Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false. $A$, $C$ and $M$ are square (not necessarily of the same size).

1. If $\mu_{M}(x) = x^2 - 1$, and the trace of $M$ is 0, then the size of $M$ is even.
2. The characteristic polynomial of $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ is the product of the characteristic polynomials of $A$ and $C$: $\chi_{M}(x) = \chi_{A}(x)\chi_{C}(x)$.
3. If $\mu_{M}(x) = x^2 - 1$, and the size of $M$ is even, then the trace of $M$ is 0.
4. If $A$ is diagonalizable, then so is $A^2$.
5. A $4 \times 4$ nilpotent is of rank 3 if and only if it is similar to $N_4$.

On Sylvester’s Matrix. A complex number $a \in \mathbb{C}$ is called an $n^{th}$ root of unity if $a^n = 1$, and it is called primitive if no smaller power of it is 1. Thus $-1$ is a primitive second root of unity, and $\omega = \frac{1}{2}(-1 + i\sqrt{3})$ is a primitive third root of unity.

1. Find all $6^{th}$ roots of unity in exact Cartesian coordinates (no decimals).
2. Find all the primitive ones.
3. Let $\lambda$ be a primitive one. Build a $6 \times 6$ matrix $S_6 = (s_{ij})$: $s_{ij} = \lambda^{(i-1)(j-1)}$.
4. Find $\mu_{S}(x)$, the spectrum of $S_6$ and $S_6^{-1}$.
5. Do $\circ$ and $\Box$ but by letting $\lambda$ be a primitive $6^{th}$ roots of unity in $\mathbb{Z}_7$.

**Bonus:** Do $\circ$ in $\mathbb{Z}_{19}$.

Consider $M = \begin{pmatrix} 4 & 4 & 1 & 1 & -2 & 1 \\ -2 & 7 & 4 & -2 & 1 & 1 \\ -1 & 2 & 5 & 2 & 2 & -1 \\ 0 & 0 & 0 & 6 & 6 & 0 & 0 \\ -1 & -1 & 2 & -1 & 8 & 2 & -1 \\ 1 & 1 & 1 & -2 & 1 & 4 & 4 \\ 2 & -1 & -1 & -1 & 2 & -1 & 8 \end{pmatrix}$.

1. Find its invariant factors. $\Box$ int: Nice eigenvalues.
2. Give its rational canonical form.

Let $\chi_{A}(x) = x^3 - ax^2 + bx - c$. Do the following:

1. Find $\chi_{B}(x)$ where $B = A - 2I$.
2. Find $\chi_{B}(x)$ where $B = A - mI$ where $m \in \mathbb{F}$.
3. Find $\chi_{B}(x)$ where $B = A^2$. 