Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false. Let \( \mathbf{u} \neq \mathbf{0} \).

1. If \( \mathbf{u} \) is an eigenvector of \( \mathbf{A} \), then so is \( 2\mathbf{u} \).
2. If \( \mathbf{u} \in \mathbf{N}(\mathbf{A}) \) and \( \mathbf{u} \neq \mathbf{0} \), then \( \mathbf{u} \) is an eigenvector for \( \mathbf{A} \) with eigenvalue 0.
3. If \( \mathbf{u} \) is an eigenvector for \( \mathbf{A} \) for the eigenvalue 1, then \( \mathbf{u} \in \mathbf{N}(\mathbf{A} - \mathbf{I}) \).
4. If \( \mathbf{A} - 3\mathbf{I} \) is not of full rank, then there is always an eigenvector of \( \mathbf{A} \) for 3.
5. If \( \det(\mathbf{A}) = 1 \), then 1 is an eigenvalue for \( \mathbf{A} \).
6. If \( \det(\mathbf{A}) = 0 \), then 0 is an eigenvalue for \( \mathbf{A} \).

Short Computations. All of the following do not involve long computations:

1. Decide if 5 is an eigenvalue of \[
\begin{pmatrix}
6 & 2 & 3 \\
4 & 10 & 6 \\
7 & 8 & 14
\end{pmatrix}
\]

2. Is \[
\begin{pmatrix}
0 \\
0 \\
-5 \\
4
\end{pmatrix}
\]
an eigenvector of \[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 4 & 6 & 8 & 10 \\
3 & 5 & 6 & 0 & 0 \\
-1 & 2 & 0 & 0 & 0
\end{pmatrix}
\]

3. Let \( \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 1 & 4 \end{pmatrix} \). The vectors \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \), \( \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} \) and \( \begin{pmatrix} -3 \\ -3 \\ 2 \end{pmatrix} \) are eigenvectors of \( \mathbf{A} \). Find a matrix \( \mathbf{P} \) such that \( \mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \).

Let \( \mathbf{A} = \begin{pmatrix} 7 & 7 & 7 \\ -5 & -7 & -9 \\ 5 & 7 & 9 \end{pmatrix} \). Decide which of the following vectors is an eigenvector of \( \mathbf{A} \). If yes, find the eigenvalue.

4. \[
\begin{pmatrix}
-1 \\
1 \\
-1
\end{pmatrix}
\]
5. \[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix}
\]
6. \[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]
7. \[
\begin{pmatrix}
-2 \\
2 \\
-2
\end{pmatrix}
\]
8. \[
\begin{pmatrix}
1 \\
-2 \\
1
\end{pmatrix}
\]

Let \( \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 0 \end{pmatrix} \). Find its characteristic polynomial, and its eigenvalues. No shortcuts here.
The female population of a country is divided into 7 age groups, and two statistics are recorded as follows: the **Survival rate** is the fraction of females in each age class expected (in other words, on the average) to survive to the next age class, and the **Fecundity rate** is the average number of daughters born to a single female while in its particular class.

<table>
<thead>
<tr>
<th>Age</th>
<th>0-10</th>
<th>10-20</th>
<th>20-30</th>
<th>30-40</th>
<th>40-50</th>
<th>50-60</th>
<th>60-70</th>
</tr>
</thead>
<tbody>
<tr>
<td>Survival</td>
<td>.953</td>
<td>.994</td>
<td>.990</td>
<td>.983</td>
<td>.963</td>
<td>.915</td>
<td>---</td>
</tr>
<tr>
<td>Fecundity</td>
<td>0</td>
<td>.259</td>
<td>1.099</td>
<td>.699</td>
<td>.154</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Suppose we had the following population distribution today.

<table>
<thead>
<tr>
<th>Age</th>
<th>0-10</th>
<th>10-20</th>
<th>20-30</th>
<th>30-40</th>
<th>40-50</th>
<th>50-60</th>
<th>60-70</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>3,546</td>
<td>3,113</td>
<td>2,897</td>
<td>2,564</td>
<td>1,987</td>
<td>1,567</td>
<td>1,123</td>
</tr>
</tbody>
</table>

Let \( \mathbf{u} = \begin{bmatrix} 3546 \\ 3113 \\ 2897 \\ 2564 \\ 1987 \\ 1567 \\ 1123 \end{bmatrix} \) represent the current populations in each of the 7 groups.

1. Compute the vector \( \mathbf{v} \) that would represent the populations in the different groups 10 years from now. For example, the second entry should be \( .953 \times 3546 \approx 3379 \).
2. Give a matrix \( \mathbf{A} \) such that \( \mathbf{Au} = \mathbf{v} \) where \( \mathbf{v} \) is as in 1.
3. Use your calculator to speculate what as happens to \( \mathbf{A}^n \) as \( n \) gets large.
4. Give an educated guess as to what is happening to the population in the long run and explain your reasoning.

Consider now a different country where the rates are as follows:

<table>
<thead>
<tr>
<th>Age</th>
<th>0-10</th>
<th>10-20</th>
<th>20-30</th>
<th>30-40</th>
<th>40-50</th>
<th>50-60</th>
<th>60-70</th>
</tr>
</thead>
<tbody>
<tr>
<td>Survival</td>
<td>.990</td>
<td>.997</td>
<td>.996</td>
<td>.993</td>
<td>.980</td>
<td>.952</td>
<td>---</td>
</tr>
<tr>
<td>Fecundity</td>
<td>0</td>
<td>.022</td>
<td>.532</td>
<td>.209</td>
<td>.009</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Give a matrix that will generate the future populations and use it to answer questions 3 and 4.