional and if $\ker(f)$ is closed, then $f$ is continuous. We use this fact in the next theorem.

**5.2 Theorem:** Every nonzero multiplicative linear functional is continuous.

**Proof -** Let $f : A \rightarrow \mathbb{C}$ be a multiplicative linear functional.

Since $\ker(f)$ is a maximal ideal, then it is closed (Theorem 3.4).

It follows from the above theorem that $f$ is continuous.

**5.3 Theorem:** If $f$ is a multiplicative linear functional, then $\|f\| = 1$.

**Proof - 1)** Show that $\|f\| \leq 1$. i.e. $|f(x)| \leq \|x\|$ $\forall x \in A$.

For suppose not, then $\exists x_o$ such that $|f(x_o)| > \|x_o\|$.

Let $\alpha \in \mathbb{C}$ such that $|f(\alpha x_o)| > 1 > \|\alpha x_o\|$

$$\Rightarrow |f[(\alpha x_o)^n]| = |f(\alpha x_o)|^n \rightarrow \infty$$

Whereas $\|(\alpha x_o)^n\| \leq \|\alpha x_o\|^n \rightarrow 0$ (by submultiplicativity of the norm)

This contradicts the continuity of the norm.

\[ \therefore \|f\| \leq 1. \]

2) Now, $f(e) = 1$ and $\|e\| = 1$.

So, $|f(e)| = \|e\|$, then $\|f\| \geq 1$.

\[ \therefore \|f\| = 1. \]
Note:
Given a multiplicative linear functional \( f : A \rightarrow \mathbb{C} \), we know that its kernel \( M_f \) is maximal.

Given a maximal ideal \( M \subset A \), we define
\[
f_M : A \rightarrow A/M \cong \mathbb{C} \text{ where } x \mapsto x + M
\]
So \( f_M \) is a multiplicative linear functional on \( A \).

This correspondence between multiplicative linear functionals and maximal ideals in injective, i.e.
\[
f_{M_f} = f \text{ and } M_{f_M} = M.
\]
We have shown that there is a 1-1 correspondence between the multiplicative linear functionals on \( A \) and the maximal ideals of \( A \).

Exercises

1. If \( f \) is a m.l.f. then \( f(e) = 1 \)

2. If \( f \) is a m.l.f. and \( x \) is invertible, then \( f(x^{-1}) = [f(x)]^{-1} \)

3. If \( M \) is a maximal ideal in \( A \), explain how the quotient map \( f : A \rightarrow A/M \) is a m.l.f.

4. A commutative Banach algebra must have at least one m.l.f. defined on it.

5. For \( t_0 \in [0, 1] \) the evaluation functional \( f_{t_0} : C[0, 1] \rightarrow \mathbb{C} \) is defined by
\[
f(x) = x(t_0). \text{ The evaluation functional is a m.l.f. What is its kernel?}
\]

6. Every m.l.f. on \( C[0, 1] \) is an evaluation functional.

7. Every m.l.f. on \( L^1(\mathbb{R}) \) is given by \( f(x) = \int_{-\infty}^{\infty} x(t)e^{int} dt \) for some \( p \in \mathbb{R} \).

8. There are no m.l.f.’s on \( L^1(0, 1) \). (Note that this algebra has no identity.)

9. Every m.l.f. in \( l^1(\mathbb{Z}) \) has the form \( f(x) = \sum_{n=-\infty}^{\infty} x_ne^{int} \) for some \( t, 0 \leq t < 2\pi \).

10. Every m.l.f. on the algebra \( A(D) \) is of the form \( f(x) = x(p) \) for some \( p \in D \).
6. Banach's Theorem on Semisimple Banach Algebras

Let \( A, B \) Banach algebras. When is a homomorphism \( \phi : A \to B \) continuous? We have already seen that if \( B = \mathbb{C} \), then \( \phi \), which is a multiplicative linear functional, is automatically continuous. In this section we see what exactly is special about \( \mathbb{C} \) that make this property of automatic continuity possible.

The Separating Space

To study continuity of homomorphisms, we consider the places they are not continuous.

**Definition:** Let \( T : X \to Y \) be a linear transformation between the Banach spaces \( X \) and \( Y \). The **separating space** of \( T \) is given by:

\[
\mathcal{S}(T) = \{ y \in Y \mid \exists (x_n) \subset X, x_n \to 0 \text{ and } Tx_n \to y \}.
\]

**6.1 Theorem:** Let \( T : X \to Y \), then
1) \( \mathcal{S}(T) \) is a closed linear subspace of \( Y \).
2) \( T \) is continuous iff \( \mathcal{S}(T) = \{0\} \).

**Proof** - 1) Let \((y_n)\) be a sequence in \( \mathcal{S}(T) \) such that \( y_n \to y \).
Since each \( y_n \) is in \( \mathcal{S}(T) \) then for each \( n \) we find \( x_n \in X \) such that \( \|x_n\| < \frac{1}{n} \) and \( \|T(x_n) - y_n\| < \frac{1}{n} \).
Clearly \( x_n \to 0 \) and \( T(x_n) \to y \). Hence \( y \in \mathcal{S}(T) \). So \( \mathcal{S}(T) \) is closed.
2) If \( T \) is continuous, then \( \mathcal{S}(T) = \{0\} \).

Suppose \( y \in \mathcal{S}(T) \), then \( \exists (x_n) \subset X \) such that \( x_n \to 0 \) and \( Tx_n \to y \).

Since \( T \) is continuous, \( Tx_n \to T(0) = 0 \). So, \( y = 0 \).

Conversely, if \( \mathcal{S}(T) = 0 \), then \( T \) is continuous by the Closed Graph Theorem.

**6.2 Theorem:** Let \( T \) be a linear transformation and \( Q \) the quotient map as follows:

\[
X \xrightarrow{T} Y \xrightarrow{Q} Y/W,
\]

where \( W \) is a closed subspace of \( Y \). If \( QT \) is continuous, then \( \mathcal{S}(T) \subset W \).

**Proof** - Since \( QT \) is continuous, \( \mathcal{S}(T) = \{0 + W\} \).

If \( x_n \to 0 \) in \( X \), then \( QT(x_n) \to 0 + W \)

If \( Tx_n \to y \), the \( y \in W \) (because \( Q(y) = 0 + W \))

\[\therefore \mathcal{S}(T) \subset W.\]
6.3 Theorem: Let $T$ and $S$ be mappings such that

$$X \xrightarrow{T} Y \xrightarrow{S} Z,$$

where $S$ is surjective. If $ST$ is continuous, then $\mathcal{G}(T) \subset \ker S$.

Proof - $Z \cong Y / \ker S$. Thus

$$X \xrightarrow{T} Y \xrightarrow{S} Y / \ker S.$$

Now apply Theorem 6.2.

\[
\]

Semisimple Banach Algebras

Definition: A Banach algebra $A$ is called semisimple iff

$$\bigcap \{M \subset A | M \text{ is a maximal ideal} \} = \{0\}.\]

(note: $R(A) = \bigcap M$ is called the (Jacobson) Radical of $A$.)

Example

$C[0, 1]$ is semisimple because if $f \in M_p$ for each $p \in [0, 1]$, then $f(p) = 0 \forall p \in [0, 1]$ which implies $f = \{0\}$.

6.4 Theorem: If $\phi$ is a homomorphism from the Banach algebra $A$ to the semisimple Banach algebra $B$, then $\phi$ is continuous.

Proof - Consider a multiplicative linear functional $f$ on $B$, and consider

$$A \xrightarrow{\phi} B \xrightarrow{f} \mathbb{C}.$$

Now $f \circ \phi$ is a multiplicative linear functional on $A$, so $f \circ \phi$ is continuous.

By Theorem 6.3, $\mathcal{G}(\phi) \subset \ker f$.

But $\ker f = M$ is a maximal ideal. Moreover, every maximal ideal is the kernel of some multiplicative linear functional. In other words, $\mathcal{G}(\phi)$ is contained in every maximal ideal of $B$. Thus, $\mathcal{G}(\phi) \subset \bigcap_{M \text{ maximal}} M = \{0\}$. So, $\phi$ is continuous.

The theorem just proved illustrates the power of the functional analysis approach. We were able to prove the continuity of a whole class of operators by considering the Banach algebra structure only, and not individual elements in the algebra. (i.e., we didn't need any $\epsilon$-$\delta$ arguments.)

Exercises

1. If $T : X \to Y$, then $\mathcal{G}(T)$ is a closed linear subspace of $Y$.
2. If $\phi : A \to B$ is an algebra isomorphism and if $A$ (or $B$) is semisimple then $\phi$ is cont.
3. If $A$ is a semisimple Banach algebra then any homomorphism $\phi : A \to A$ is cont.
4. The Banach algebra $C[0, 1]$ is semisimple.
5. The Banach algebra $A(D)$ is semisimple.
7. The Gelfand Theory

The goal is to represent any Banach algebra $A$ as a subalgebra of the Banach algebra $C(X)$ where $X$ is a compact Hausdorff space. Recall that $C(X)$ was given as an example of a Banach algebra. We want to show that any Banach algebra is a subalgebra of $C(X)$. Thus, we need to find $X$ and a homomorphism $A \rightarrow C(X)$. Recall, we defined the maximal ideal space of $A$ as:

$$M(A) = \{ M | M \text{ is a maximal ideal in } A \}$$
$$= \{ f | f \text{ is multiplicative linear functional on } A \}$$

It is possible to equip $M(A)$ with a topology so that it becomes a compact topological space. The topology is the $w^*$-topology inherited from the $w^*$-topology on the dual $A'$ of $A$. Note that $M(A) \subset A'$. The $w^*$-topology is described in Appendix 1.

Note that if $(f_n)$ is a sequence in $A'$ then $f_n \rightarrow f$ in the $w^*$-topology iff $f_n(x) \rightarrow f(x)$ for each $x \in X$. (This is because each $x \in X$ represents a functional on $A'$ in $J(A)$.)

7.1 Theorem: Let $A$ be a Banach algebra. Then $M(A)$ with the $w^*$-topology is a compact Hausdorff space.

Proof - By the Banach-Alaogolu theorem the unit ball $U$ of $A'$ is $w^*$-compact. We show that $M(A)$ is a closed subset of the $U$. Clearly $M(A) \subset U$ because for every multiplicative linear functional $\|f\| \leq 1$. Now suppose $(f_n)$ is a sequence in $M(A)$ and $f_n \rightarrow f$. We need to show that $f \in M(A)$; so we need to show that $f$ is a multiplicative linear functional. We show that $f$ is multiplicative and leave the other ($f$ is linear as an exercise). For any $x, y \in X$ we have $f_n(x) \rightarrow f(x), f_n(y) \rightarrow f(y)$ so

$$f_n(x)f_n(y) \rightarrow f(x)f(y)$$

We also have $f_n(xy) \rightarrow f(xy)$ and since $f_n(xy) = f_n(x)f_n(y)$ so we must have

$$f_n(x)f_n(y) \rightarrow f(xy)$$

By the uniqueness of limits we must have $f(xy) = f(x)f(y)$. Thus $f$ is multiplicative. In the same way we show that $f$ is linear. Thus $f \in M(A)$ so $M(A)$ is closed in $U$.

Finally, since $A'$ is Hausdorff in the $w^*$-topology, it follows that $M(A)$ is also Hausdorff. \qed
**Definition:** Let $x \in A$. Define the Gelfand transform $\hat{x}$ of $x$ to be the function $\hat{x} : M(A) \to \mathbb{C}$ where $f \mapsto \hat{x}(f)$, where $\hat{x}(f) = f(x)$.

**7.2 Theorem:** Each $\hat{x}$ is a continuous function on $M(A)$. So, $\hat{x} \in C(M(A))$.

Proof - If $f_n \to f$ in $M(A)$ then $f_n(x) \to f(x)$ for each $x \in X$. But this last statement means that $\hat{x}(f_n) \to \hat{x}(f)$. So $\hat{x}$ is continuous.

**Example**

Let $A = C[0, 1]$. $M(A) = \{f_p| p \in [0, 1]\}$ where $f_p(x) = x(p)$.

So we can identify $M(A)$ with $[0, 1]$ as $p \mapsto f_p$. With the $w^*$-topology, $M(A)$ is actually homeomorphic to $[0, 1]$. This shows we are able to recover the space $[0, 1]$ from only the algebraic structure of $A$. So, we can replicate this procedure with any Banach algebra.

i.e. start with the maximal ideals, impose a topology on $M(A)$, etc...

Now, $\hat{x} : M(A) \to \mathbb{C}$, where $f \mapsto \hat{x}(f) = f(x)$. In particular, $\hat{x}(f_p) = f_p(x) = x(p)$.

i.e. $\hat{x} : p \mapsto x(p)$, where $\hat{x}(p) = x(p)$. So that $\hat{x} = x$.

Thus, we are able to recover $C[0, 1]$ from the algebraic structure of $A$.

It can be observed that $\hat{x}$ is a bounded function on $M(A)$.

Proof - $|\hat{x}(f)| = |f(x)| \leq \|x\|$ since $f$ is continuous and $\|f\| = 1$.

Thus, $\hat{x}(f)$ is bounded by $\|x\|$.

**7.3 Theorem:** The map $\Gamma : A \to C(M(A))$ where $x \mapsto \hat{x}$ is a continuous algebra homomorphism from the Banach algebra $A$ into the Banach algebra $C(M(A))$ with the sup norm. i.e. $\|\hat{x}\| = \sup_{f \in M(A)} |\hat{x}(f)|$. Also, if $A$ is semi-simple, then $\Gamma$ is injective.

Proof - 1) Show $\Gamma$ is a continuous algebra homomorphism.

\[
\begin{align*}
(x + y)\hat{}(f) &= f(x + y) = f(x) + f(y) \\
&= \hat{x}(f) + \hat{y}(f) \\
(xy)\hat{}(f) &= f(xy) = f(x)f(y) \\
&= \hat{x}(f)\hat{y}(f) \\
(\alpha x)\hat{}(f) &= f(\alpha x) = \alpha f(x) \\
&= \alpha\hat{x}(f)
\end{align*}
\]

$\|\hat{x}\| = \sup_{f \in M(A)} |\hat{x}(f)| = \sup_{f \in M(A)} |f(x)|$

$\leq \|x\|$ since $|f(x)| \leq \|x\|$ $\forall f$.

i.e. all m.l.f. have norm 1.

2) Show if $A$ is semi-simple, then $\Gamma$ is injective.

Suppose $\hat{x} \equiv 0$. i.e. $\hat{x} = 0 \forall f \in M(A)$

$\Rightarrow f(x) = 0 \forall f \in M(A)$

$\Rightarrow x \in \ker(f) \forall f \in M(A)$

$\Rightarrow x \in M \forall$ maximal ideals of $M$

$\Rightarrow x \in \text{Rad}(A)$

$\Rightarrow x = 0$
7.4 Theorem: $x$ is invertible iff $\hat{x}(f) \neq 0 \forall f \in M(A)$.

Proof - $(\Rightarrow)$ Suppose $\hat{x}(f) = 0$ for some $f \in M(A)$
Then $f(x) = 0 \Rightarrow x \in \ker(f)$, which is a maximal ideal
$\Rightarrow x$ is not invertible

$(\Leftarrow)$ Suppose $x$ is not invertible.
Then $x \in$ some maximal ideal $M$ (By Theorem 3.6).
But then $x \in \ker(f_M)$. i.e. $f_M(x) = 0$. □

Exercises
1. Let $X$ and $Y$ be compact Hausdorff spaces. If $C(X)$ is algebraically isomorphic to $C(Y)$ then $X$ and $Y$ are homeomorphic.
2. $X$ is a connected topological space iff $C(X)$ has no zero divisors.
3. Banach algebra $A$ has no zero divisors if and only if $M(A)$ is connected.
4. If the Gelfand representation of a semisimple Banach algebra $A$ is complete in the spectral norm, then the original norm of $A$ is equivalent to the spectral norm.
5. If $\tau$ is any compact Hausdorff topology on $M(A)$ for all the Gelfand transforms $\hat{x}$ are continuous then $\tau$ coincides with the Gelfand topology on $M(A)$.
6. If $X$ is compact then the maximal ideal space of $C(X)$ is homeomorphic with $X$. [Hint: Use Exercise 5].
7. If $A$ has no identity and $A_1$ is the algebra obtained by adjoining an identity then $M(A_1)$ is the one-point compactification of $M(A)$.
8. The Gelfand transform is an isometry if and only if $\|x^2\| = \|x\|^2$ for every $x \in A$.
9. The following are equivalent: (1) $\Gamma$ is 1-1, (2) $M(A)$ separates the point of $A$, (3) $r(x)$ is a norm on $A$.
10. The $hk$-topology is weaker than the $w^*$-topology, so the two topologies coincide if the $hk$-topology is Hausdorff.
8. Wiener's Theorem on Absolutely Convergent Trigonometric Series

Let $W$ be the collection of all absolutely convergent trigonometric series, that is, series of the form

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{int} \quad t \in [0, 2\pi)$$

with $\|x\| = \sum_{n=1}^{\infty} |a_n| < \infty$. With this norm $W$ is a Banach space. It becomes an algebra if we define

$$x(t)y(t) = \left(\sum_{n=-\infty}^{\infty} a_n e^{int}\right)\left(\sum_{n=-\infty}^{\infty} b_n e^{int}\right) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} a_{n-k} b_k\right) e^{int} \quad \text{(Cauchy product)}$$

This norm is an algebra norm:

1) $\|x\| = 0$ iff $x = 0$.

2) $\|\alpha x\| = |\alpha|\|x\|$

3) $\|x + y\| = \sum_{n=-\infty}^{\infty} |a_n + b_n|$

$$\leq \sum_{n=-\infty}^{\infty} |a_n| + \sum_{n=-\infty}^{\infty} |b_n| = \|x\| + \|y\|$$

4) $\|xy\| = \sum_{n=-\infty}^{\infty} \left|\sum_{k=-\infty}^{\infty} a_{n-k} b_k\right|$

$$\leq \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |a_{n-k}| |b_k|$$

$$= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |a_{n-k}| |b_k|$$

$$= \sum_{k=-\infty}^{\infty} |b_k| \sum_{n=-\infty}^{\infty} |a_{n-k}|$$

$$= \sum_{k=-\infty}^{\infty} |b_k| \left(\sum_{n=-\infty}^{\infty} |a_{n-k}|\right)$$

$$= \|x\|\|y\|$$

$\therefore W$ is a Banach algebra with this norm called the Wiener algebra.
**8.1 Theorem:** Every multiplicative linear functional on $W$ is of the form

$$f_{t_o}(x) = x(t_o)$$

where $t_o \in [0, 2\pi)$.

**Proof - ( ⇒ )** We first show that $f_{t_0}$ is indeed a multiplicative linear functional on $W$.

$$f_{t_0}(x + y) = (x + y)(t_o) = x(t_o) + y(t_o) = f_{t_0}(x) + f_{t_0}(y)$$

$$f_{t_0}(\alpha x) = \alpha x(t_o) = \alpha f_{t_0}(x)$$

$$f_{t_0}(xy) = xy(t_o) = x(t_o)y(t_o) = f_{t_0}(x)f_{t_0}(y)$$

Next we show that every multiplicative linear functional is of this form.

Let $f$ be a m.l.f. on $W$.

Let $x_o(t) = e^{it} \in W$ and let $\gamma = f(x_o)$.

Then $|\gamma| = |f(x_o)| \leq \|x_o\| = 1$

$$|\gamma^{-1}| = |f(x_0^{-1})| \leq \|x_0^{-1}\| = 1$$

Since $\gamma \in \mathbb{C}$, then $\gamma = e^{it_o}$ for some $t_o \in [0, 2\pi)$

Thus, $f(x_0) = e^{it_o}$

$$f\left(\sum_{n=-N}^{N} a_n x_0^n\right) = f\left(\sum_{n=-N}^{N} a_n e^{int}\right) = \sum_{n=-N}^{N} a_n f(e^{int}) = \sum_{n=-N}^{N} a_n e^{int_o}$$

By continuity of $f$, if we let $x(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{int_o} = x(t_o)$$

i.e. $f = f_{t_o}$

**8.2 Theorem: (Wiener’s Theorem)** Let $x \in W$, if $x(t) \neq 0 \ \forall t \in [0, 2\pi)$, then $\frac{1}{x(t)} \in W$. i.e. if $x$ is an absolutely convergent trigonometric series that never vanishes, then its reciprocal is also an absolutely trigonometric series.

**Proof -** By the Gelfand Theorem, suppose $x(t) \neq 0 \ \forall t \in [0, 2\pi)$. Consider $\hat{x}$. Since each $f \in M(W)$ is of the form $f_{t_o}$.

Thus, $\hat{x}(f_{t_o}) = f_{t_0}(x) = x(t_o)$

So, if $x$ never vanishes, $\hat{x}$ never vanishes. So by Theorem 7.4, $x$ is invertible in $W$.

**Exercises**

1. Show how the Cauchy product is obtained by directly multiplying the two series and collecting like terms.
9. Finitely-Generated Banach Algebras

Let \( A \) be a Banach algebra. A subset \( K \subseteq A \) is called a set of generators for a subalgebra \( A_0 \) if \( A_0 \) is the smallest subalgebra of \( A \) containing \( K \) and \( e \). We write \( A_0 = \langle K \rangle \). If \( A = \langle K \rangle \) we say that \( A \) is generated by \( K \). If \( A \) is generated by \( K = \{x\} \), a single element, we say that \( A \) is singly generated.

**Theorem 9.1:** If \( A \) is finitely-generated by \( \{x_1, \ldots, x_n\} \) then \( M(A) \) is homeomorphic to a compact subset of \( \mathbb{C}^n \).

Proof - The map

\[
\phi : M(A) \to \mathbb{C}^n
\]

\[
f \mapsto (\hat{x}_1(f), \hat{x}_2(f), \ldots, \hat{x}_n(f))
\]

is continuous because it is a map into a product space and the composition with each projection is continuous (Appendix 1, Section 4, Exercise 2). It follows that \( \phi \) is a homeomorphism onto its image by (Appendix 1, Section 5, Exercise 2).

Exercises

1. \( A \) is generated by \( K \) if and only if the set of all finite sums of the form

\[
\sum a_{k_1,k_2,\ldots,k_n} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}
\]

is dense in \( A \). Here \( a_{k_1,k_2,\ldots,k_n} \in \mathbb{C} \) and \( x_1, x_2, \ldots, x_n \in K \).

2. The Banach algebra \( C[0, 1] \) is singly-generated.

3. The Banach algebra \( C(\mathbb{T}) \) is not singly-generated.