Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false.

1. If $A$ is diagonalizable, then $r(A) = r(A^2)$.
2. If some power of $A$ is the identity, then $A$ is diagonalizable.
3. If $A^3 = A$, then $A$ is diagonalizable.
4. It is possible that the only nonzero idempotent in $\langle A \rangle$ is $I$.
5. If $A$ is nilpotent, then all of its eigenvalues are 0.

**On Nilpotent Matrices.**

1. Let $F$ be arbitrary. Show that if $A \in F^{n \times n}$ is nilpotent then $\text{tr}(A^i) = 0$ for $i = 1, 2, \ldots$.
2. Show that the converse fails in $\mathbb{Z}_p$ for any prime $p$.
3. Let $F = \mathbb{C}$. Let $A \in \mathbb{C}^{n \times n}$ be such that $\text{tr}(A^i) = 0$ for $i = 1, \ldots, n$. Show that if $n = 2$, then $A$ is nilpotent.
4. Let $F = \mathbb{C}$. Let $A \in \mathbb{C}^{n \times n}$ be such that $\text{tr}(A^i) = 0$ for $i = 1, \ldots, n$. Show that if $n = 3$, then $A$ is nilpotent.

**Bonus:** Prove 4 for arbitrary $n$.

Find the rational canonical form for $M^2$ where $M$ is the companion matrix to the polynomial $x^3(x - 1)^2(x + 1)$.


1. Find its rational canonical form.
2. Find its Jordan canonical form.