Consider the veracity or falsehood of each of the following statements. For bonus, argue for those that you believe are true while providing a counterexample for those that you believe are false.

1. It is possible for $A$ to be a nonsquare matrix and $AB$ to be invertible for some matrix $B$.

2. If $A$ is $10 \times 12$, and has 10 pivots, then $Ax = b$ has infinitely many solutions for any $b$.

3. A solution to $Ax = b$ where $A = \begin{bmatrix} 2 & 8 & 0 & 6 & 8 \\ 8 & 0 & 6 & 8 & 1 \\ 12 & 8 & 0 & 4 & 5 \\ 11 & 12 & 0 & 0 & 9 \end{bmatrix}$ and $b = \begin{bmatrix} 23 \\ 23 \\ 29 \\ 32 \end{bmatrix}$.

4. It is possible for $Ax = b$ to have infinitely many solutions for some $b$, and $A$ be $5 \times 3$.

5. If $A$ is square, then $Ax = b$ always has a solution.

6. If $r(A) = 8$ and $A$ is $10 \times 12$, then $Ax = b$ cannot have a unique solution.

7. If $r(A) = 10$ and $A$ is $10 \times 12$, then $Ax = b$ will have infinitely many solutions for any $b$.

Consider the system of equations

$$
5x_1 + 2x_2 + 8x_3 + x_4 + 3x_5 = 13
$$

$$
3x_1 = 5x_3 
$$

$$
3x_1 + 3x_5 = 9 + 3x_4 
$$

$$
2x_2 + 2x_5 = 7 + 3x_1 + x_3 
$$

$$
4x_3 = x_1 
$$

and write it in the form $Ax = b$ making clear what $A$, $x$ and $b$ stand for.

Each of the following is the augmented matrix of a system of linear equations. Proceed in the following order of decisions.

A. If at any time, you can decide that a system has no solutions, just state so, and stop—you need not go further.

B. If a system is not reduced, then finish reducing.

C. If when reduced, the system has a unique solution, give that solution as a vector.

D. If when reduced, the system has multiple solutions, give two solutions as vectors.
A store sells almonds for $6 per pound, cashews for $5 per pound and peanuts for $2 per pound. One week the manager decides to prepare 100 1-pound packages of nuts by mixing the peanuts, cashews and almonds. Each package will be sold for $4. The mixture is to produce the same revenue as selling the nuts separately.

1. Write down the system of equations which captures this information and use Gaussian elimination to reduce it.

2. Write down at least 3 solutions which make sense to this system.

A message has been encoded using the matrix

\[
M = \begin{pmatrix}
2 & 1 & 3 & 1 \\
5 & -1 & -1 & 1 \\
2 & 2 & 1 & 1 \\
1 & 3 & 2 & 1
\end{pmatrix}
\]

The encoded message is

\[
\begin{array}{cccccccccccccc}
X & H & L & T & A & B & U & T & F & M & Y & Q \\
G & D & R & E & S & V & G & E & Z & X
\end{array}
\]

Your assignment (should you choose to accept it) is to decode it.

**Extra Problems**

6. Find all solutions to \( Ax = b \) where

\[
A = \begin{pmatrix}
1 & 2 & 4 & 3 \\
3 & 6 & 14 & 11 \\
11 & 22 & 50 & 39 \\
12 & 24 & 58 & 46
\end{pmatrix}
\]

and \( b = \begin{pmatrix}
29 \\
101 \\
361 \\
418
\end{pmatrix} \).

7. A psychologist is preparing some white rats for an experiment. The experiment is to determine the effect of diet on learning, and the specific question is whether a low carbohydrate-high protein diet will produce different learning curves than a high carbohydrate-low protein diet. Two groups of test animals will be used. Each group is to receive 10 grams (g) of food per day. For the sake of simplicity and manageability, we will assume the diet consists of three foods: wheat meal, fish meal and tomatoes (partially
dehydrated). Each gram of wheat meal contains 0.6 g of carbohydrates and 0.05 g of protein. Each gram of fish meal contains 0.8 g of protein and basically no carbohydrates. Each gram of tomatoes contains 0.4 g of carbohydrates and 0.05 of protein. The animals in group I are to receive 2 g of carbohydrates and 5 g of protein per day while those of group II are to receive 5 g of carbohydrates and half a gram of protein. What should the composition of the diets be?