Homework #3

1. Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false. As usual, \( a \) is a positive integer.
   ① If \( a^n - 1 \) is a prime for some positive integer \( n \), then \( a = 2 \).
   ② \( 4444^{3333} + 3333^{4444} \equiv 0 \mod 7 \).
   ③ Any palindrome is a multiple of 11.
   ④ \( 933354738245187864 \) is a multiple of 36.
   ⑤ If \( m | n \) then \( a^m - 1 \) is a factor of \( a^n - 1 \).
   ⑥ \( 5x \equiv 30 \mod 40 \) if and only if \( x \equiv 6 \mod 8 \).

2. The twins Alex and Jamie start jogging from the same place at the same time on a circular track. Alex, the faster one, circles the track every five and a half minutes while Jamie is half a minute slower. They will keep jogging that way until they meet again at the start.
   ① How long will they jog?
   ② Including the starting point, in how many different places along the track will they meet?

3. Induction vs. Mods.
   ① Use induction to prove for every integer \( n \geq 0 \), 27 divides \( 2^{5n+1} + 5^{n+2} \).
   ② Use congruences to prove the same claim.

4. Use the squaring algorithm to do each of the following computations (you have to show your work, so forget the TI-89’s):
   ① The last two digits of \( 7^{30} \).
   ② The last three digits of \( 11^{40} \).
   ③ \( 135 \mod (123^{135}, 97) \).