Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false. \( a \in \mathbb{Z}^+ \).

1. \( |a|_m^k = 2^{k+1} \).
2. If \( |a|_m = k \), then \( \varphi(m) \) is a multiple of \( k \).
3. Let \( \chi(n) \) denote the number of solutions to \( x^2 + 1 \) mod \( n \). Then \( \chi(p) > 1 \) for only finitely many primes.
4. If \( \varphi(n) = 4 \), then \( n = 5 \).
5. There are exactly 4 idempotent mod 100.

Alex is given the following problem: let \( n \in \mathbb{Z}^+ \). Then \( \alpha(n) \) is the number of ways of writing \( n \) as a sum of consecutive positive integers. Compute \( \alpha(1,000,000) \). Emulate Alex by answering the following:

1. Prove \( \alpha(9) = 3 \).
2. Compute \( \alpha(8) \), \( \alpha(12) \) and \( \alpha(15) \).
3. Let \( n = k \times d \) where \( k \) is odd. Show that one can write \( n \) as a sum with \( k \) consecutive integers and \( d \) in the middle.
4. Revisit 2 from the 3 point of view.
5. Show \( \alpha(n) \) is the number of odd divisors of \( n \).
6. Compute \( \alpha(1,000,000) \).

On Exponents.

1. Find \( \xi(30) \) and find an element of that order.
2. Find \( \xi(30^p) \) and find an element of that order for each \( n \).
3. Find \( \xi(6300) \) and \( \xi(630000) \).

On Nilpotents. Consider modulo \( m \). If a residue \( x \) satisfies \( x^k \equiv 0 \) for some \( k \), then \( x \) is called nilpotent. Certainly 0 is nilpotent.

1. Find all nilpotents mod 36.
2. Find all nilpotents mod 100.
3. Find all nilpotents mod \( p^\ell \) where \( p \) is a prime.
4. Let \( m = p_1^{\ell_1} p_2^{\ell_2} \cdots p_\ell^{\ell_\ell} \) be factored into primes. Which residues are nilpotent? Prove your answer.
5. Is the number of nilpotents a multiplicative function?
6. How many nilpotents mod \( p^\ell \)?

Bonus. Sylvester Revisited. Let \( a, b > 1 \) be relatively prime integers. Most of the proof of Sylvester’s theorem can come from studying the function:
\[ T(x) = \frac{1}{1-x} - \frac{1-x^{ab}}{(1-x^a)(1-x^b)}. \]

1. Prove that \( T(x) \) is a polynomial of degree \( ab - a - b \).
2. Use \( \text{L'Hôpital's Rule} \) to compute \( T(1) \).
3. Compute \( T'(1) \), the derivative at 1.