A First Course in the History of Mathematics

by

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CSULB
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Preface

The measurement of everything is human—Protagoras

Before we start we would like to make some disclaimers.

Disclaimer 1 Although the style of the course is one of focusing on some of the individuals in the history of mathematics, we deeply believe that although individuals matter greatly in history, there are other forces—intellectual, material or economical—that are much more powerful. Explicitly, for example, we will state that Calculus was discovered (or created, or invented) by Newton and Leibniz independently, but actually we do not have the time to indicate how much their ideas were based on previous work by a multitude of others, and that these ideas behind their creation were around for at least half a century before either was born. Ultimately, we believe that if Leibniz and Newton had not existed, Calculus would have been discovered (or created, or invented) in any case.

Disclaimer 2 Mathematics is one of the oldest intellectual pursuits of mankind, and as such, we do not have the time to explore its complete history. We will stop at the end of the 17th century, but even so we do not cover all the cultures equally. Some ancient cultures have more pertinent materials readily accessible to us than other cultures. This is particularly true of the Hindus and Islam, who made major contributions to Western Mathematics, yet what is accessible to us is rather limited. Furthermore, throughout the times all cultures have had to develop and use mathematical tools, yet we do not have time in one semester to discuss, except briefly, many of them, including the Chinese who have a long and illustrious history in mathematical activity. We regret this shallowness.

Disclaimer 3 There are no women among the list of personalities that we study in the course. First, we restrict ourselves to roughly the top 50 names in the history of this old subject during the period, hence the sieve is very severe. Second, social conditions have occurred through the ages where women were not encouraged to participate actively in the pursuit of mathematics and other intellectual activities (for example, if we did the history of art, we would be in a similar dilemma.) Fortunately, some of these conditions have changed, and this course has a sequel that covers the nineteenth and twentieth centuries, where more of a gender diversity occurs.

Disclaimer 4 Notation has been crucial in the development of mathematics. We will pay very shallow attention to it. Although we will try to understand some of the modes of thinking in the past, we will not have the time, often, to present proofs that are in the original mode of thinking, nor in the original notation.
There are to be many proofs in this course, and a wide variety of techniques will be used. **What is a proof?** Very loosely put, a proof is a process by which the speaker or writer attempts to convince others of some claim. It naturally does depend on the audience, and the proponent should be aware of this. The basic structure of a proof is: *given these premises, let me show you why this (a new, perhaps unexpected) consequence follows.* Although most of the time the conclusion is clear, the assumptions may be hard to pin down. However, as long as the audience accepts the steps we may be safe. Personally, I have usually found that if I can honestly convince myself, then I can convince others. The key word is **honestly.** It is often the case that the frustration that goes hand-in-hand with the process of discovery will gnaw at one's core of integrity, and then crucial points will be sloughed off just to get the process over. A brief example of the venting of this frustration is the statement by Saccheri (1667-1733) who at the time was very close to developing non-Euclidean geometry, yet frustrated he proclaims:

**The Hypothesis of the Acute Angle is absolutely false, being repugnant to the nature of a straight line.**

The lack of objectivity in the claim is apparent since no explanation for it is given except **being repugnant.** The nature of argumentation has changed through the years and although some seem comical (like the one given below), we should always keep the perspective that what we are doing is not absolute. We mention a historical case. **Galileo Galilei** (1564-1642) was one of the first scientists to use the telescope. He used it both to make money with and to look up in the skies. An amazing thing happened when he looked up. From ancient times people had believed that there was the Earth, the fixed stars and seven wandering stars (or planets): **the Sun, the Moon, Mercury, Venus, Mars, Jupiter and Saturn.** But when Galileo looked up at Jupiter, he saw that it had satellites of its own. When he proclaimed this to his world, many people were upset and even refused to look up through the telescope. A contemporary astronomer of Galileo gave the following **proof** of why Jupiter had no satellites:

There are seven windows in the head, two nostrils, two ears, two eyes and one mouth; so in the heavens there are two favorable stars, two unpropitious, two luminaries, and Mercury alone undecided and indifferent. From which and many others similar phenomena of nature such as the seven metals, etceteras, which it were tedious to enumerate, we gather that the number of planets is necessarily seven... Moreover, the satellites are invisible to the naked eye and therefore can have no influence on the earth and therefore would be useless and therefore do not exist.
Chapter 1
Roots

As one begins to contemplate the history of mathematics, one begins to appreciate the vastness of its panorama. Mathematics, together with music and art, must be among the first intellectual pursuits of mankind. Mathematics certainly antedates writing and the dawn of history—but even further, there is evidence that it antedates language. In this chapter, ideas and notions that occurred early, in prehistory, before any written records that have survived, are discussed. Mathematics has been referred to as the study of number, shape, arrangement, change and chance. It is, certainly, the first two of these, number and shape, that occur earliest in the human experience. It is known, for example, that evidence exists of counting and tallying before writing.

In Nigeria, Africa, a bone (with 29 scratches) was found, which indicates tallying activity as far back as the year 6,000 BC. It has been speculated that the Ishango Bone, as it is known by, represents the tallying of a lunar month. The picture on the right should help visualize how old this evidence is as we compare it with two other monumental events in the history of mankind, the advent of writing in the third millennium BC, and the invention of printing, approximately 500 years ago. Recently, even older evidence of counting has been found in Europe.

A little reflection will in fact make one realize that measurement of both space and time must have provided early and powerful incentives for mathematical activity. In fact, hunters if interested in hunting at night must have been soon interested in the motion of the moon, as darker nights made the hunting easier. Thus, one can speculate that the measurement of time led possibly to some of the first concepts of numbers. Or maybe, it was space, and the counting of children, or sheep, or soldiers that led to the idea of number. In any case, evidence exists of very early arithmetical activity, and it may even be true that symbols for numbers appeared before any other written language.

Since the measurement of time, in particular, the calendar, was one of the most universal stimulators of arithmetical activity, we will take a small detour and discuss some of the basic facts about it.
The Calendar

The Sun and the Moon have played a pivotal role in nearly all cultures. Most of the early units for the measurement of time are involved with at least one of these two very visible astral bodies. We will review these words one at a time in the order of ease of perception.

The Day is the most easily perceived as the motion of the Sun around the Earth. Thus it is Solar in origin, and there is a universal understanding of day. We have speculated before that the counting of days is one of the first origins of counting.

Next in ease of perception is the Month, which, as its name alone indicates, is Lunar in origin. The Month is the cycle of the Moon as it travels from Full to New and Full again. It is roughly 29 ½ days. Many cultures have based their calendar on the month, and then, these are called Lunar Calendars. Most importantly, the Mesopotamians used such a calendar, and their influence persists today in both the Jewish and the Arabic calendars, but independently, the Chinese also use a lunar calendar. More will be said about Lunar Calendars once the most important of the words has been discussed.

The Year is more easily perceived in higher latitudes where the length of days has the higher variation. Nevertheless, the ancient Egyptians, as well as many other ancient cultures, developed a firm understanding of its duration. The importance of the year for them was the yearly, regular flooding of the Nile. With some thought, and considerable patience, one can estimate the length of the year by using a stick firmly planted, and each day when the Sun is at its highest, marking the shadow of the stick on the floor. If one does this regularly, one will see a pattern of shadows on the floor that which will eventually cycle—return to its starting point, and thus one would have calculated the length of the year, namely the duration of the long cycle of the sun around the earth from the ancient point of view.

This would give the well-known 365 days without too much difficulty, but to approximate further takes significantly more energy and patience. We know now that a year is approximately 365 ¼ days. The Egyptians indeed were aware that the year was 365 days long—but for a long time did not have the quarter of a day fraction. A vast amount of computation and mathematics has been developed to deal with the approximations caused by the lengths of both the Month and the Year. For many centuries, the leading mathematicians in many cultures were the leading astronomers. This was the case in India, for example, as we will see below.

Of course, we have 12 months in a year because 12 is the closest one can come to fitting
in 29 ⅝ days into 365 days. The importance of the seasonal return of the weather and how to adjust it to the Lunar Calendar has been dealt with in a variety of ways. In some cultures an extra month is added to the year to balance the year, and hence some years will have 13 months in the Jewish and Arabic calendar.

The leap year adjustment goes back to Julius Caesar. By his time, approximately 50 BC, the seasons were so out of tempo that it was snowing in July (which was called Quintillius before him). Thus, he sought the counsel of scientists in Alexandria (which is in Egypt), and of Sosigenes in particular. Julius, then, in typical imperial fashion, introduced the addition, every four years, of an extra day in February (which was the last month of the year in the Roman calendar). It was then that the year changed from 365 days to 365 ¼ days. He also declared the correct day to be March 1 (the first day of the Roman calendar), so the months would be in harmony with the seasons. However, the decree caused that specific year to last more than 440 days! But then the calendar was correct for quite an extended period of time—for more than 1500 years!

However, by 1582, the calendar was again running out of tune, and it was adjusted again, this time by the Pope, Gregory XIII, and thus our calendar is the Gregorian Calendar. The Gregorian modification is simple. A year is actually slightly less than 365 ¼ days. And so not every fourth year should be a leap year, and thus by the Gregorian command, century years will not be leap years unless the first two digits are a multiple of four. Thus, the year 1700 was not a leap year, but the year 2000 was. This makes the duration of the year to be 365.2425 days, which is closer to the astronomically more accurate 365.2422 days, and it is accurate up to more than 3,000 years. But it will have to be adjusted in the future!

By the time of the Gregorian adjustment, 9 days had to be deleted from the calendar to correct it. The Gregorian calendar was not adopted in the United States until 1752 since it was a Catholic idea, and by then 11 days had to be deleted. Perhaps it is not well known that George Washington was actually born on February 11, 1732?

We can see that all this needed calculation leads to dexterity with numbers. Calendric considerations have played a role with developing notation to improve the computations among other things. We discuss briefly the Mayans of Southern Mexico and Guatemala.

The Mayans did not use the motions of the Moon as a foundation for their months. Their months had 20 days—and 20 was also their base. Base 20 is a rare occurrence since, as expected, the most common base among cultures is 10. Yet, one can conjecture that the hot climate led to such a basis, and, not coincidentally, cultures in West Africa, the Yoruba, for example, also used base 20.
The Mayans had an outstanding calendar and excellent number notation. They inherited ingredients for both from the Olmecs (the Aztecs inherited their calendar from the Olmecs also), but by the year 250 BC, the Mayans had already made significant improvements on both of them. In fact, some modern day authors give the Mayan calendar a duration of 365.2420 days, which is even closer to the current measurement that the Gregorian 365.2425.

The Mayan calendar consisted of two parts, the Tzolkin and the Haab. Their word for day was kin. The Tzolkin was a cycle of 260 days—which is close to the length of human gestation. It was divided into 13 groups of 20. The Haab consisted of 365 days divided into 18 months of 20 days each with an additional 5 days of omen and religious significance. They adjusted the calendar by using their very accurate placement of Venus in the sky. They were aware that Venus is both the morning star and the evening star—a fact that many cultures had no knowledge of.

Hence to the Mayans, who believed in the cyclical essence of nature, one of their cycles consisted of 18,980 days. This is the length of time it will take for both calendars to restart—in modern days one would notice that 18,980 is the least common multiple of 260 and 365. But they had other and longer cycles, one of over 1 million days. Eventually, they believed the world would end on December 24, 2011.

Since the Mayans could manipulate large numbers, one would expect them to have had a good number notation. Indeed they did. Their number system used two symbols:  stood for 1 (think of a finger perpendicular to one's eye) and the symbol: | or — which stood for 5, (think of a hand perpendicular to one's eye). Thus to write 7, they would simply write ___. They also used a positional notation, thus ___ stood for thirteen of the base, 20, plus seven units, in other words, it represented 267. However, they did not use the positional notation as consistently as we do. The third digit, instead of $20^2 = 20 \times 20 = 400$, represented the more calendrically correct $18 \times 20 = 360$. Thus, ___ __ __, stood for $14 \times 360 + 13 \times 20 + 7 = 5,307$. But, after that digit they would go back to the traditional times 20, so the fourth digit stood for $20 \times 18 \times 20$.

More surprisingly, the Mayans had a placeholder—they had a zero. Although we obtained ours from India (see the table below), we have to pay compliment to a rare achievement indeed, one that many cultures failed to reach. They used a shell-like symbol for 0. We will use ⨇, although their glyphs were much more elegant. For the Mayans, then, ___ ⨇ ___ stood for

$$13 \times 360 + 7 = 4,687.$$  

We have just begun to understand the subtleties of Mayan mathematics since their hieroglyphics were not decoded until the 1980’s. One of the reasons for the delay in deciphering was the lack of Mayan documents. Most of them were destroyed by Bishop Landa shortly after the conquest of Yucatan.
The table on the right illustrates the ubiquitous search for the length of the year by every culture, and at all times, from ancient to modern.

<table>
<thead>
<tr>
<th># of days</th>
<th>The number of days in the year in</th>
<th>Where</th>
<th>When</th>
</tr>
</thead>
<tbody>
<tr>
<td>365</td>
<td>most ancient solar calendars</td>
<td>Egypt</td>
<td>Ancient</td>
</tr>
<tr>
<td>365.25 or 365 1/4</td>
<td>the Julian Calendar</td>
<td>Rome-Alexandria</td>
<td>First Century BC</td>
</tr>
<tr>
<td>365.244407 or 365 743/3640</td>
<td>the Chinese Calendar</td>
<td>China</td>
<td>Eighth Century</td>
</tr>
<tr>
<td>365.2420</td>
<td>the Mayan Calendar</td>
<td>Mexico-Guatemala</td>
<td>Ninth Century</td>
</tr>
<tr>
<td>365.2424… or 365 1/11</td>
<td>Omar Khayyam’s Calendar</td>
<td>Persia</td>
<td>Eleventh Century</td>
</tr>
<tr>
<td>365.2425 or 365 37/400</td>
<td>the Gregorian Calendar</td>
<td>Rome</td>
<td>Sixteenth Century</td>
</tr>
<tr>
<td>365.242193</td>
<td>today’s measurements</td>
<td>The World</td>
<td>Twentieth Century</td>
</tr>
</tbody>
</table>

The Roman influence on the calendar is much more pervasive than just the addition of leap years.

In fact, the names of the months were originally given to us by the Romans. Of course, the month of March was named after their most powerful god, Mars, and it indicated the beginning of their year. Other months named after some gods and deities in the Roman Pantheon are April, May, June and January. The last month of their year, February, was considered a somber period of self-examination and reflection.

July and August are named after Julius and Augustus (his avenger and successor), but the last four carry number names September, 7; October, 8; November, 9; and December, 10. One may notice that the numbers are not correct for us, but they are indeed correct in the Roman count—in fact, if the year were to start in March, December would be the tenth month.

Of the remaining units of measurement of time, week, hour, minute and second, most were created for arithmetical convenience. The hour was a twelfth of the night and a twelfth of the day. The reasons for 12 are not known. Perhaps the Egyptians chose 12 because of 12 months in a year, or perhaps because several crucial fractions, such as 1/12, 1/3 and 1/4, of 12 can be taken painlessly. Thus the day ended with 24 hours. The minute and the second are Babylonian in nature, and 60 was their base (see the next chapter).

Finally, we arrive at the week. The week of seven days is far from universal across the planet. To the Romans, it was 8 days. To the Revolutionary French, it was 10 days. But in our society it consists of 7 days. Why 7?

Could it be that that each week corresponds to a fourth of the lunar cycle? We have, of course, the Judeo-Christian tradition of a week, and certainly this is a major component for its survival to our times. Further back are the Babylonians, and the very ancient belief
of the seven planets in the skies. As we mentioned in the Preface, from ancient times people had believed that there was the Earth, the fixed stars and seven wandering stars (or planets): the Sun, the Moon, Mercury, Venus, Mars, Jupiter and Saturn.

As we saw before, there was great deal of reluctance to add to the list of wandering stars. We saw in the Preface an argument that was launched against Galileo by a contemporary scholar. A more complete version of the argument is:

There are seven windows in the head, two nostrils, two ears, two eyes and one mouth; so in the heavens there are two favorable stars, two unpropitious, two luminaries, and Mercury alone undecided and indifferent. From which and many others similar phenomena of nature such as the seven metals, etceteras, which it were tedious to enumerate, we gather that the number of planets is necessarily seven... Moreover, the satellites are invisible to the naked eye and therefore can have no influence on the earth and therefore would be useless and therefore do not exist. Besides, the Jews and other ancient nations, as well as modern Europeans, have adopted the division of the week into seven days and have named them after the seven planets. Now if we increase the number of planets, this whole and beautiful system falls to the ground.

Note the reference to the planets and the week. To be more specific, the belief from ancient times until the Copernican revolution was that the planets went around the Earth. Furthermore, they believed the distance from each planet to the Earth was inversely proportional to the planet’s apparent velocity in the skies as viewed from Earth—the faster they moved, the closer they were. Given this, their placement of the planets was

The names of the days in English come from the Saxon, so the translation from the planets to the days of the week is not so obvious. Nevertheless, the Saxons had identified their gods and goddesses with Roman deities, and this identification makes the connection clearer. Even better, if one were to use one of the romance languages such as French, Italian or Spanish, the connection is apparent.
In the table, a list of the English and Spanish names for the days of the week together with their connections to the planets is given. As we saw above, each of the planets had a sign associated with it.

<table>
<thead>
<tr>
<th>Planet</th>
<th>Day</th>
<th>Spanish</th>
<th>Saxon</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun</td>
<td>Sunday</td>
<td>Domingo</td>
<td>Sun</td>
</tr>
<tr>
<td>Moon</td>
<td>Monday</td>
<td>Lunes</td>
<td>Moon</td>
</tr>
<tr>
<td>Mars</td>
<td>Tuesday</td>
<td>Martes</td>
<td>Tiw</td>
</tr>
<tr>
<td>Mercury</td>
<td>Wednesday</td>
<td>Miércoles</td>
<td>Woden</td>
</tr>
<tr>
<td>Jupiter</td>
<td>Thursday</td>
<td>Jueves</td>
<td>Thor</td>
</tr>
<tr>
<td>Venus</td>
<td>Friday</td>
<td>Viernes</td>
<td>Frigg</td>
</tr>
<tr>
<td>Saturn</td>
<td>Saturday</td>
<td>Sábado</td>
<td>Saturn</td>
</tr>
</tbody>
</table>

But if the order of the planets in the skies was Saturn, Jupiter, Mars, the Sun, Venus, Mercury and the Moon, how do we arrive to our present order for the days of the week? To understand this last ingredient to the puzzle, we have to be aware that each hour of the day was associated with a deity, and that the day was named after the god or goddess assigned to the first hour of that day. Suppose then our day starts with an hour of the Sun. Let us look to which deity each of the following hours will it be dedicated to:

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun</td>
<td>Venus</td>
<td>Mercury</td>
<td>Moon</td>
<td>Saturn</td>
<td>Jupiter</td>
<td>Mars</td>
<td>Sun</td>
<td>Venus</td>
<td>Mercury</td>
<td>Moon</td>
<td>Saturn</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jupiter</td>
<td>Mars</td>
<td>Sun</td>
<td>Venus</td>
<td>Mercury</td>
<td>Moon</td>
<td>Saturn</td>
<td>Jupiter</td>
<td>Mars</td>
<td>Sun</td>
<td>Venus</td>
<td>Mercury</td>
</tr>
</tbody>
</table>

So the first hour of the next day would be dedicated to the Moon, thus, Monday follows Sunday. If we repeat this with each of the following days we get our traditional order for the days of the week!

We cannot leave this entertaining section on the Calendar without discussing briefly a more modern attempt to modify time measurement. One of the last major attempts to modify all the units of time occurred during the French Revolution—and while they succeeded with their changes concerning measurement for distance and mass, namely the Metric System, their attempts at time measurement modification lasted but a few decades.

Their changes were indeed radical: the year was to consist of 12 months of 30 days each (with 5 special days similar to the Mayan tradition). The month would consist of three weeks each of 10 days. The National Convention (who took over the running of the government) declared September 22, 1792 (the fall equinox) to be the first day of Year 1 of the Republic of France. What perhaps would be one of the ingredients that would inevitably doom the system were the names of the months, all which would be connected with weather or agricultural conditions, with no ties to the Romans or to the Pope.
The French Revolutionary Calendar

<table>
<thead>
<tr>
<th>Autumn</th>
<th>Winter</th>
<th>Spring</th>
<th>Summer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vendémiaire</td>
<td>Nivôse</td>
<td>Germinal</td>
<td>Messidor</td>
</tr>
<tr>
<td>Brumaire</td>
<td>Pluviôse</td>
<td>Floréal</td>
<td>Thermidor</td>
</tr>
<tr>
<td>Frimaire</td>
<td>Ventôse</td>
<td>Prairial</td>
<td>Fructidor</td>
</tr>
</tbody>
</table>

Note that the ending of the name of the month indicated the season. We now return to our main task at hand, that of looking at the deep roots of the earliest components of mathematics, number and shape, arithmetic and geometry.

**Arithmetic**

God made integers, all else is the work of mankind—Kronecker

One could perhaps improve on the nineteenth century German mathematician’s saying by exchanging the word *integers* by the words *counting numbers*. Let us start by reviewing briefly the complex history of the different types of numbers that inhabit our modern zoo.

As aforementioned, the first numbers to occur are naturally the *positive integers*, also called the *counting numbers*, or the *natural numbers*—without a zero. The idea of such numbers as mentioned above is old, and different cultures have had very different notations for representing these, the most basic of all mathematical ingredients. We have seen on such notation already, and we will briefly study two others later on.

It is impossible to tell when these numbers first appeared, but they are much older than any historical records. As mentioned above, recent evidence suggests that the first symbolic writing of mankind may have been to indicate counting. Additionally, the management not only of numbers, but also of very large numbers is old. We mention the Incas of South America who kept great accounting records of a large empire without ever developing a written language!
Some may be surprised to learn that the zero comes about roughly fifteen hundred years ago—thousands of years after the positive integers. One reason may be that there was no need for such a concept, and much less for a symbolic representation of it. Our zero comes from India, although as we saw before, the Mayans also had the concept.

\( \mathbb{N} \) is the set of counting numbers, a very natural, intuitive set. We will agree to include 0, \( \mathbb{N} = \{0,1,2,3,4,\ldots\} \), although historically that is far from essential. The notation we use to indicate collections of numbers is very common in mathematics courses, but some of its use is limited to such courses. Most of the notation comes from nineteenth century Germany. \( \mathbb{N} \) could be for number, but it could also be for natural. As observed above, the human intuition of these numbers comes from very ancient times, but that is not the case for the marvelous notation we use to describe them.

We refer to our numeral system as Hindu-Arabic, after the two cultures from which we inherited it in the late Middle Ages. We have already encountered two of the most crucial ingredients to our numeral system: base & position. The notion of base is ancient, and as in our system, base ten is among the most common through various cultures, although base 2, base 5, base 12 and base 20 also occur in one form or another, as we saw with the Mayans. The more subtle power is in the position idea, namely that 15 does not mean the same as 51—which although harder to learn, is much more powerful in the long run. We will discuss this idea further later on.

The next collection of numbers to appear in the human landscape, the positive rationals, lie but a small jump from the positive integers. They also occur in pre-history, and they are easily justified as a change-of-units idea. Hence, when we say \( \frac{1}{3} \), we call 2 the numerator, or counter, and 3 the denominator, or labeler. Our notation for the ratio of two positive integers is not ancient, and should not be taken for granted. Both the Egyptians and the Greeks would have benefited from such notation. There is no traditional symbol to denote the collection of positive rationals, although \( \mathbb{Q}^+ \) could be used—see below.

The Babylonians, more than 3,000 years ago, introduced something similar to our decimal notation, 3.14159 for example, except that their base was 60, and they are called hexagesimals. Hexagesimals will be one of the prevalent notations in Europe until the modern version of decimals is introduced roughly 400 years ago. We will look at hexagesimals in more detail in the next chapter.

The harsh realization that rational numbers were not enough to express all ideal lengths came early in the history of mathematics (more than 2000 years ago). Namely, if one wants to associate a number with each point on the line, then there are some natural lengths that aren't rational. In particular, as in the picture, the diagonal of a square of side 1. And thus irrational numbers were born. Among the earliest ones are \( \frac{1 + \sqrt{2}}{2} \), and \( \sqrt{2} \), which is the length of the diagonal in the picture. So in
order to correspond a number to each point of the line, the real numbers were created. But that does not really occur until the seventeenth century. However, satisfactory understanding of the real number system does not occur until the nineteenth century! One way to think of the real numbers: real numbers are points of the line.

Negative numbers are even more recent, and although operating in the red appears early in the history of Western accounting (14th century), and even earlier in the Chinese culture, negative numbers are not universally accepted even by the end of the seventeenth century. Brilliant minds—including Pascal and Descartes—would dispute their need or their existence as late as the 1600’s. To Descartes, one of the creators of coordinate axes, there would be only one quadrant of what we refer to as the Cartesian plane, the first quadrant. He was so opposed to them he even referred to negative numbers as imaginary. Later on, many would have problems with what a minus times a minus is in the eighteenth century.

Once negative integers are accepted, then we have the integers, or whole numbers. One uses $\mathbb{Z}$ for this collection, $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$. $\mathbb{Z}$ is for zahlen, which means numbers in German. And then all rational numbers, positive and negative is denoted by $\mathbb{Q}$. $\mathbb{Q}$ could stand for quotient, but it could also be conjectured that $\mathbb{Q}$ comes before $\mathbb{R}$. And $\mathbb{R}$ is the set of real numbers, the real line.

Those pesky numbers that most people ignore, those embarrassing objects called complex numbers started appearing in Europe as early as the 1500’s. But, it was not until the 1800’s that they became widely accepted by the mathematical community. And by the 20th century, they have become necessary for engineering. $\mathbb{C}$ denotes the set of complex numbers.

Finally, hypernumbers (now more commonly referred to as matrices) appear in the middle part of the nineteenth century, and become overwhelmingly important in the second half of the 20th century.

The following table summarizes the history of the different components of the number system:

<table>
<thead>
<tr>
<th>NUMBERS THROUGH the AGES</th>
</tr>
</thead>
<tbody>
<tr>
<td>From Oldest to Youngest</td>
</tr>
<tr>
<td>Counting Numbers: 1, 2, 3, 4, 5, 6, ...</td>
</tr>
<tr>
<td>Positive Fractions: $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \ldots$</td>
</tr>
</tbody>
</table>

$^1$These times are grossly simplified.
$^2$Very rough geographical approximations. And certainly many ideas have flourished in many places if not simultaneously, independently, of each other.
We end our section concerning arithmetic with a deeper, but short, discussion of the real numbers and complex numbers. Another way to think of the real numbers is: real numbers are arbitrary decimals, for example: 0.101001000100001000001 is a real number.

The criterion for a decimal to be a rational number is well known: a decimal is rational if it is eventually periodic, in other words, some string of digits repeats indefinitely. For example, 1.23333333… is rational since 3 is a repeating block, and indeed $1.23333333… = \frac{37}{30}$. Similarly, $17.714285714285714285\ldots$ is also a fraction, $\frac{124}{7}$.

The fundamental property that the real numbers satisfy, that the rationals do not, is the following:

- **if** $x_1 > x_2 > x_3 \cdots$ is an infinite sequence of positive real numbers, then there is necessarily a real number $x$ to which this sequence is converging, that is, the limit of the sequence exists: $\lim_{n \to \infty} x_n = x$ for some real number $x$.

This principle, not clearly enunciated until the 1800's, is one of the building blocks of modern mathematical analysis.

In the same way as $\sqrt{2}$ was necessary to solve $x^2 = 2$, the complex numbers were necessary to solve $x^2 + 1 = 0$. We let $i$ be such a solution: $i^2 = -1$. Then we begin to do arithmetic: $3i$ is just $i + i + i$, and if we add 2 to it we get $2 + 3i$, and this is a typical complex number, and it is as complex as one needs to get. Suppose we want to add 1 to $2 + 3i$, it's easy: we get $3 + 3i$. Similarly, if we want to add $i$ to $3 + 3i$, we get $3 + 4i$. In short, $(2 + 3i) + (1 + i) = 3 + 4i$. 

<table>
<thead>
<tr>
<th>Some Positional Notation</th>
<th>Hexagesimals: $\frac{1}{60}, \frac{1}{60^2}, \ldots$</th>
<th>4,000</th>
<th>Mesopotamia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Some Irrationals: $\sqrt{2}, \frac{1 + \sqrt{5}}{2}, \sqrt{2}, \pi \ldots$</td>
<td>2,500</td>
<td>Greece</td>
<td></td>
</tr>
<tr>
<td>O and Positional Notation</td>
<td>1,500</td>
<td>India</td>
<td></td>
</tr>
<tr>
<td>Negative Numbers: $-1, -2, -\frac{1}{2}, \ldots$</td>
<td>500</td>
<td>Europe</td>
<td></td>
</tr>
<tr>
<td>Decimals &amp; Real Numbers: $2.46, e, \ln(2)$</td>
<td>350</td>
<td>Europe</td>
<td></td>
</tr>
<tr>
<td>Imaginary &amp; Complex Numbers: $i, 1 + i, \sqrt{i}, \ldots$</td>
<td>200</td>
<td>Europe</td>
<td></td>
</tr>
<tr>
<td>Hypernumbers &amp; Matrices:</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{bmatrix}, \begin{bmatrix} 0 &amp; 1 \ -1 &amp; 0 \end{bmatrix}, \ldots$</td>
<td>100</td>
<td>Europe</td>
</tr>
</tbody>
</table>
What about multiplication? Just use the distributive law: 
\[(2 + 3i)(1 + i) = 2(1 + i) + 3i(1 + i) = 2 + 2i + 3i + 3i^2.\]
The only confusing term is \(3i^2\), but recalling what \(i\) is all about, \(i^2 = -1\), so \(3i^2 = -3\), hence \((2 + 3i)(1 + i) = -1 + 5i\).

Furthermore, what is very important, but yet not often stressed enough, is that complex numbers have as much a geometrical reality as real numbers. Just as every real number corresponds to a point in the line, each complex number corresponds to a point in the plane.

We have, for example, \(2 + 3i\) corresponding to the point whose Cartesian coordinates are 2 and 3 respectively.

So the \(x\)-axis corresponds to the real numbers while the \(y\)-axis are the pure imaginary, which are numbers of the form \(i, 2i, -i\), etcetera.

We now consider the historical roots of the other ancient branch of mathematics:

**Geometry**

God ever geometrizes—Plato

Together with number, shape runs deep in the history of mankind. The roots of geometry are also very old (perhaps even older than those of arithmetic). And the oldest geometrical questions had to do, again with measurement. This time of space, thus the notions of length, angle, distance, area and volume are among the first to occur.

Of course, length is very much associated with counting. Once we have a unit of linear measure, we count how many times it fits around the room, or whatever we are attempting to take the length of, and we have arrived at an estimate. Of course, fractions occur naturally in this context.

One can only use common sense to speculate what did happen, and one can easily conjecture that the oldest area to be computed was the rectangle: the base \( \times \) height formula for the area could easily be deduced via multiplication from brick laying or tiling examples.

Naturally, we use the contemporary language of formulas to express this idea; however, it is good time to remind ourselves that **formulas are not the only way of expression or communication**. There are other forms to express the result for the area of a rectangle such as **two rectangles with the same base and height have equal areas**.

Similarly, the first volume to be achieved was, most probably, that of a rectangular...
parallelepiped (a rectangular box) with the well-known \( \text{length} \times \text{width} \times \text{height} \) expression for its volume.

The next area, after the rectangle, to be computed was, probably, that of a parallelogram, which is also \( \text{base} \times \text{height} \). That this was done early follows from the easy rearrangement of any parallelogram into a rectangle.

And then the triangle could not be far behind since two of them make a parallelogram:

\[
\text{Area} = \frac{1}{2} (\text{base} \times \text{height})
\]

The more sophisticated idea of gliding a vertex of a triangle (or the side of a rectangle), so that the area does not change as we glide, was probably more recent. Thus, the three shaded triangles all have the same area because they have the same base and the same height. Whether the idea was applied first to triangles or to parallelograms is impossible to tell.

Another fundamental notion in early geometry is that of similarity. Two figures are similar if they have the same shape.

The need and realization for such a notion arises from religious ornaments and artifacts, and its prevalent occurrence in nature:

It is not clear whether ancient civilizations completely understood the relation between the area (or volume) of a figure and its linear dimensions.

Thus, for example, when the linear dimensions are doubled, the area is quadrupled. Similarly, if the linear dimensions are halved, then the volume is one eighth of the original volume. How that fundamental fact was first discerned is, probably, due again to the tiling example, as the figure shows.
Chapter 2
Mesopotamia

What science can there be more noble, more excellent, more useful for men, more admirably high and demonstrative, than this of the mathematics—B. Franklin

The two oldest major civilizations, which are often referred to as the cradle of our own Western Civilization, are the Mesopotamians and the Egyptians.

We will start our study of the history of mathematics with the study of the land between the rivers, Mesopotamia. The two rivers are the Tigris and the Euphrates, and the area roughly corresponds to present day Iraq. But civilization in the area is very ancient, and from early times the area experienced many upheavals and changes, where one group was conquered by another, yet the conquering group incorporated many of the norms, mores and culture of the conquered. Among the cultures are Sumer, Akkad, Elam and Assyria, and among the ancient cities are Ur, Nineveh, Susa and Babylon.

Mesopotamian History Highlights

<table>
<thead>
<tr>
<th>Date</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>15,000</td>
<td>Wild grains are harvested.</td>
</tr>
<tr>
<td>7,000</td>
<td>Agriculture and livestock rearing become widespread.</td>
</tr>
<tr>
<td>3,300</td>
<td>Pictographic script occurs in Uruk.</td>
</tr>
<tr>
<td>2,100</td>
<td>Ur, the Sumerian capital is at its zenith.</td>
</tr>
<tr>
<td>1,800</td>
<td>Hammurabi becomes king of Babylon.</td>
</tr>
<tr>
<td>1,750</td>
<td>The Code of Hammurabi is carved on a stele.</td>
</tr>
<tr>
<td>1,700</td>
<td>Horses revolutionize warfare.</td>
</tr>
<tr>
<td>1,600</td>
<td>The Phoenicians, in today’s Lebanon, begin to use the first alphabetic script.</td>
</tr>
<tr>
<td>1,550</td>
<td>The Hittites sack Babylon.</td>
</tr>
<tr>
<td>1,300</td>
<td>Hittites and Egyptians engage in war.</td>
</tr>
<tr>
<td>1,100</td>
<td>Nebuchadnezzar I rules Babylon.</td>
</tr>
<tr>
<td>700</td>
<td>Assyrians move their capital to Nineveh and war with Egypt.</td>
</tr>
<tr>
<td>600</td>
<td>Babylon rules an empire that reaches to the borders of Egypt.</td>
</tr>
<tr>
<td>500</td>
<td>Cyrus of Persia captures Babylon</td>
</tr>
</tbody>
</table>

All times are BC.
Mesopotamian mathematics is quite sophisticated, and their influence persists until today. When we talk about 360°, we are using a Mesopotamian idea. The Babylonians (we will use the terms interchangeably) never developed an alphabetic language like the Phoenicians, instead they used a pictographic one, and they wrote by wedging in clay tablets that were very abundant in the area. Our knowledge of their culture comes from thousands of those clay tablets, in which they wrote using their peculiar writing that is wedge-shaped (cuneiform). We will briefly discuss four Mesopotamian topics:

1. Notation
2. Pythagorean Triples
3. Quadratic Equations
4. Square Roots

The Mesopotamian notation for numbers was quite sophisticated, and it is possible that our own notation stems indirectly from theirs via meaningful and sophisticated contributions by the inhabitants of the Indian subcontinent. Writing on clay was, of course, not conducive to elaborate symbolism. Thus, not surprisingly, they used only two symbols, ▼ which had the value 1 and ▶ with the value of 10 (we are using notation that resembles their cuneiform writing). Two things however were remarkable.

As most numeric systems, they had a base (recall the Mayans base 20 notation), but their base was 60. Why 60? Possibly because 60 has so many divisors—it is the smallest multiple of 2, 3, 4, 5 and 6. Also, they would use position to indicate value. Thus, ▶▶▼ ▶▼▼▼ would represent the number twenty-one of the base and thirteen units, that is, \((21)_{60} + 13 = 1273\). However, since they had no symbol for 0, meaning had to be gathered from context often. Thus, ▶ could represent 10 (ten units), or 600 (ten of the base), or 36,000 (ten of the base to the power 2), and the meaning had to be deduced elsewhere.

But even more admirably, their notation included sophisticated decimals (in reality, they are called hexagesimals since their base was 60), and thus ▶ could also mean \(\frac{10}{60}\) (ten of the base to the power \(-1\) ), or \(\frac{10}{3600}\) (ten of the base to the power \(-2\) ), and, similarly, ▶▼▼ ▶▶▼ could represent \((12)_{60} + 21 = 741\), or \((12)_{60}^2 + (21)_{60} = 44,460\), or, \(\frac{12}{60} + \frac{21}{60^2} = \frac{741}{3,600}\). But, more enigmatically, it could mean \((12)_{60}^2 + 21 = 43,221\) since there is no symbol for holding the place for the base. The notation of hexagesimals was so useful that it was used in Europe well past the thirteenth century.

The Babylonians (as well as possibly the Egyptians, Chinese, and Hindus) were aware of the Pythagorean Theorem, and they probably knew how to generate Pythagorean Triples.
A Pythagorean triple is a trio of numbers \((a,b,c)\) of, naturally, positive integers, such that they can constitute the legs and hypotenuse of a right triangle, namely, they satisfy:
\[ a^2 + b^2 = c^2. \]

Indeed, among their tablets is a list of such triples. They used those triples to among other things solve quadratic equations by taking square roots.

The easiest way to generate triples is by taking two numbers \(\alpha\) and \(\beta\) with \(\alpha > \beta\) (in modern days, these are called parameters), and then letting:
\[
\begin{align*}
    a &= 2\alpha\beta, \\
    b &= \alpha^2 - \beta^2, \\
    c &= \alpha^2 + \beta^2
\end{align*}
\]

That they form a Pythagorean triple is clear with modern algebra:
\[
\begin{align*}
    a^2 + b^2 &= 4\alpha^2\beta^2 + \alpha^4 - 2\alpha^2\beta^2 + \beta^4 = (\alpha^2 + \beta^2)^2 = c^2.
\end{align*}
\]

Thus, a few examples are given in the following table, the first being the best known of all Pythagorean triples, the one used from very ancient times for the construction of perpendiculares:

<table>
<thead>
<tr>
<th>(\beta)</th>
<th>(\alpha)</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>8</td>
<td>15</td>
<td>17</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>12</td>
<td>5</td>
<td>13</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>20</td>
<td>21</td>
<td>29</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>24</td>
<td>7</td>
<td>25</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>40</td>
<td>9</td>
<td>41</td>
</tr>
</tbody>
</table>

It will be proven in a later section (Diophantus) that all Pythagorean triples arise from this construction.

The Babylonians also knew how to solve quadratic equations by methods not too dissimilar to completing the square and the quadratic formula. Thus we could reasonably state that the quadratic formula is 4,000 years old!

We give one of their examples in modern notation, but still using hexagesimals, rather than decimals to retain some of the flavor of the times. Since the base is 60, we need markers to indicate separation of places. We will use commas to do this. The Babylonian problem was:

Find a number so that when added to its reciprocal we get \(b=2.0,0,33,20.\)

Note that \(2.0,0,33,20\) means \(\frac{33}{60} + \frac{20}{60^2}.\)
Their solution was as follows:

**Take half of** \( b \) \( \frac{1.0,0,16,40}{1.0,0,16,40} \), which stands for \( \frac{16}{60^4} + \frac{40}{60^5} \);

**Square it**

\[
\left( \frac{b}{2} \right)^2 = (1.0,0,16,40)^2 = 1.0,0,33,20,4,37,46,40
\]

As we perform the multiplication with a modern algorithm, and needing to regroup because of base 60:

<table>
<thead>
<tr>
<th>1</th>
<th>0</th>
<th>0</th>
<th>16</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>16</td>
<td>40</td>
</tr>
</tbody>
</table>

Then, **Subtract 1**, \( \frac{33}{60^4} + \frac{20}{60^4} + \frac{4}{60^5} + \frac{37}{60^5} + \frac{46}{60^6} + \frac{40}{60^7} \).

**Find the square root.** To do this, one would have to convert the first term to an even exponent in the denominator so as to be able to take the square root easily. Thus, \( \frac{33}{60^4} + \frac{20}{60^4} \) becomes \( \frac{2000}{60^5} \). So we can estimate its square root to be \( \frac{44}{60^2} \) with a remainder of \( \frac{64}{60^5} \). And we have performed the first step of an algorithm that was much later developed in India (and further enhanced by Islamic mathematicians).

Converting the next two terms to a common (even exponent) denominator, and adding the remainder we have \( \frac{64}{60^5} + \frac{4}{60^5} + \frac{37}{60^5} = \frac{230677}{60^6} \). Knowing the next term will lead to a number of the form \( \frac{x}{60^5} \), then we would know that \( \left( \frac{44}{60^2} + \frac{x}{60^5} \right)^2 \) is approximately the sum of the first four fractions, and so \( \frac{230677}{60^6} \approx \frac{88x}{60^5} + \frac{x^2}{60^6} \). But dividing by the larger term, \( \frac{88}{60^5} \), to estimate \( x \), we get the estimate \( x = 43 \).

Thus, the square root should be of the form \( \frac{44}{60^2} + \frac{43}{60^5} + \frac{y}{60^5} \), and squaring this and equating to \( \frac{33}{60^4} + \frac{20}{60^4} + \frac{4}{60^5} + \frac{37}{60^5} + \frac{46}{60^6} + \frac{40}{60^7} \), we get that \( \frac{6439600}{60^6} \approx \frac{5365y}{60^7} + \frac{y^2}{60^8} \), and so dividing to estimate, we get \( y = 20 \) for an estimate, and indeed \( \frac{44}{60^2} + \frac{43}{60^5} + \frac{20}{60^5} \) is the exact square root of \( \frac{33}{60^4} + \frac{20}{60^4} + \frac{4}{60^5} + \frac{37}{60^5} + \frac{46}{60^6} + \frac{40}{60^7} \). Thus \( \sqrt{\left( \frac{b}{2} \right)^2} - 1 = 0.0,44,43,20 \).
Add to \( \frac{b}{2} \) to obtain the answer: \( 1.0,0,16,40 + 0.0,44,43,20 = 1.0,45 \).

In fact, to verify that the computation is correct, take the reciprocal of 1.0,45, \( 1 + \frac{45}{60^2} \), to obtain 0.59,15,33,20, and so, as was desired,

\[
1.0,45 + \frac{1}{1.0,45} = 1.0,45 + 0.59,15,33,20 = 2.0,0,33,20.
\]

As mentioned above, there isn’t a large intellectual jump from this example to the quadratic formula as symbolized by:

\[
x + \frac{1}{x} = b, \; x^2 - bx + 1 = 0, \; x = \frac{b + \sqrt{b^2 - 4}}{2} = \frac{b + \sqrt{\frac{b^2}{4} - 1}}{2}.
\]

In the previous example, we used a rather modern (about 1000 years old) process for obtaining the square root of a number, the Babylonians were quite capable of computing square roots, and in fact we end our discussion of them with the nice algorithm for taking square roots.

Let \( n \) be a positive integer. The idea behind the computation of its square root is simple. Take any guess, let us call it \( a \). Then a companion guess is \( \frac{n}{a} \) since \( a \cdot \frac{n}{a} = n \), and that is the property we are looking for. So any time we have a guess, we really have two guesses. When will we have succeeded? When our two guesses are close to one another, for then we are indeed close to a square root. What should we do with the two guesses then? What is more reasonable than to average them? Indeed, that is what we do. This is an iteration, a never-ending process. This idea did not particularly seem troublesome to the Babylonians, but to their successors, the Greeks, with their more rigorous requirements, infinite processes indeed seemed untidy.

Returning to the algorithm, more precisely, given \( n \) take \( a_0 = a \) to be any (positive) initial guess for its square root. As mentioned above, we are going to keep refining this guess by defining a sequence that actually does converge to \( \sqrt{n} \).

Define

\[
a_{k+1} = \frac{1}{2} \left( a_k + \frac{n}{a_k} \right).
\]

For example, let \( n = 257,941 \), \( a = a_0 = 1,000 \). An initial guess that is very bad will just prolong the algorithm, but not deter it. The following table illustrates how quickly we will have quite a few decimal places correctly computed. Remember, what we want is the two guesses to be close to each other, which is equivalent to consecutive guesses being close.
to one another:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
<td>1000</td>
</tr>
<tr>
<td>$a_1$</td>
<td>628.9705</td>
</tr>
<tr>
<td>$a_2$</td>
<td>519.534074</td>
</tr>
<tr>
<td>$a_3$</td>
<td>508.0096871</td>
</tr>
<tr>
<td>$a_4$</td>
<td>507.8789394</td>
</tr>
<tr>
<td>$a_5$</td>
<td>507.8789226</td>
</tr>
<tr>
<td>$a_6$</td>
<td>507.8789226</td>
</tr>
</tbody>
</table>

What we see in these numbers of course depends on who we are. A Mesopotamian would perhaps accept this answer as finished, while some Greeks would perhaps not accept it as ever finished.

Thus, in the example for the quadratic equation, if one converted the original expression
\[
\frac{33}{60^2} + \frac{20}{60^2} + \frac{4}{60^2} + \frac{37}{60^2} + \frac{46}{60^2} + \frac{40}{60^2}
\]
to a common denominator, the answer would have been
\[
\frac{25921000000}{60^4},
\]
so it would suffice to compute the square root of 25,921:

\[
\text{And then } \frac{161000}{60^2} = \frac{44}{60^2} + \frac{43}{60^2} + \frac{20}{60^2}.
\]

We end the chapter with a proof (à la nineteenth century) that this sequence does converge to $\sqrt{n}$. First, assume that we have shown the sequence converges to some number. Let's call it $A$. Hence we have
\[
\ell = \lim_{k \to \infty} a_k = \lim_{k \to \infty} a_{k+1} = \lim_{k \to \infty} \frac{1}{2} \left( a_k + \frac{n}{a_k} \right) = \frac{1}{2} \left( \lim_{k \to \infty} a_k + \lim_{k \to \infty} \frac{n}{a_k} \right) = \frac{1}{2} \left( \ell + \frac{n}{\ell} \right)
\]
and thus
\[
\ell = \frac{1}{2} \left( \ell + \frac{n}{\ell} \right),
\]
and we get by algebraic manipulation, $2\ell = \ell + \frac{n}{\ell}$, which gives $\ell = \sqrt{n}$, as desired.

But we have yet to show the sequence converges. And this is where that real number principle mentioned in the previous chapter becomes very important. Since all the terms of the sequence are positive, if we can show it is a decreasing sequence (see the example above), we will have the existence of the limit.

We have
\[
0 \leq \frac{(a_k - \sqrt{n})^2}{2a_k} = \frac{a_k^2 - 2a_k \sqrt{n} + n}{2a_k} = \frac{a_k}{2} - \sqrt{n} + \frac{n}{2a_k} = a_{k+1} - \sqrt{n},
\]
and so
\[
\sqrt{n} \leq a_{k+1}.
\]
But then, since \( n \leq a_{k+1}^2 \),
\[
a_{k+2} = \frac{1}{2} \left( a_{k+1} + \frac{n}{a_{k+1}} \right) \leq \frac{1}{2} \left( a_{k+1} + a_{k+1} \right) = a_{k+1},
\]
and we have a decreasing sequence.

We should observe, again, that once we can take square roots of whole numbers, we can take the square root of any rational. Furthermore, the algorithm above works for an arbitrary positive real number \( n \).
Chapter 3

Egypt

The Nile is nothing else than the year, because the numbers expressed by the letters ΝΕΙΛΟΣ, Nile, are in Greek arithmetic, \( N = 50, E = 5, I = 10, \)
\( \Lambda = 30, O = 70, \Sigma = 200; \) and these figures make up together 365 —

Heliodorus

The other major ancient civilization that had major impact in our Western world is that of the Egyptians. The other two ancient cultures, India and China, have had also considerable, but more recent, influence, and will be discussed later.

The Egyptians were influential in that their impact on most ancient civilizations, including the Greeks, was considerable. As with the Mesopotamians, the limited availability of writing materials had a definite effect on their mathematics and notation. They wrote on stone, and on papyri, which was manufactured from plants that grew near the great Nile. Although the word papyrus gives rise to our modern word paper, the manufacture of it was quite different, and much more problematic—thus, it did not have the impact that the much more modern Chinese invention would have several thousand years later.

The highlights of Egyptian history of the period are given in the following table:

<table>
<thead>
<tr>
<th>Date</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>3100</td>
<td>Menes unifies lower and upper Egypt.</td>
</tr>
<tr>
<td>2900</td>
<td>Memphis is capital.</td>
</tr>
<tr>
<td>2700</td>
<td>Hieroglyphic and Hieratic writing appear.</td>
</tr>
<tr>
<td>2600</td>
<td>Great Pyramid of Cheops is built.</td>
</tr>
<tr>
<td>2100</td>
<td>Thebes is capital.</td>
</tr>
<tr>
<td>1700</td>
<td>Rhind Papyrus is written (later copied by Ahmose).</td>
</tr>
<tr>
<td>1300</td>
<td>Tutankhamen lives.</td>
</tr>
<tr>
<td>600</td>
<td>Persian Conquest.</td>
</tr>
<tr>
<td>300</td>
<td>Alexandria is built, Ptolemy I becomes king.</td>
</tr>
<tr>
<td>30</td>
<td>Cleopatra is the last queen, Rome conquers.</td>
</tr>
</tbody>
</table>

All times are BC.

Our knowledge of Egyptian mathematics stems mainly from a few papyri that are mathematical in nature. One of the most famous of these is the Ahmose or Rhind
Papyrus; another one is called the Moscow Papyrus. The Ahmose Papyrus consists of a list of problems with accompanying solutions as a guide to instruct the reader. Ahmose was a scribe.

We will briefly discuss four Egyptian mathematical topics:

1. Notation
2. Multiplication
3. Volume of the Frustum of the Pyramid
4. Unit Fractions

In both of their written languages, Hieroglyphic and Hieratic, the Egyptian number system was a simple grouping system on base 10. In other words, they had symbols for 1, 10, 100, and so on as high as 10,000,000 illustrating without a doubt their ability to handle very large numbers. Their symbols resembled:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>l</td>
<td>1</td>
</tr>
<tr>
<td>μ</td>
<td>10</td>
</tr>
<tr>
<td>s</td>
<td>100</td>
</tr>
<tr>
<td>$\text{𓊫}$</td>
<td>1,000</td>
</tr>
<tr>
<td>$\text{𓊳}$</td>
<td>10,000</td>
</tr>
<tr>
<td>ⲱ</td>
<td>100,000</td>
</tr>
<tr>
<td>ⲳ</td>
<td>1,000,000</td>
</tr>
<tr>
<td>ⲱ</td>
<td>10,000,000</td>
</tr>
</tbody>
</table>

Thus to write 231, they would simply write $\text{𓊳} \text{𓊳} \text{𓊳}$, or possibly, $\text{𓊳} \text{𓊳} \text{𓊳} \text{𓊳}$, since neither order nor position were relevant.

Addition in this system was very simple, and so was subtraction. Simply, all one had to do was collect terms every time the base 10 was reached. Of course one had to know, for example, that 10 of $\text{𓊳}$ made one of $\text{𓊫}$.

On the other hand, multiplication was a little more challenging, and we will illustrate their doubling algorithm with an example. Below we will see a variation of their doubling algorithm that survived until our century. One of the great advantages of Egyptian multiplication is that one does not have to memorize multiplication tables in order to be able to multiply. It is based on the anciently known fact that every natural number can be written uniquely as a sum of powers of 2.

For example, to multiply $67 \times 51 = 3417$, we would start with the larger multiplicand, 67 in our case, and double every time (note that doubling is just an addition), keeping track of the factor as we double. But we would double until we reach a number, which when doubled would exceed our other multiplicand, 51 in our case.

We will also need to write our lower multiplicand as a sum of powers of 2, and this is readily accomplished by subtracting at each stage the highest power of 2 possible. Thus, for 51, we start by subtracting 32, which leaves 19, from which we subtract 16, leaving 3, from which we subtract 2, and we are finished. Thus,
Now to the doubling algorithm:

<table>
<thead>
<tr>
<th>Doubling</th>
<th>Collecting</th>
<th>In Our Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>67</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>67</td>
<td>134</td>
</tr>
<tr>
<td>4</td>
<td>134</td>
<td>268</td>
</tr>
<tr>
<td>8</td>
<td>268</td>
<td>536</td>
</tr>
<tr>
<td>16</td>
<td>536</td>
<td>1072</td>
</tr>
<tr>
<td>32</td>
<td>1072</td>
<td>2144</td>
</tr>
</tbody>
</table>

To arrive at our answer then we collect the terms that correspond to the decomposition of 51:

| 1 | 67 |
| 2 | 134 |
| 16 | 1072 |
| 32 | 2144 |

which after collecting becomes

As mentioned above, a variation of this algorithm has survived until our times. It is commonly referred to as the peasant algorithm for multiplication. We exemplify with the same computation $67 \times 51 = 3417$. In this algorithm, one writes both numbers, and one doubles one column, but the other column is halved (rounding down to a whole number if necessary):

| 67 | 51 | ✓ |
| 134 | 25 | ✓ |
| 268 | 12 | |
And a row from the doubling column is included in the total only when the corresponding entry in the halving column is odd, which have been marked with a check mark. So we have then that, \(67 \times 51 = 67 + 134 + 1072 + 2144\), just as before.

Not surprisingly, one of the greatest geometric accomplishments of the Egyptians was the estimation of the volume of the frustum of a pyramid (or a truncated pyramid) with a square base, which is given in modern notation by the formula

\[
\text{Volume of frustum} = \frac{1}{3} h \times (a^2 + ab + b^2)
\]

where \(h\) is the height of the truncated pyramid and \(a\) and \(b\) are the sides of the upper and lower bases respectively.

We have no idea on how they arrived at that surprising (and important) coefficient: \(\frac{1}{3}\), but one can speculate that it was empirically arrived at. Naturally, this result implies knowledge of the volume of the pyramid being

\[
\text{Volume of Pyramid} = \frac{1}{3} H \times \text{area of the base},
\]

where \(H\) is the height of the pyramid, since \(a = 0\) in this case. Equivalently, one can state three pyramids fill in the prism with the same dimensions as the pyramid.

Knowing the volume of the pyramid, one can use similarity to arrive at the volume of the frustum, via the modern tool of calculus, as follows:

\[
\text{Volume of frustum} = \int_0^h A(z) \, dz,
\]

where \(A(z)\) stands for the area of the square at height \(z\). If we let \(x\) denote the side of the square at height \(z\), then by looking in one of the faces of the pyramid, we have the following picture:

Thus, we have that \(x = a + 2w\), and by similarity of triangles:

\[
\frac{w}{h - z} = \frac{b - a}{h},
\]

which after clearing gives:

\[
x = b - \frac{(b - a)z}{h}.
\]

Since \(A(z) = x^2\), we are reduced to having to integrate \(\int_0^h \left(b^2 - \frac{2b(b-a)z}{h} + \frac{(b-a)^2 z^2}{h^2}\right) \, dz\),

which gives \(b^2z - \frac{b(b-a)z^2}{h} + \frac{(b-a)^2 z^3}{3h^2}\) evaluated from 0 to \(h\). Thus we have

\[
\text{Volume of frustum} = b^2h - b^2h + bzh + \frac{(b-a)^2 h}{3},
\]
and so factoring \( \frac{1}{3}h \), we have

\[
\text{Volume of frustum} = \frac{1}{3} h (3ab + b^2 - 2ab + a^2),
\]

which is as the Egyptian knew.

We end our brief discussion of the Egyptians with a discussion of a topic that was, appropriately, exclusively theirs. Possibly because of the lack of writing materials, or their desire to preserve their writing in stone, the Egyptians had only one type of fraction, the reciprocals of positive integers, or equivalently, fractions with numerator equal to 1, or \textbf{unit fractions} as we will refer to them.

Roughly, their notation would be \( \overline{m} \) for the reciprocal of the integer \( m \)—the only exception was a special symbol for the fraction \( \frac{2}{3}, \overline{3} \). Hence they looked at ways of representing other fractions in terms of the ones they would write, and they would want \textbf{different} unit fractions to represent a given fraction. Especially, they were mainly interested in representing \( \frac{2}{m} \) as a sum of distinct unit fractions. They would use these representations to perform division, as well as addition and subtraction of fractions. For example, their list includes \( \frac{2}{9} = \frac{1}{6} + \frac{1}{18}, \frac{2}{15} = \frac{1}{10} + \frac{1}{30} \) and \( \frac{2}{57} = \frac{1}{38} + \frac{1}{114} \). Much scholarship since then has been devoted as to the why of their \textbf{specific} selections. Several of the papyri include tables of such representations.

But before we dismiss their preferences as idiosyncratic, let us consider it a little deeper. For example, consider the fraction \( \frac{4}{5} \). An Egyptian mind might have chosen to represent this division, \( 4 \div 5 \), by \( \frac{4}{5} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} \). If we concentrate more on the division, and less on the fractions, we may speculate that their motives were practical, as a way of accomplishing distribution in reality.

For example, going back to the example above, suppose we have 4 loaves of bread to be distributed among 5 people; then, the Egyptian expression

\[
4 \div 5 = \overline{2} + \overline{5} + \overline{10}
\]

tells us to divide 3 loaves into halves, and give each person one half, the other loaf into fifths and give each person a fifth, and finally the remaining half is to be divided into fifths and give one piece to each individual. On the other hand, our modern description \( \frac{4}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} \) could be interpreted as having to cut each loaf into fifths,
and give four of those pieces to each of the individuals, (which entails more cuts: 16 cuts versus 11, or 50% more cuts). In an alternate interpretation of \( \frac{4}{5} \), we would cut a fifth out of each loaf, and give the four small pieces to one individual, but again this is not as satisfactory since one person is being treated differently from the rest.

But is every fraction a sum of different unit fractions? For example, consider \( \frac{7}{23} \). What is the largest unit fraction that is less than or equal to it? Note that \( 23 = 3 \times 7 + 2 \), so \( 23 > 3 \times 7 \), and so \( \frac{1}{3} > \frac{7}{23} \). But on the other hand, since 2 is the remainder (and thus \( < 7 \)), we have that \( 23 < 4 \times 7 \), which gives \( \frac{1}{4} \) as the largest unit fraction we can subtract from \( \frac{7}{23} \). Let us do that, and obtain \( \frac{5}{92} \). By doing the division again, we have \( 92 = 18 \times 5 + 2 \), we have that \( \frac{1}{19} \) is the largest unit fraction we can subtract, getting \( \frac{3}{1748} \). Are we going anywhere with this approach? Yes! Note the numerators are decreasing, hence we will have a 1 eventually. In fact, \( 1748 = 582 \times 3 + 2 \) and subtracting \( \frac{1}{583} \) from \( \frac{3}{1748} \), we indeed get a unit fraction, \( \frac{1}{1019084} \), and thus,

\[
\frac{7}{23} = \frac{1}{4} + \frac{1}{19} + \frac{1}{583} + \frac{1}{1019084}.
\]

We end our Egyptian section with a formalization of our example. This proof has been given by many people including Fibonacci (who gave several proofs of the fact) and the British mathematician of the nineteenth century, J.J. Sylvester.

**Theorem.** Take any fraction \( \frac{m}{n} < 1 \). Then it can always be expressed as a sum of distinct unit fractions.

**Proof.** By induction on the numerator of the reduced fraction. Actually, as before in the example, being greedy works. As before, what is the largest unit fraction which is less than or equal to \( \frac{m}{n} \)? Easily, let us divide \( m \) into \( n \), and get a remainder (if there is no remainder, then we already have a unit fraction.) So \( n = qm + r \) where \( q \) is the quotient and \( r \) is the remainder. By necessity, \( 0 < r < m \), so \( qm < n \), so \( \frac{m}{n} < \frac{1}{q} \), but also \( (q + 1)m > n \), and so \( \frac{m}{n} > \frac{1}{q + 1} \), so if we let \( p = q + 1 \), then \( \frac{1}{p} \) is the largest unit fraction
that is less than \( \frac{m}{n} \). Now, \( \frac{m}{n} - \frac{1}{q+1} = \frac{mq + m - n}{n(q+1)} = \frac{m-r}{n(q+1)} \), which is excellent since \( m-r < m \), and thus, \( \frac{m-r}{n(q+1)} \) is, by induction, a sum of distinct unit fractions, and by the construction all of them are smaller than \( \frac{1}{q+1} \).
Chapter 4
Greek Beginnings

... there was far more imagination in the head of Archimedes than in that of Homer—Voltaire

As we start discussing the truly wonderful and imaginative world of Greek mathematics, there is a basic idea to keep in mind. The basic Greek perception of a number had a geometric attachment to it, it could represent a line segment, or an area, or a volume. They derived much of their strength from this, and most of their weaknesses too.

This does not mean that the Greeks did not have notation for numbers, but theirs was not nearly as powerful as the Mesopotamian notation. The oldest of the Greek enumeration systems used letters to represent numbers: \( \alpha = 1 \), \( \beta = 2 \), \( \gamma = 3 \), \( \delta = 4 \), \( \kappa = 20 \), etcetera. Thus, \( \kappa \alpha = 21 \). In order to add, many rules had to be memorized, and multiplication was also demanding from the symbolic point of view. Yet, as we mentioned above their power was obtained from their geometric visualization.

Later on they improved their notation to one similar to the Egyptian grouping system with symbols for 1, 5, 10, 100, etcetera. But, we know, again, this is not as powerful as the Babylonian notation. It is indeed a short trek from the latter Greek notation to the more familiar Roman notation for numbers, I, V, X, and so on.

We will retain the geometric point of view throughout most of our discussion of the Greek mathematical world. From this point of view, addition and subtraction are extremely easy to both visualize and perform. The picture on the left shows the addition of segments. Similarly, there was addition of areas by just putting them together, and of volumes in a similar fashion. For example, the addition of two rectangles with a side in common would represent no difficulty as the picture exemplifying the distributive law in the next page exemplifies. However, other interesting issues occurred naturally within this context.

The addition of two squares to form a square is equivalent to the Pythagorean Theorem. Thus,
At all times, homogeneity had to be maintained. **We could not add a segment to a rectangle!**

On the other hand, multiplication acquires a very different connotation than just repeated addition—unless we consider it as a continuously repeated addition.

In multiplying two numbers one obtains not a line, not a linear number, but an area. The product of two segments was basically the rectangle with one of the sides as the base, and the other as the height. Thus, **the product of three numbers would be a volume and the product of four or more numbers would not necessarily make sense!** This was one of the prices to be paid by this approach to numbers, but there are other ways to see the multiplication of several numbers as we will see in a future section.

However, on the positive side, other algebraic facts such as commutativity of both addition and multiplication become very intuitively obvious. Think, commutativity is nothing but the flipping of the rectangle. Another important algebraic fact, the distributive law of multiplication over addition:

\[
ab + ac = a(b + c)
\]

becomes as clear as day.

Certainly, this geometric algebra of the Greeks had limitations as we already pointed above and our more symbolic modern algebra has admittedly much more power, nevertheless, there are many advantages to their way of thinking, especially when one first encounters algebra.

Another fundamental identity that is clearly seen from this point of view is:

\[
(a + b)^2 = a^2 + 2ab + b^2
\]

which many students, even today, miss. As the picture shows, the result is, again, very intuitive, and a similar consideration on the cube shows that

\[
(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3
\]

At the same time, we are exposed to one aspect of the weakness of the geometric thinking since geometric intuition abandons us when trying to expand \((a + b)^4\).

We have seen the Greek views on addition, subtraction and multiplication. We have also seen, that to the Egyptians, division was closely connected to the idea of distribution, a
view which is valid until today. But what could **division** mean to the Greeks? Actually, there are at least two very different ways of viewing division.

One of the views is clear.

Suppose $a$, $b$ and $c$ are numbers, then $\frac{ab}{c}$ has the following meaning: $ab$ is a rectangle, and $c$ is the side of a new rectangle, thus we are looking for the side $x$ to go with $c$, so that the rectangle with sides $a$ and $b$ has the same area as the rectangle with sides $x$ and $c$. In algebraic terms we are solving the equation: $cx = ab$.

**Note the respect of homogeneity.**

In pictures:

where $x$ is the desired, but unknown, amount.

The problem is solved as follows:

- Extend both sides of the original rectangle with sides $a$ and $b$, and in one of these extended sides mark off $c$. Then from that extreme draw the line going through the corner of the rectangle, and extend that line until it hits the other extended side. Then $x$ is the segment from the lower corner of the rectangle to that point of intersection. Thus the rectangle with sides $c$ and $x$ has the same area as the rectangle with sides $a$ and $b$. The proof that this is so is immediate since the right triangle with legs $a$ and $x$ is similar to the right triangle with legs $b$ and $c$, thus

$$\frac{c}{a} = \frac{b}{x}.$$  

But there is another, even more important, meaning for division, that of **measurement** or **ratio**.

Namely if $a$ and $b$ are segments, then we have the measurement of $b$ by $a$, which we would think of as $b$ divided by $a$, $\frac{b}{a}$, or in another (and more relevant) notation: $b:a$ (which reads $b$ is to $a$). The end result of the division is now not a line, but an abstract, dimensionless number.

For example, the circumference of a circle is to the diameter in the ratio of $\pi$. The Greek knew such a ratio existed, but they did not refer to it by $\pi$.

In general this comparison of quantities would not have to be constrained to linear quantities, and it could be extended to areas or volumes, and it would make sense as long
as the two quantities compared were of the same quality (one would have difficulties comparing a rectangle with a segment, for example). E.g., the basic statement about similar rectangles would be that if the ratio of their sides is 2, then the ratio of their areas is 4. We will be expanding on this meaning throughout the Greek era as it becomes very important after the discovery of incommensurability.

It’s time to catch up with history. We will be staying with Greek mathematics for a long time: from roughly **600 BC to 300 AD**—a nine hundred year period. Although we will refer to this period as Greek mathematics, the location varies from **Asia Minor** (present day Turkey), to present day Greece and **Athens**, and then the longer stay will be across the Mediterranean Sea, in the African coast, in the Delta of the Nile in Egypt, in the greatest scientific city of antiquity, **Alexandria**.

The political structure of the Mediterranean world during this period will vary a great deal. When we start in Asia Minor, Persia is the greatest power of the region. The king Cyrus of Persia had conquered Babylon, and soon afterwards the Persians also conquered Egypt. As the Persians tried to advance their conquest into Greece, they are defeated by the Greek city-states of **Athens**, **Corinth**, **Thebes** and **Sparta** in a rare unified stance. The Cities gained power and prestige, but soon they would fight among themselves in the Peloponnesian war. Approximately, in the year 300 BC, the Macedonian king Philip and his famous son, **Alexander** conquered Greece. Alexander then conquered a large empire including Egypt, Mesopotamia and Persia, parts of India. This conquest created a great exchange of knowledge from the Greek world to the outside and vice versa. After Alexander died at the age of 33, the empire was divided among his generals. Most importantly, Ptolemy I became king of Egypt, and it was his descendant, Cleopatra VII who would surrender Egypt (and Alexandria) to the Romans. The Romans ruled the Mediterranean from 250BC to 450AD, but their contribution to mathematics will be of a technical and ephemeral nature.
### Greek History Highlights

<table>
<thead>
<tr>
<th>Date</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>2,200</td>
<td>The Minoan civilization flourishes in Crete.</td>
</tr>
<tr>
<td>1,000</td>
<td>Colonists migrate from mainland Greece to Asia Minor.</td>
</tr>
<tr>
<td>776</td>
<td>First athletics festival.</td>
</tr>
<tr>
<td>750</td>
<td>Sparta and Athens become powerful.</td>
</tr>
<tr>
<td>700</td>
<td>Syracuse is founded by Greeks in Sicily, and so is Tarentum in Southern Italy.</td>
</tr>
<tr>
<td>594</td>
<td>Solon reforms the laws of Athens.</td>
</tr>
<tr>
<td>490</td>
<td>Darius, the Persian king is defeated at Marathon by the Athenians.</td>
</tr>
<tr>
<td>480</td>
<td>Xerxes, the Persian king is defeated by the Greeks.</td>
</tr>
<tr>
<td>469</td>
<td>Socrates is born.</td>
</tr>
<tr>
<td>443</td>
<td>Pericles rules Athens.</td>
</tr>
<tr>
<td>431</td>
<td>Peloponnesian War begins.</td>
</tr>
<tr>
<td>429</td>
<td>Plato is born.</td>
</tr>
<tr>
<td>404</td>
<td>Athens is defeated by Sparta.</td>
</tr>
<tr>
<td>399</td>
<td>Socrates is sentenced to die.</td>
</tr>
<tr>
<td>384</td>
<td>Aristotle is born.</td>
</tr>
<tr>
<td>350</td>
<td>Philip II becomes king of Macedonia, and in 338 conquers Greece.</td>
</tr>
<tr>
<td>323</td>
<td>Alexander dies.</td>
</tr>
<tr>
<td>100</td>
<td>Rome annexes Macedonia and Greece.</td>
</tr>
<tr>
<td>30</td>
<td>Alexandria is annexed by Rome.</td>
</tr>
</tbody>
</table>

All times are BC.

We choose to divide the long history of Greek mathematics into 3 periods that are basically geographic in nature. The first two periods concern the area that is traditionally associated with modern Greece and its neighboring area known as Asia Minor:

1. The Ionic period: 600 BC to 500 BC
2. The Athenian period: 500 BC to 300 BC
3. The Alexandrine period: 300 BC to 400 AD

The dates are greatly simplified as usual, but we can appreciate the length of mathematical activity in the Greek world.

In this chapter, we will discuss the two major figures in the Ionic period: Thales of Miletus and Pythagoras of Samos. Both Miletus and Samos belong to Asia Minor or Ionia.

## Thales

Thales (625-545 BC) has traditionally been referred to as the first Greek mathematician. Although none of his writings survive, his reputation as a wise man survives. He is attributed with having stressed the need for supplying proof of our claims, a characteristic of Greek mathematics that perhaps has set the tone for Western mathematics until the present day.
He was certainly acquainted with some of the most fundamental facts in geometry such as:

1. **Two triangles are congruent if they have two sides and the angle in between congruent**—often referred to as side-angle-side (or SAS).

2. **An isosceles triangle has equal base angles.**
   How he reasoned this fact we do not know. Many centuries after Thales, Pappus is going to give the following elegant, yet sophisticated argument: Think of the triangle as being seen from the other side, or as in a mirror. Since the two sides are equal, and the angle in between is the same, by SAS, we have congruence of the two triangles, but corner $c$ corresponds to corner $b$, hence the two angles are the same.

3. **The sum of the angles in a triangle is $180^\circ$.**
   Let us recall how an elementary proof of this fact goes: Start with an arbitrary triangle, and draw the parallel to one side from the opposing vertex. Then, since the lines are parallel, the opposing angles are equal, on either side, and thus the three angles in the triangle are equivalent to the angles on a straight line, or $180^\circ$. Let us caution to the fact that this argument is only seemingly elementary in that it uses a deep, but innocent looking requirement: opposing angles in a transversal to parallel lines are equal.

4. **The Pythagorean Theorem.** Of which we will have much more to say below.

5. **Similar triangles have proportional sides.**
   Below, when we discuss the Pythagoreans, we will see a possible argument for this important fact. What is clear in any case is that, already, by the time of Thales, this theorem is of great importance, computational, theoretical and practical. It was used in many applications such as the construction of tunnels.
Thales himself is attributed with using it for the measurement of quantities such as the distance from shore-to-ship.

To accomplish the measurement, start from any point on shore, and call it $A$. Then draw a segment of arbitrary length perpendicular to the line between shore and ship, and call it $AB$.

Then draw another arbitrary segment $BC$ perpendicular to $AB$. From $C$, draw the line to the ship until it intersects the original $AB$ at the point $O$. Now, since triangle $OBC$ is similar to the triangle with vertices $O$, $A$ and the ship, and since all the sides of the triangle $OBC$ can be measured as well as the side $OA$ of the larger triangle, we know the larger triangle completely.

Specifically: if we let $x$ denote the distance from $A$ to the ship, then $x = \frac{OA}{BC} \cdot OB$.

The reader may recognize the rudiments of trigonometry, which will be a Greek creation a few centuries later.

We also have a wonderful theorem that is still often referred to as Thales' Theorem:

Any triangle inscribed in a circle with the diameter as the base is a right triangle.

Equivalently, any of the triangles in the picture is a right triangle.

*Proof.* Consider the radius joining the center $O$ to the point $C$. Then the $\triangle COA$ is isosceles, so $\angle OAC = \angle OCA = \beta$, similarly $\angle OCB = \angle OBC = \alpha$. But then $180^\circ = \angle CAB + \angle ACB + \angle ABC = 2\alpha + 2\beta$.

And hence $\alpha + \beta$ is a right angle.

Thales was certainly aware that any two circles are similar.

He was also probably aware of the connection between the area of a circle and its circumference via the following easy visualization. As mentioned above, no written records exist, but we can speculate his thinking:
Start with an arbitrary circle and cut it by some diameters:

Think of the pieces as slicing an orange and thus break it up into two halves and arrange them until they fit neatly together into an almost parallelogram whose base is half the circumference and whose height is the radius, hence the area is given by

$$\frac{1}{2}Cr$$

where $C$ is the circumference of the circle and $r$ is the radius.

Of course the more the pieces, the closer we would be to a parallelogram.

The Greeks would however never use a formula such as $A = \frac{1}{2}Cr$ since this is algebraic language, which was not available to (nor needed by) them. Instead they would rather use expressions such as:

*The areas of two circles are to each other as the squares of their diameters.*

In pictures:

Or, in yet another fashion, as Archimedes accurately stated:

*The area of a circle is the same as the area of the triangle whose base is the circumference and whose height is the radius.*

**Pythagoras**

Despite being an early figure in the history of mathematics, **Pythagoras** (581-497 BC)-or more appropriately the **Pythagoreans**-remains one of the most influential, not only in the Greek mathematics period, but until the present day. Some of the philosophical issues raised at that time will probably remain interesting for many centuries to come. They gave us the word **mathematics**, as well as the word **philosophy**.
When we hear the name, naturally, the first thing that comes to mind is the important **Pythagorean Theorem**. How they proved this theorem is not known but the following argument is very much in the Greek style.

Start with an arbitrary right triangle with sides \( a \) and \( b \) and hypotenuse \( c \). We need to show that

\[
    a^2 + b^2 = c^2.
\]

Of course, to the Greeks this meant that if you build the square on segment \( a \), and the square on segment \( b \), and put them together you will have the square on segment \( c \). Start with a square with side \( a + b \), then it can be cut into six pieces as in the picture. The six pieces are the square on \( a \), the square on \( b \), and 4 triangles all congruent to the original one. By rearranging the pieces, as in the picture, then we obtain still 4 triangles as before, but in addition we get the square on segment \( c \). From which we conclude the theorem.

The Pythagoreans with their motto **All is Number** (where number meant a natural number, a counting number) gave great impetus toward the idea of the quantification of nature, and the rational pursuit of such quantification. Thus, modern science owes them a considerable debt.

At the same time they also promoted the too-simplistic quantification of nature and its superstitious offsprings such as **numerology**. Some of their traditions still endure until today such as the pentagram being a magic form, while others have not endured as well such as odd numbers being male while even are female. And lest we judge them, it is not at all clear we may not yet return to those high degrees of superstition in the future.

In any case, their passionate belief in patterns and their fascination with their discoveries are still exciting today, both in a collective and individual basis. For example, they looked at numbers as having shapes—**figurate** numbers.

There were **triangular** numbers, **square** numbers, **pentagonal** numbers, **hexagonal** numbers, and etcetera.
The **triangular** numbers were so called because they could be made into a triangular shape, and they are: 1, 3, 6, 10, 15, 21, … . Note that as we build the triangular numbers, every new level has the respective number of points, thus the third level line has three points, the fourth line has 4 points, and so on. Thus, the triangular numbers arise from the sum of the first consecutive numbers: 1=1, 3=1+2, 6=1+2+3, 10=1+2+3+4, and so on.

An easy algebraic expression for the \( n^{th} \) triangular number, \( \Delta_n \), is achieved as follows:

\[ \Delta_n = 1 + 2 + 3 + \cdots + n. \]

We have already twice used a common symbol \( \ldots \), which is called **ellipsis**, and which means: *reader, beware, a pattern has been written, it is your responsibility to discern it*. When seeing an ellipsis, the reader needs to keep three things in mind:

- Do I understand where the pattern starts?
- Do I know what comes next?
- Do I know where it ends (if at all)?

It is customary to write sequences in the form 1,3,5,...,17. Here we would answer the three questions raised above with: it starts with a 1, it goes by two’s, and it ends at 17.

We will adopt a modern way to write sequences. It uses two rows, one to denote the register (indicating the position in the sequence), and a second row indicating the sequence (or what is in the register). Thus our example above would be rewritten as

<table>
<thead>
<tr>
<th>register</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>sequence</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>...</td>
<td>17</td>
</tr>
</tbody>
</table>

Note that we already were forced to understand our sequence better since now we know there are nine terms in it. Or the triangular number would be written as

<table>
<thead>
<tr>
<th>register</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>triangulars</strong></td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
</tr>
</tbody>
</table>

And clearly we would like more—we would like a ready expression for any triangular number, namely answer the question mark on the table. We can do this as follows: we know \( \Delta_n = 1 + 2 + 3 + \cdots + n - 2 + n - 1 + n \), hence \( \Delta_n = n + n - 1 + n - 2 + \cdots + 3 + 2 + 1 \), also. If we now add the two expressions we get:

\[
\begin{align*}
\Delta_n &= 1 + 2 + 3 + \cdots + n - 2 + n - 1 + n \\
\Delta_n &= n + n - 1 + n - 2 + \cdots + 3 + 2 + 1
\end{align*}
\]

and, thus, we get the fact that

\[ 2\Delta_n = (n+1)n, \]

which gives

\[ \Delta_n = \frac{(n+1)n}{2}. \]
A similar technique for finding $\Delta_n$ is going to be independently discovered by young Gauss at the age of 10 when supposedly he was given to add $1 + 2 + \cdots + 99 + 100$, and wrote, quickly, the answer: 5050. It is not completely clear what the arithmetic progression was that he was given to add, and accounts differ on how old he was when he was given the problem—but there is no dispute on how quickly he solved it.

Although the algebraic language was not available to them, this is the kind of pattern recognition that would excite the Pythagoreans.

Similarly, one would consider the squares: 1, 4, 9, 16, 25, 36, … And again from the geometric shape we can see that we add an odd number to each of the squares to arrive to the next square. Thus $1=1$, $4=1+3$, $9=1+3+5$, $16=1+3+5+7$, and so on.

Naturally, after 1, 5 is the first true pentagonal number, then 12, 22, 35, … If we observe how we obtain a new pentagonal number from an old one we see that we do it by consecutively adding first 4, then 7, then 10, and then 13. Hence, if we start with the arithmetic progression: 1, 4, 7, 10, 13, 16, …; then the pentagonal numbers are obtained from adding the first so-many terms of this sequence.

Many centuries after the Pythagoreans, Fermat is going to observe that every number is the sum of three (or fewer) triangular numbers, and that every number is the sum of at most 4 squares, and that the sum of at most 5 pentagonal numbers will give any number, etcetera. Again, a fact that would have fascinated the Pythagoreans.

**But the most dramatic, and deepest, observation of the Pythagoreans concerns the irrationality of the square root of 2.**

We saw above the notion of the ratio of two numbers by the idea of one measuring the other. This, as well as many other mathematical ideas, have stemmed from the human need to measure our reality. Again, what does one mean when one says that a segment measures some other segment? The idea is very simple. We say a segment $\mathcal{M}$ measures $\mathcal{L}$ if $\mathcal{M}$ (which is the smaller of the two) goes into $\mathcal{L}$ a whole number of times, or in other words goes into $\mathcal{L}$ evenly. What could be simpler? Thus, for example, a foot measures a yard since it goes 3 times into it, or similarly a centimeter measures a meter since 100 of the former make one of the latter.

Suppose we have two segments $\mathcal{L}$ and $\mathcal{M}$, and suppose that neither measures the other. There is hope yet. Perhaps we can find some segment $\mathcal{N}$, smaller than the two that will
measure both. What does this amount to? Suppose indeed \( N \) measures both. Then regardless of what \( L \) and \( M \) are, their ratio will be a quotient of positive integers, since \( M \) will be so many \( N \)'s and so will \( L \). Conversely, if under some measurement, the ratio of \( L \) and \( M \) is a quotient of integers, then they are commensurable by some segment. Thus, commensurability is equivalent to rational quotients.

It was a very important and deep observation of the Greeks (in particular the Pythagoreans) that this naive view of measurement was not sufficient. They could argue that the side and the diagonal of a square are not commensurable (and, naturally, we will see this argument below) and hence that their ratio, \( \sqrt{2} \), is not a quotient of positive integers, and hence not a rational number. Yet this realization was quite devastating to their basic beliefs, and suddenly basic and important needs like the proportionality of the sides of triangles with equal angles could not be successfully argued anymore.

Although no record exists of how they would argue that similar triangles had proportional sides, we can perhaps imagine such an argument. Take two triangles with equal angles—as in the picture. We would like to show all the sides are proportional.

First find a length that measures two of the respective sides. We took the left sides in our example. Measure both sides by this unit of common length—in our particular example the sides measure 5 and 9 units, respectively.

Before we proceed we need two readily acceptable facts from geometry:

1. In a parallelogram, opposite sides are equal.
2. If two triangles have equal angles, and a common side, then they are congruent.

We intend to show the other two sides are also in the ratio 9:5. Take the right-hand sides. Draw the parallels to the base through the markings for the units. And close the parallelogram that these lines make.

Consider any of the triangles on the side. Obviously, one triangle will have 5 of them and the other will have 9, since these are determined by the parallels to the base. If we can show that these triangles are all congruent, then we will have shown that the two sides are also in the 9:5 ratio, and we would be finished. Because their angles are all cut from parallels by transversals, all the small triangles have the same angles. And the left sides of all these triangles are equal,
since they are all opposite sides in a parallelogram to a segment with length one unit.

Note that by similarity, as before, the areas should be in the ratio 81:25, since the lengths are in the 9:5 ratio. Indeed that is the case. If we finish tiling by the unit triangle, the smaller of the two triangles has: $1 + 3 + 5 + 7 + 9 = 25$ triangles, while the larger of the two has: $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = 81$ triangles. All is as expected.

But, now we return to our main task at hand, the irrationality of $\sqrt{2}$, or equivalently, that the side and the diagonal of a square are not commensurable. As we prepare to give arguments for this fact, one obvious, yet fundamental principle we will use over and over is the following:

If $N$ measures both $L$ and $M$, then it will certainly measure their difference.

We will, in addition, use the more subtler yet also-clear principle that:

If $M$ is measurable by $N$, then we cannot indefinitely be measuring smaller segments of $M$ by the same measurement $N$ since we started with a whole number of $N$'s in $M$.

We will give two proofs of the irrationality of the square root of 2. One proof is more symbolic than geometric, and some authors may give preference to the second proof as possibly more Greek in spirit.

Both proofs are by contradiction, and they both start by assuming something happens that we intend to prove does not happen indeed.

For the first proof we will need the fact that if the product of numbers is even, then at least one of the factors is also even, or equivalently

Lemma. The product of two odd numbers is odd.

Proof. What does it mean to say a number is odd—it means that it is not even. What does it mean to be even? It means you can break the number into two equal pieces, or equivalently it is of the form $x + x$ where $x$ is a whole number, that is, an even number is of the form $2x$. If a number is odd, then we must not be able to break it into two equal pieces, or equivalently, we have a remainder when we divide by 2, and that remainder is necessarily 1, hence an odd number is of the form $2x + 1$. If we multiply two odd numbers: $(2x + 1)(2y + 1)$, we get $4xy + 2x + 2y + 1$ which equals $2(2xy + x + y) + 1$, and hence it is odd.

In particular, as consequence, we get, if the square of a number is even, the number must have been even.
Theorem. $\sqrt{2}$ is irrational.

Proof. Suppose then that $\sqrt{2}$ is a rational number, a quotient of integers. Then $\sqrt{2} = \frac{a}{b}$ where $a$ and $b$ are positive integers, and without loss we can assume that at least one of $a$ and $b$ is odd, for if not we could divide both of them by 2 and obtain yet smaller numbers, and proceed until at least one of them is odd. Remember, then, that at least one of $a$ or $b$ is odd. But since $\sqrt{2} = \frac{a}{b}$, $2 = \frac{a^2}{b^2}$, hence $2b^2 = a^2$, and so we have that $a^2$ is even, but then it follows, by the lemma, that $a$ is even, or equivalently, that $a = 2x$ for some positive integer $x$. But then $2b^2 = a^2 = (2x)^2 = 4x^2$, and canceling a 2, we get that $b^2 = 2x^2$, and contradictorily we get that $b^2$, and hence $b$, is also even.

Another Proof of the Irrationality of $\sqrt{2}$.

But as mentioned above, it is the second proof that many consider closer to the Greek heart. Naturally, it is more geometric in nature. Start with a square.

Let $ABCD$ be the vertices of the square and consider the diagonal $AC$. Suppose that both the side and the diagonal are measurable by some common unit of length.

Take the point $E$ on the diagonal $AC$ such that $AE = AB$ and draw the perpendicular at $E$ until it meets the side $BC$ at $F$.

We argue that since the angle $ACB$ is $45^\circ$, $EF = EC$, hence $CE$ and $EF$ are two sides of a square with diagonal $CF$.

The crucial point is that both the side and the diagonal of this new square are measurable by the same unit of measurement as the original square.
We proceed to argue this crucial point. Start by drawing the line $EB$. 

That the side of the new square is measurable by the old unit is very simple since $AC$ and $AE$ can both be measured, it follows that their difference, $EC = EF$, is also. We next show the diagonal of the new square: $CF$ is also measurable. To do this all we need is to show that $FB$ is measurable since then $CF$ would be the difference of two measurable segments. We show $FB$ is nothing else but $EC = EF$. The triangle $BAE$ is isosceles since $AE = AB$. Hence its angle at $E$ is the same as its angle at $B$. But then in the triangle $EFB$, the angle at $E$ is also the same as the angle at $B$ (they are both complements of the same angle). Hence as promised, $FB = EC$. Hence we have arrived at a new square that has both its side and its diagonal measured by the same old unit of measurement, yet it is smaller than the old one.

This leads to a contradiction, since nothing could stop us from continuing in this fashion, and building yet another new and smaller square in the same fashion by first finding the point $H$ and then building yet another square. We then get but all these squares we make have side and diagonal measurable by our original common unit of length. This unit of length never changes. This will give us a contradiction since we keep building squares with shorter and shorter sides yet their sides can be measured with a whole number of our units, and eventually we will run out of units. To make this last point clearer, suppose our unit is feet, and we started with a side that measured a million feet. The next square can have at most 999,999 feet for its side (it is in actuality much smaller), and the next only 999,998, etceteras. Eventually we have an impossibility!

We cannot help but look at the idea behind the second proof in a more modern context$^1$.

---

$^1$The reader unacquainted with matrix arithmetic may skip the remaining discussion.
Let \[ \begin{bmatrix} s \\ d \end{bmatrix} \] denote the side \( s \) and diagonal \( d \) of the smaller square. How are the dimensions \( S \) and \( D \) of the original square related to these? A little observation will convince you that the following are true:

\[
S = s + d,
D = s + S = 2s + d.
\]

Thus the vector \[ \begin{bmatrix} S \\ D \end{bmatrix} \] is obtained from the vector \[ \begin{bmatrix} s \\ d \end{bmatrix} \] by multiplication by the matrix \[ \mathbf{M} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \]. By applying the geometric argument above in reverse, we have that if \( \frac{d}{s} = \sqrt{2} \), then \( \frac{D}{S} = \sqrt{2} \) also.

But to us, a trivial algebraic argument is also available: to say \( \frac{d}{s} = \sqrt{2} \) is equivalent to \( \frac{d}{s} = \sqrt{2} \) and \( s \) being solutions to \( x^2 - 2y^2 = 0 \), and then when we substitute \( D \) and \( S \) into this equation, we get

\[
(2s + d)^2 - 2(s + d)^2 = 4s^2 + 4sd + d^2 - 2s^2 - 4sd - 2d^2 = 2s^2 - d^2 = 0,
\]

thus another solution. Since we are hence producing infinitely many solutions to the polynomial \( z^2 = 2 \), we are led to a contradiction. This argument is reminiscent to one given by Fermat, many centuries later, which is referred to as the method of infinite descent.

Yet we can also use the idea in an slightly different way. What we can do is keep multiplying by \( \mathbf{M} \), and see what occurs. For example, if we start with \[ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \], then we get consecutively,

\[
\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 3 \end{bmatrix} \mapsto \begin{bmatrix} 5 \\ 7 \end{bmatrix} \mapsto \begin{bmatrix} 12 \\ 17 \end{bmatrix} \mapsto \begin{bmatrix} 29 \\ 41 \end{bmatrix} \mapsto \begin{bmatrix} 70 \\ 99 \end{bmatrix} \mapsto \begin{bmatrix} 169 \\ 239 \end{bmatrix} \mapsto \begin{bmatrix} 408 \\ 577 \end{bmatrix} \mapsto \begin{bmatrix} 985 \\ 1393 \end{bmatrix} \mapsto \begin{bmatrix} 2378 \\ 3363 \end{bmatrix} \mapsto \begin{bmatrix} 5741 \\ 8119 \end{bmatrix} \mapsto \begin{bmatrix} 13860 \\ 19601 \end{bmatrix} \mapsto \begin{bmatrix} 33461 \\ 47321 \end{bmatrix} \mapsto \begin{bmatrix} 80782 \\ 114243 \end{bmatrix} \]

and, finally, we stop at with \[ \begin{bmatrix} 195025 \\ 275807 \end{bmatrix} \]. What we are obtaining are closer and closer approximations to our goal. Namely, if we build a rectangle where one side is the first entry of the vector and the diagonal is the second, then we would be approximating indeed a square figure. For example, in the last pair, the length of the other side squared is, by the Pythagorean Theorem, equal to:

\[
275807^2 - 195025^2 = 76069501249 - 38034750625 = 38034750624,
\]

which can not get much closer to 195025, or equivalently,

\[
\sqrt{38034750624} = 195024.999997.
\]
Chapter 5
Athens
Let no one ignorant of geometry enter my door—Plato’s Academy

It was perhaps the devastating discovery of the irrationality of the square root of 2 that sent such quivers down the Greek mathematical spine that they are, for several centuries, going to insist on geometric constructions as the only source of true knowledge—a view that will be held by some until Newton's time. Since they could do much with their tools: *the straightedge and compass*, their reliability on them wasn't an overwhelming limitation. And even perhaps, just a few centuries after Pythagoras, some Greek minds (including Archimedes) would realize the foolishness of the insistence on straight edge and compass alone, without even one single marking on the ruler.

From the historical point of view, however, there were some constructions they could not accomplish, and these became a famous and important part of their legacy, as well as a meaningful component of their mathematical activity of the period. In particular, three problems and a general construction have been left to posterity. These impossible constructions were not successfully disposed of until the nineteenth century:

1. **The Trisection of the Angle.** When any angle is given, one was required to trisect it. Naturally, as we will see below, they could bisect any angle. The key word in the problem is the word *any*. In other words, they desired a technique for trisection applicable to any angle. Some angles, such as 90°, could be trisected easily—while others such as 60° could not.

2. **The Quadrature of the Circle.** Given a circle, to build a square with the same area. As we will see below they could square any polygonal area, and some curvilinear forms, but not the circle. Archimedes would later even square the parabola.

3. **The Duplication of the Cube.** Given a cube, to construct a cube with twice the volume. As we will see, they could double the square, but not the cube.

Of course, as mentioned above, all of these constructions were to be done with straightedge and compass alone. Their insistence on that—narrow as it was—**forbade even one arbitrary marking on the straightedge.**
Along with the three problems mentioned above, there is a fourth one:

4 Which regular polygons were constructible?

The heptagon, the 7-sided polygon, was particularly fascinating to them since it was the one with the fewest number of sides that could not be done. Around 1800, the young Gauss would explain completely which polygons are constructible and which are not, and why.

But in order to appreciate these problems, as well to study the Greeks attempts, we need to review what is constructible by their tools: straightedge and compass.

1. To bisect a segment, or equivalently, to draw the perpendicular bisector. Draw the two circles centered at each of the end points and going through the other, then join the intersection of the two circles, and this line will actually will the perpendicular bisector of our original segment.

2. To drop the perpendicular to a line from a point not on the line. First draw any circle centered at the point and crossing the line. The two intersections mark a segment than when bisected by the previous construction will give the desired perpendicular.

3. To drop the perpendicular to a line at a point on the line. Draw any circle centered at the point and bisect the segment that is formed by the two intersections of this circle with the line.

4. To build the equilateral triangle on a segment, or to build the square on a given segment. Both of these were easy extensions of previous constructions and we will let the pictures speak for themselves.

5. To bisect any angle. Starting with any angle. Draw an arbitrary circle centered at the vertex of the angle and draw the corresponding chord. Bisect the chord. The angle has also been bisected.

Surprisingly, this does not work for trisections.
6. Given a line and a point not on it, to draw the parallel to the line through the point. This was a common construction, and there were several ways to accomplish it. One of them was as follows. Start with a line and a point off it. Choose an arbitrary point on the line, and centered there, draw a circle that goes through the point not on the line. From the intersection of the circle and the line, draw a circle with the same radius, and also draw a circle with the same radius centered at the point off the line. Consider the point where these two circles intersect. Then the line going through this point of intersection and the point off the line is parallel to the original line.

An alternative way is to draw two perpendiculars.

7. To divide a segment into \( n \) equal parts: Given a segment, divide it into \( n \) equal parts. We will exemplify the general procedure with \( n = 5 \).

Take an arbitrary segment. Draw an arbitrary line through one of the end points of our segment to be divided. And mark off 5 equal segments of arbitrary length in that line. Draw the line connecting the end point of our fifth segment to the other end point of our original segment. Draw the parallels to that line from each of the other points. Similarity of triangles then gives the proportionality of the segments.

8. Doubling the square: Given a square to build the square with twice the area. This was explained by Plato in one of his dialogues. Take your original square and build a square with twice the side. This new square has four times the original area. If we then join the midpoints of the sides, we have exactly half of the big square, or equivalently, twice the little one.

9. Given two segments, to find their arithmetic mean, in other words given \( a \) and \( b \), find \( \frac{a + b}{2} \). This is just the bisection of the segment \( a + b \).

10. Given two segments, to find their geometric mean: Given two segments \( a \) and \( b \), find \( \sqrt{ab} \). This construction was very important to them since it involved one of their dearest means. Starting with the two segments \( a \) and \( b \), build a semicircle with diameter \( a + b \), and erect the perpendicular where the two segments meet until it touches the circle. This segment is the geometric mean.
To prove that this line is indeed of the appropriate length, construct the triangle, which by Thales' Theorem we know it is a right triangle, so by Pythagoras' Theorem,
\[ a^2 + x^2 = y^2, \quad x^2 + b^2 = z^2, \text{ and } y^2 + z^2 = (a + b)^2, \]
so adding the first two equations and substituting, we get
\[ a^2 + x^2 + x^2 + b^2 = (a + b)^2 = a^2 + 2ab + b^2, \]
hence \( x^2 = ab \), and this is what we wanted.

11. Squaring a rectangle: Given a rectangle, to find the square with the same area. This is just the geometric mean of the two sides of the rectangle, for if \( b \) is the base and \( h \) is the height, and then the square with side \( \sqrt{bh} \) has the same area as the rectangle with base \( b \) and height \( h \).

12. Squaring a triangle: Given a triangle, to find a square with the same area. Start with a triangle and draw its altitude. Then take the geometric mean of its altitude and its base, and build the square on that geometric mean. If \( b \) denotes the base of the triangle and \( h \) its altitude, then the geometric mean is \( \sqrt{bh} \), so the square has area \( bh \), which is twice what we want, hence if we take half of the square we will have a square of the same area as our original triangle.

13. Squaring any polygonal area, or equivalently, transforming any polygonal area to a triangle (without changing the area): Given any polygonal area, to find the square with the same area. The idea was to cut off one vertex at a time until we are down to a triangle and then square the appropriate triangle. The key was to exchange two triangles with the same base and height and thus of equal area. We illustrate by transforming a pentagonal polygon into a quadrilateral.

1. Take any three consecutive vertices.
2. Draw the chord joining the two outer ones.
3. Draw the parallel through the middle vertex to that chord.
4. Any triangle with the same base (the chord) and the same height will have the same area.
5. Extend one of the sides in order to form a new triangle.
6. Make the swap, and, presto, a corner has disappeared, and we have a quadrilateral of the same area.
14. **Inscribing the hexagon, or the triangle, or the square.** Given a circle, inscribe these regular polygons in it. To inscribe a polygon what one needs is to build the appropriate angle at the center of the circle, thus the chord that subtends that angle will be the side of the polygon. The hexagon, triangle and square were the easiest of all regular polygons to inscribe. First the hexagon. We need to break $360^\circ$ into 6 equal parts so we need $60^\circ$. That is, we need an equilateral triangle built at the center (two radii and the side of the hexagon) hence that side is the radius, and it will fit exactly 6 times around the circle. For the triangle, once we have the hexagon, just join every other side. Finally for the square (we need $90^\circ$), so just draw two perpendicular radii, and the hypotenuse will be the desired side of the square.

We come now to the polygon that was dearest to the Greeks:

15. **Inscribing the pentagon:** Given a circle, inscribe the regular pentagon in it. This construction is basically equivalent to that of the construction of the golden mean (see construction 16).

1. Start with a circle.
2. Construct two perpendicular diameters.
3. Then bisect any of the radii to obtain the point $X$.
4. Construct a circle with center $X$ and going through one of the points of the perpendicular diameter in the circle, for example the point $A$.
5. Find the point of intersection $Y$ of this circle with the diameter that $X$ is on.
6. Then the segment $AY$ is the side of our pentagon.
We can use modern analytic geometry to analyze the pentagon further. Assuming we are starting with the unit circle with equation \( x^2 + y^2 = 1 \). Then \( X \) has coordinates \((0, \frac{1}{2})\) and \( A \)'s are \((1, 0)\). The distance between \( A \) and \( X \) is thus \( \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2} \), so the point \( Y \) has coordinates \((0, \frac{1}{2} - \frac{\sqrt{5}}{2})\). This implies that the distance between \( A \) and \( Y \) is given by

\[
\sqrt{1 + \left(\frac{1 - \sqrt{5}}{2}\right)^2} = \sqrt{1 + \frac{1 - 2\sqrt{5} + 5}{4}} = \sqrt{\frac{5 - \sqrt{5}}{2}}.
\]

Thus the two vertices \( B \) and \( E \) of the pentagon adjacent to point \( A \) are given by the intersection of the unit circle with the circle centered at \( A = (1, 0) \) with radius \( \sqrt{\frac{5 - \sqrt{5}}{2}} \).

In other words, the intersection of \( x^2 + y^2 = 1 \) with \( (x-1)^2 + y^2 = \frac{5 - \sqrt{5}}{2} \). This leads readily to \( 2 - 2x = \frac{5 - \sqrt{5}}{2} \), or \( x = \frac{\sqrt{5} - 1}{4} \). So the coordinates of the point \( B \) are

\[
\cos 72^\circ = \frac{\sqrt{5} - 1}{4} \quad \text{and} \quad \sin 72^\circ = \sqrt{\frac{5 + \sqrt{5}}{8}}.
\]

Note that both expressions involve square roots (and only square roots). More than 2,000 years later, young Gauss would construct the heptadecagon (17-sided) regular polygon in the Greek way, except his tools were algebraic. He in fact showed that

\[
\cos \left(\frac{360^\circ}{17}\right) = \frac{1}{16} \left(-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + \sqrt{68 + 12\sqrt{17} - 16\sqrt{34 + 2\sqrt{17} + 2}(-1 + \sqrt{17})\sqrt{34 - 2\sqrt{17}}}\right)
\]

—quite an expression solely populated by square roots.

16. **To break an arbitrary segment into the Golden Ratio.** A segment was said to be broken into the **Golden Ratio**

if the short piece was to the long piece as the long piece was to the whole.

Starting with the segment \( AB \).

1. We find the midpoint \( M \) and the perpendicular at \( A \).
2. Then we find the point \( P \) on the perpendicular whose distance to \( A \) is the same as the distance from \( A \) to \( M \). Then centered at \( P \), we find the point \( Q \) whose distance to \( P \) is the same as the distance from to \( P \) to \( B \).
3. Then we find the point \( X \) in the segment \( AB \) is the same as the distance from \( A \) to \( Q \). We claim

\( X \) is the desired point. That is, we need to prove that
From the construction we know that
\[ \frac{BX}{AX} = \frac{AX}{AB}. \]

And from the Pythagorean Theorem: \((BP)^2 = (AB)^2 + (AP)^2\).

Substituting we get: \((AP + AQ)^2 = (AB)^2 + (AP)^2\), or
\[ (AP)^2 + 2(AP)(AQ) + (AQ)^2 = (AB)^2 + (AP)^2, \]
hence \(2(AM)(AQ) + (AQ)^2 = (AB)^2\). Since \(AQ = AX\) and \(2(AM) = AB\),
\[ (AX)^2 = (AB)^2 - (AB)(AX), \]
so
\[ (AX)^2 = (AB)(BX), \]
which is what we wanted.

As mentioned above, the Greeks were particularly fascinated with the pentagon. One of the reasons for this interest was the fact that the Golden Ratio naturally occurred inside the pentagon. Namely, if we look at any two diagonal of the pentagon then they cut each other by the Golden Ratio—in other words, in the picture, we should have that
\[ AX:XB = XB:AB. \]

The proof of this is not hard. Consider the picture on the left. We know that \(\angle AOB = 108^\circ\).

We also know
\[ \angle OAB = \angle OBA = \alpha. \]

But then, \(2\alpha = 180^\circ - 108^\circ = 72^\circ\), so \(\alpha = 36^\circ\). Furthermore, by symmetry, \(\angle AOX = \angle XAO = \alpha\), and \(\beta = 36^\circ\) also. Thus, \(\triangle OXB\) is isosceles, and we have \(XB = OB\). Also, the two triangles \(\triangle OXA\) and \(\triangle AOB\) are similar, hence \(AX : AO = AO : AB\), but \(BO = AO = XB\), and so
\[ AX : XB = XB : AB, \]
which is what we desired.

17. Inscribing other regular polygons: Given a circle inscribe other regular polygons in it, in particular the decagon (10 sides) and the pentadecagon (15 sides). Both of these constructions are easy by-products of previous constructions. The decagon is directly obtainable from the pentagon, and the general observation that once we could get an \(n\)-gon, then one can easily get the polygon with twice the number of sides by just bisecting a side and going on a radius on that bisection. The example from the pentagon to the decagon is typical.

\(\text{Start with a pentagon.}\)
Bisect any side.
Extend the radius through the middle point.
Join the new point of the circle to one of the two vertices of the pentagon.
That is the side of the decagon.

The pentadecagon was more interesting and consisted on imposing a 3-gon (a triangle) and a 5-gon (a pentagon) in the right fashion: and thus obtaining the side of our 15-gon:

In closing, then, what was remarkable to the Greeks was that:
they could bisect any angle, yet they could not trisect every angle;
they could square any polygonal area, yet they could not square the circle;
they could double the square, yet they could not double (duplicate) the cube;
and
while they could build the pentadecagon, they could not construct the heptagon (7 sides).

Naturally we must not lose sight of the arbitrariness of the restrictions, and later in their history they express reservations about the limitations. For example, Archimedes will, in the near future (next chapter), give a trisection by putting a single arbitrary mark on the straightedge.
We must also not lose sight of the tremendous mathematical energy that these problems have generated through the centuries, well into the nineteenth century. The Athenians themselves contributed in many ways. For example, Hippocrates (c. 440 BC) succeeded in squaring a curvilinear area, a shape commonly referred to as a lune. An achievement he was proud of. Naturally the word lune is connected with lunar (or moon-shaped) and we are considering shapes that resemble a moon. Actually the construction is quite simple. Start with an arbitrary semicircle, and on it inscribe an arbitrary triangle.

Recall that from Thales’ theorem such a triangle is necessarily a right triangle so if we label the diameter $c$ and the other two sides $a$ and $b$, then $a^2 + b^2 = c^2$. On each of the sides of the triangle construct a semicircle obtaining a figure resembling our picture. In our picture we can see two moon-shaped figures that are formed from the smaller semicircles when the big semicircle is removed. Remarkably, the area of the two lunes is the same as the area of the triangle.

In other words, the shaded areas in the two pictures are the same. The proof is quite simple. Let $a$ and $b$ denote the sides of the triangle, and let $c$ denote the hypotenuse. Then we know that $a^2 + b^2 = c^2$. But the area of the large semicircle is $\frac{\pi c^2}{8}$, and the area of the smaller semicircles are $\frac{\pi a^2}{8}$ and $\frac{\pi b^2}{8}$, hence the two smaller semicircles when put together amount to the larger semicircle. Although we have used modern algebraic notation, the reasoning is identical to that of Hippocrates, except he would have said something like:

since the area of a circle is proportional to the square of its diameter, and the two squares on the smaller diameters equal the square on the larger diameter, the two smaller semicircles when put together form the same area as the larger semicircle.

Once we have that fact we can easily finish. What is the larger semicircle made out of: the triangle and the two slivers. What are the smaller semicircles made out of: the two lunes and the two slivers, hence if we subtract equals from equals, namely if we take the slivers from our considerations we are left with the two lunes being the triangle.
Another typical Athenian contribution was the curve known as the **quadratrix** attributed to the sophist **Hippias** (c. 420), one of the many curves the Athenians were interested in.

We will discuss it from the modern point of view.

1. Let us start with a square of unit length and vertices $A$, $B$, $C$ and $D$.

2. Suppose that at time $t = 0$, the top horizontal line—the line $DC$—starts moving down with uniform velocity reaching the bottom—line $AB$—at time $t = 1$.

3. Simultaneously, the left vertical line—line $AD$—starts rotating anchored at the bottom—at point $A$—also with uniform velocity until at time $t = 1$, it hits the bottom too—line $AB$.

4. At any one time the two moving lines are intersecting, and it is the path of this intersection that we are interested in.

5. As we connect the points of intersection we will be tracing a curve:

This curve is called the **Quadratrix of Hippias**. His original purpose was to solve one of the three classical problems in Greek geometry: **the trisection of any angle**. We illustrate his technique for this trisection, and we will prove that it is in fact a trisection below, at the end of this section, once we have computed the path of the quadratrix.
Suppose we want to trisect the angle \( \text{MON} \).

1. Draw the quadratrix.
2. Drop the vertical from the point of intersection \( T \) to the horizontal at \( S \).
3. Find the point \( H \) that trisects the segment \( ST \) and draw the horizontal at \( H \).
4. Finally draw the line from \( O \) to the point \( P \) where the quadratrix intersects the horizontal line drawn at \( H \). It is a fact (proven below) that the angle \( \text{PON} \) is a third of the angle \( \text{MON} \).

Actually the quadratrix could be used not just to trisect an angle but also to break it up any way we would like to by taking the appropriate fraction of the segment.

Hippia was extremely pleased when he found out he could use the quadratrix not only to solve one of the classical problems, but two—namely he was able to also square the circle.

What does it take to square the circle? We want to build a square with the same area as a circle. Without loss we can assume the circle has radius 1 hence its area is \( \pi \). Hence the square with the same area will have side \( \sqrt{\pi} \). Since we can do square roots by the geometric mean construction, all we need is to be able to build some segment of length \( \pi \), or since all arithmetic operations could be performed by straight edge and compass alone, all we need is some arithmetic expression involving \( \pi \). As it turns out the length \( \frac{2}{\pi} \) is readily available in the quadratrix! However we need calculus and analytic geometry to derive that result (the original argumentation is not as readily available to us as it was to Hippia).

Of course, by now, it should be clear that one can not construct a quadratrix by straightedge and compass alone. However, Hippia built a mechanical contraption that traced the quadratrix.
In modern terms, and after the advantages of calculus, what do we know about the path of the quadratrix? Without loss of generality, we can assume that the origin is located at point A (so B is the point (1,0), and D=(0,1)).

Let \( x(t), y(t) \) and \( \theta(t) \) denote the coordinates indicated in the diagram below at any time \( t \):

We know: \( y(0) = 1 \), \( y(1) = 0 \), and since \( y(t) \) has uniform velocity, we know \( \frac{dy}{dt} = c \), a constant. Thus \( y(t) = ct + d \), and by using \( t = 0 \), and \( t = 1 \), we have that \( y(t) = 1 - t \) at any time \( t \).

Similarly \( \theta(0) = \frac{\pi}{2} \) and \( \theta(1) = 0 \) and \( \frac{d\theta}{dt} = a \), a constant, so \( \theta(t) = at + b \) for some constants \( a \) and \( b \), so \( \theta(t) = \frac{\pi}{2}(1-t) = \frac{\pi}{2}y(t) \) at any time \( t \).

But then what is \( x(1) \)? Namely what is the \( x \) coordinate of the bottom point of the quadratrix? Since \( \tan(\theta) = \frac{y}{x} \), for any time \( t \),

\[
x(t) = \frac{1-t}{\tan\left(\frac{\pi}{2}(1-t)\right)},
\]

or equivalently,

\[
x = \frac{y}{\tan\left(\frac{\pi}{2}y\right)}.
\]

But then

\[
x(1) = \lim_{t \to 0} \frac{1-t}{\tan\left(\frac{\pi}{2}(1-t)\right)} = \lim_{s \to 0} \frac{s}{\tan\left(\frac{\pi}{2}s\right)}.
\]

But by L'Hôpital rule,

\[
x(1) = \lim_{s \to 0} \frac{1}{\frac{\pi}{2}\sec^2\left(\frac{\pi}{2}s\right)} = \frac{2}{\pi}
\]

Thus a length related to \( \pi \) could be obtained, and thus one could use the quadratrix to square the circle. The table of values on the right just confirms our abstract calculations:

We finish our discussion of the quadratrix with a modern argument that it indeed trisected angles as it was supposed to. Referring to the previous process for trisecting the angle \( \theta = \text{MON} \).

Then, since \( t = 1 - \frac{\theta}{\pi} \), the coordinates of T are (as functions of \( \theta \)):

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>0.000</td>
<td>1.000</td>
</tr>
<tr>
<td>0.050</td>
<td>0.075</td>
<td>0.950</td>
</tr>
<tr>
<td>0.100</td>
<td>0.143</td>
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</tr>
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<td>0.551</td>
<td>0.400</td>
</tr>
<tr>
<td>0.650</td>
<td>0.571</td>
<td>0.350</td>
</tr>
<tr>
<td>0.700</td>
<td>0.589</td>
<td>0.300</td>
</tr>
<tr>
<td>0.750</td>
<td>0.604</td>
<td>0.250</td>
</tr>
<tr>
<td>0.800</td>
<td>0.616</td>
<td>0.200</td>
</tr>
<tr>
<td>0.850</td>
<td>0.625</td>
<td>0.150</td>
</tr>
<tr>
<td>0.900</td>
<td>0.631</td>
<td>0.100</td>
</tr>
<tr>
<td>0.950</td>
<td>0.635</td>
<td>0.050</td>
</tr>
<tr>
<td>1.000</td>
<td>0.637</td>
<td>0.000</td>
</tr>
</tbody>
</table>
\[ y_1 = \frac{2\theta_1}{\pi} \text{ and } x_1 = \frac{2\theta_1}{\pi \tan(\theta_1)}. \]

Then the coordinates of \( H \) are \[ y_H = \frac{2\theta_0}{3\pi} \text{ and } x_H = \frac{2\theta_0}{\pi \tan(\theta_0)}. \]

But then at \( P \), \[ y_P = \frac{2\theta_0}{3\pi} \]

so \[ t_P = 1 - \frac{2\theta_0}{3\pi} \]

hence \[ \theta_P = \frac{2}{3}\left(\frac{2\theta_0}{3\pi}\right) = \frac{\theta_0}{3}, \] as was promised.

Before we discuss the premier mathematician of the Athenian age, Eudoxus, since Athens occupies a remarkable place in the history of philosophy, we mention two philosophers that had a great impact on our subject, Plato and Zeno. The latter first.

\section*{Zeno}

Zeno (c. 450 BC) is best known for his interest on the discrete vs. continuous nature of reality. And he successfully provoked thought about these deep issues by proposing some well-known paradoxes that posed difficulties from both sides of the issue. These issues have stayed relevant until our time. We will illustrate with one of them that argues against time and space being indefinitely divisible, but he could just as successfully argue against a discrete time-space.

\subsection*{Achilles \& the Tortoise}

Achilles and the Tortoise are going to engage in a race. Suppose we give the tortoise a head start. Then by the time Achilles has caught up to where the tortoise was, the latter has gone further, and so when Achilles gets to where the tortoise was, the tortoise will have gone further, and so when Achilles gets to where the tortoise was, the latter will have gone further, and so…

Hence, Achilles never catches the tortoise. What is wrong with this argument?

For several centuries now, the way to successfully elucidate this paradox is to point out that an unbounded number of numbers do not necessarily add to an unbounded number—for example, if we add all of \( 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots \), we just get 2, hence although there is an infinite number of occasions that Achilles will be behind the Tortoise, the total time taken by all these occurrence is very much finite.
Plato

Plato (427 BC-347 BC) has a distinguished place in the history of Western philosophy. In the history of mathematics, his position is not as clear where he had a definite influence in the subject through his famous school the Academy. However, some authors consider that he failed to realize the deeper aspects of the subject and that Plato emphasized some rather simplistic aspects in a narrow fashion. In particular, his insistence in straightedge and compass alone, as well as the view that simple beauty (such as the circle being the most perfect of curves) is more important than reality, are both considered by some as truly detrimental.

The Platonic subject we choose to pursue deeper is that of the regular or Platonic solids. What is a regular solid? We are all familiar with a die or cube that consists of 6 squares making a nice regular box. This is a typical regular solid, which is a 3-dimensional convex solid where each face is a regular polygon, all faces have the same number of sides, and every corner has the same number of faces in it. Convex means the solid is in all places like the outside of a sphere as opposed to like the inside of a bowl. Are there others besides the cube? As it turns out there is a grand total of 5 convex regular solids, and let us give some ideas of why it is so.

Let us start by considering the angle at a corner of a regular polygon. In a triangle, the angle is 60°, in a square it is 90°, in a pentagon it is 108°, and in general if we draw a triangle from one of the sides to the center, the central angle is $\frac{360°}{n}$ where $n$ is the number of sides, hence the angle at the corner is $180° - \frac{360°}{n}$.

Hence for a hexagon, it is 120°, and as the number of sides increases, the angle increases.

Think of a corner of our solid. In order to be a corner, there must be at least three faces coming together at that corner, and each of those faces must be a regular polygon, and all of them have the same number of sides. The total angle at the corner from all the faces have to add up to less than 360° for otherwise there wouldn't be a corner. Hence, since 6 triangles form a flat surface, there could be at most 5 triangles, and similarly there could be at most 3 squares (4 squares add up to 360°), and also 3 pentagons. Finally since the angle at a hexagon is 120° and we need at least 3 faces, there cannot be a regular solid with hexagons for its sides, nor heptagons, nor anything with more than 5 sides.

Hence the possibilities are:
The Greeks constructed a regular solid for each of these possibilities.

### The Tetrahedron.

Tetra means four in Greek. This solid consists of four equilateral triangles and 3 meet at each corner. Of the 4 elements, Plato associated this solid with Fire.

<table>
<thead>
<tr>
<th>Type of Face</th>
<th>Faces at a Corner</th>
<th>Angle at a Corner</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangles</td>
<td>3</td>
<td>180°</td>
</tr>
<tr>
<td>Triangles</td>
<td>4</td>
<td>240°</td>
</tr>
<tr>
<td>Triangles</td>
<td>5</td>
<td>300°</td>
</tr>
<tr>
<td>Squares</td>
<td>3</td>
<td>270°</td>
</tr>
<tr>
<td>Pentagons</td>
<td>3</td>
<td>324°</td>
</tr>
</tbody>
</table>

### The Hexahedron.

This is the common die with 6 (hexa) sides, each a square, and 3 squares meet at each corner. The element Earth corresponded to this solid.

### The Octahedron.

Octa is from the Greek Okto for 8, and hence this solid has 8 triangles with 4 triangles meeting at a corner. The octahedron corresponded to Air.

### The Dodecahedron

This consists of 12 pentagons, and each corner has three faces. This was the last of the regular polyhedra to be discovered and Plato associated it with the Universe itself.

Finally,

### The Icosahedron

has 20 triangles for faces, and 5 triangles come together at a corner. The element Water corresponded to this solid in the Platonic view.

It has been considered by some that the Greek interest on the regular solids stemmed from a gambling interest as they searched for other dice.
Eudoxus

The greatest mathematician of the Athenian age was a disciple of Plato, Eudoxus (c. 370 BC) who actually resolved the problem of the incommensurable that had disturbed his master so much. His contributions are very deep, and some are technical in nature. Alas, none of his writings survive, and our discussion mainly follows references to him by Euclid and Archimedes, both of whom we discuss in the next chapter.

According to Archimedes, Eudoxus was the first one to prove that the volume of a pyramid is \( \frac{1}{3} \) the volume of the circumscribing prism—a fact the Egyptians had known, but did not care to or know how to prove. Eudoxus was also the first one to prove that the volume of a cone was \( \frac{1}{3} \) the volume of the circumscribing cylinder, and this fact was probably discovered by him.

How did he prove these and other facts of a similar sort? He used a technique he devised called the method of exhaustion that is going to become the major method of proof by many of the mathematicians that followed him, including Euclid and the great Archimedes.

Before he could set this new standard of rigor, Eudoxus had to address the problem of the incommensurable. What is the problem indeed? Let us, for an example's sake, take two squares and their diagonals. We would like to say that one diagonal is to its side as the other diagonal is to its respective side. From our modern point of view since both ratios are equal to the \( \sqrt{2} \), we would not have any problem with this assertion. From the point of view of commensurability again this would be easily resolved. But as we know the side and the diagonal are not commensurable. The definition that we find in Euclid (but, probably given by Eudoxus) is as follows:

*Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and the third, and any equimultiples whatever of the second and the fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.*
Using the wonderful (and powerful) language of modern algebra we can clarify the meaning of the definition. Suppose we let $D$ and $S$ be the diagonal and side of one of the squares, while $d$ and $s$ are the diagonal and side of the other square. We want to argue that $D$ is to $S$ ($D : S$) as $d$ is to $s$ ($d : s$). Then the definition would say that they are in the same ratio provided if we take any two positive integers $m$ and $n$, and if we add $D$ to itself $m$ times, and $d$ to itself $m$ times, and we compare the first amount with $S$ added to itself $n$ times, and the second amount (namely, $d$ added to itself $m$ times) with $s$ added to itself $n$ times, then the comparisons are the same way, specifically, $mD < nS$ if and only if $md < ns$. In other words, we are approximating the real number $\frac{D}{S}$ by the rational number $\frac{m}{n}$, and we are saying that

$$\frac{D}{S} \text{ is the same as } \frac{d}{s} \text{ if the same collection of rational numbers is below one as the other.}$$

Let us observe that this definition is absolutely rigorous, and that for except missing the algebraic language to express it, it would be comparable to any definition of a real number found in a modern analysis textbook.

To pursue the method of exhaustion to any depth would take us too far afield. We will be content with stating one of its underlying ideas:

If initially a magnitude is given, and it is diminished by at least half of itself, and the remainder is diminished again by at least half of itself, and so on, a point will be reached where we will have a magnitude smaller than any other prescribed magnitude.

In modern notation, $\frac{a}{1}, \frac{a}{2}, \frac{a}{4}, \frac{a}{8}, \ldots$ converges to 0 regardless of what $a$ is.
Chapter 6
Early Alexandria

Geometry that held acquaintance with the stars,
And wedded soul to soul in purest bond
Of reason, undisturbed by space or time—
Wordsworth

After Plato's Academy had lost some luster, the Museum in Alexandria became the most renowned school in the Greek world. It is in this city founded on the African coast by young Alexander that Greek mathematics and science truly have their Golden Age. Alexandria will be the mathematical center through the existence of the Roman Empire, and although the Roman capital will be the political center of the world, Alexandria will remain its educational center when it comes to Greek (and Roman) science and mathematics. We review some of the history of the period, extending to the history relevant for the next chapter.

Alexandria History Highlights

<table>
<thead>
<tr>
<th>Date</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>332 BC</td>
<td>Alexander founds Alexandria.</td>
</tr>
<tr>
<td>305 BC</td>
<td>Ptolemy I, a general under Alexander, declares himself King of Egypt.</td>
</tr>
<tr>
<td>280 BC</td>
<td>First Roman coinage is developed.</td>
</tr>
<tr>
<td>264-241 BC</td>
<td>First Punic War between Rome and Cartage.</td>
</tr>
<tr>
<td>218-201 BC</td>
<td>Second Punic War. Hannibal invades Northern Italy.</td>
</tr>
<tr>
<td>149-146 BC</td>
<td>Third Punic War. Final Carthaginian defeat. Rome conquers Macedonia and Greece.</td>
</tr>
<tr>
<td>30 BC</td>
<td>Rome conquers Egypt and Alexandria.</td>
</tr>
<tr>
<td>27 AD</td>
<td>Augustus becomes Emperor of Rome.</td>
</tr>
<tr>
<td>114 AD</td>
<td>The Roman Empire reaches its greater extent under Trajan.</td>
</tr>
<tr>
<td>324 AD</td>
<td>Constantine becomes sole Emperor of Rome.</td>
</tr>
<tr>
<td>337 AD</td>
<td>Constantine is baptized in his deathbed.</td>
</tr>
<tr>
<td>391 AD</td>
<td>Theodosius I suppresses pagan worship and makes Christianity the religion of the empire.</td>
</tr>
<tr>
<td>415 AD</td>
<td>Hypatia is murdered in Alexandria.</td>
</tr>
</tbody>
</table>

As aforementioned, we will remain in Alexandria for a long time, roughly more than 600 years. But the most glorious mathematical period is the first century after its founding. This first century of mathematical activity has true giants of the subject and we will consider 4 of them: Euclid, Apollonius, Eratosthenes and the latter's friend, the best scientific mind of antiquity, Archimedes. We will start with the most influential author in mathematical history.
Euclid

Euclid (c. 300 BC) is the author of the most important book in mathematical history. The book is called the Elements and consists of most of the Greek mathematical knowledge at the time it was written. Actually, many of the ideas previously discussed in other sections are documented only in Euclid.

The Elements has gone through more editions, translations and commentaries than any other book except the Bible, and its impact on mathematics is tremendous—hence, we will devote some consideration to this important book.

The Elements consists of 13 Chapters (usually referred to as books).

Book I starts with definitions, common notions and postulates from which a long list of propositions and their proofs follow. This axiomatic approach has been a model for mathematical writing throughout the centuries (despite the fact that many consider it the wrong style). It has been often cited as the ideal presentation, and the epitome of mathematics truth. The first definition is:

A point is that which has no part.

The second one is:

A line is breadthless length.

And so on for a total of 23 definitions. Then come 5 postulates (or axioms). These are things that will be assumed, and after that come 5 common notions (these are milder than postulates in the sense that they are supposedly just common sense).

We will look at the common notions first, and then the postulates.

**Common Notions**

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equal, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

Since all of the Elements have received much commentary through the ages, the common notions have naturally received some. We give an example of a comment to 5, the last common notion, which is worthy of Zeno. If indeed the whole is greater than the part, then how come both have the same number of points? For suppose two segments are given, then draw the lines joining the extremes. Any line drawn from the point of intersection of these lines to either one of the segments contains exactly one point from the other one, and thus a one-to-one correspondence between the two sets of points is set up.
Postulates

1. To draw a straight line from any point to any point. Namely, given two points there is one (and Euclid implies tacitly, only one) line that contains both points.

2. To produce a finite straight line continuously in a straight line. As the picture indicates, this postulate or axiom allows us to replicate any given segment on any line we choose.

3. To describe a circle with any center and distance. Plainly put, given a center and a radius, we can construct the circle with that center and that radius.

4. That all right angles are equal to one another. Self-explanatory.

And last, but most importantly,

5. That if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

The picture—where $\alpha + \beta < 180^\circ$—should clarify the meaning of this important postulate.

It is obvious to the naked eye that the Fifth Postulate is in a different league just by its length and complexity, from all other postulates or common notions.

From time immemorial, mathematicians (literally thousands of them) have tried to prove the fifth postulate as deducible from the other postulates and the common notions since they felt it was not needed.

It was not until the first half of the nineteenth century that some (namely Gauss, Bolyai and Lobachevsky) realized that the fifth postulate may not hold and non-Euclidean geometry was born. This discovery had tremendous impact on both mathematics and physics, and introduced relativistic thinking to almost all branches of human knowledge; in particular, many stopped arguing for what was truth, and instead started considering models that apply at certain times. Thus, statements like the following by eighteenth century philosopher Kant became insupportable:
the concept of [Euclidean] space is by no means of empirical origin, but it is the inevitable necessity of thought.

Curiously, Euclid avoids the use of the fifth postulate until Proposition 29 in the Elements, postponing its use until the inevitable.

Many of the Propositions in Book I are familiar, and we have looked at many of them in previous sections. Proposition 47 of Book I is the Pythagorean Theorem, and Book I basically ends with it. Euclid's proof is as follows:

Start with an arbitrary right triangle $\triangle ABC$ and build the squares on each of the sides. We would like to prove that the two smaller squares when put together make up the larger square. We should observe that since $\triangle ACB$ is right, the side $BC$ is collinear with the side $CD$, and similarly $AC$ is collinear with $CF$.

Draw the line $EB$ forming the $\triangle EAB$, draw the line $CP$ forming the $\triangle CAP$, and draw the perpendicular from $C$ to $PQ$, forming the rectangles $APYX$ and $BXYQ$. First observe that the square $ACDE$ has the same base and height as the $\triangle EAB$, and that rectangle $APYX$ has the same base and height as $\triangle CAP$. Also observe that by side-angle-side, $\triangle EAB$ and $\triangle CAP$ are congruent. Hence we conclude that rectangle $APYX$ has the same area as square $ACDE$.

By going to the other side, we can similarly conclude that square $BGFC$ has the same area as rectangle $BXYQ$. Since the two smaller rectangles make up the large square $APQB$, we have succeeded in proving our theorem.

Book II of the Elements deals with geometric algebra in the Greek style, and we have seen many of these ideas when we first introduced Greek mathematics.
However, we have not seen how a quadratic equation was solved by geometric tools and so we will do that at present. In our modern world, we see only one quadratic equation, the one of the form

$$x^2 + ax + b = 0.$$ 

As we write it, this equation would not make sense to a Greek since we are adding two areas (the first two terms) to a length (the third term), so the first adjustment to be made is to make all the terms have the same dimension (in modern days, we refer to **homogeneous polynomials** when we want to emphasize all terms sharing a dimension). Thus, more in the Greek style would be an equation of the form:

$$x^2 + ax \pm b^2 = 0.$$ 

In addition, since no negative perception of a number could be made in the Greek world, in reality we have three different types of quadratics depending on the sign of $a$ and the sign of $b^2$. If we write them in the form where all coefficients are to be positive, we have actually 3 different quadratics:

$$x^2 + ax = b^2,$$

$$x^2 + b^2 = ax,$$

$$x^2 = ax + b^2,$$

and each of them had slightly different geometric meaning, and hence were thought of in very different contexts.

In the equation $x^2 + ax = b^2$, we are given the square on $b$ and the side $a$, and we are looking for the length $x$ such that the square on $x$ when added to the rectangle with sides $a$ and $x$ gives us the square on $b$.

In the equation $x^2 + b^2 = ax$, while the same square on $b$ and the length $a$ are given, instead we are looking for $x$ so that the rectangle with sides $(a - x)$ and $x$ has the same area as the square on $b$.

and finally

in the equation $x^2 = ax + b^2$, we are looking for $x$ so that the rectangle with sides $a$ and $x$ when added to the square on $b$ has the same area as the square on $x$—or equivalently, the rectangle with sides $(x - a)$ and $x$ has the same area as the square on $b$.

Since there were 3 different quadratics, there were three different methods for solving them. We are just going to look in detail at one of them:
To solve the first type, \( x^2 + ax = b^2 \), we start by building a perpendicular of length \( b \) at the end of a segment of length \( a \). Then with center at the midpoint of the segment of length \( a \), and going through the end of the perpendicular with length \( b \) draw a circle. Extend the segment of length \( a \) until it reaches the circle, the length of this extension is our desired \( x \). To prove this, consider the right triangle with vertices the center of the circle, the end point of the segment with length \( a \) and the end point of the perpendicular. Then the length of one leg is \( \frac{a}{2} \), while the length of the other is \( b \). What is the length of the hypotenuse? Clearly it is \( \frac{a}{2} + x \). By the Pythagorean Theorem then, \( \left(\frac{a}{2}\right)^2 + b^2 = \left(\frac{a}{2} + x\right)^2 \), which is easily seen to be tantamount to our original equation.

Books III and IV of the Elements have many wonderful facts about circles, and regular polygons. We have mentioned some, such as the construction of the pentagon, in a previous chapter. Let us review some other useful ones.

Proposition 18 of Book III states that

If a straight line touch a circle, and a straight line be joined from the center to the point of contact, the straight line so joined will be perpendicular to the tangent.

In a more modern version, A radius at a point of tangency is perpendicular to the tangent line.

Before we look at the proof, consider a point not on a line. What is the shortest distance from the point to that line? It has to be the perpendicular by the Pythagorean theorem since any other line is the hypotenuse of a right triangle with that perpendicular as one of the legs. Now for the proof: Since the tangency point is clearly the nearest point on the tangent to the center of the circle—any other point of the tangent line is outside the circle, hence, further than the radius—the radius has to be perpendicular to the tangent.

Proposition 20 is another useful proposition:

In a circle the angle at the center is double of the angle at the circumference, when the angles have the same circumference as base.
Again, a modern phrasing is:

Consider any arc on a circle, then the angle subtended by the arc at the circle is half the angle subtended by the arc at the center.

In symbols, for the picture below:

\[ \angle AOB = 2\angle APB. \]

This is easily proven if we draw the line OP obtaining 2 isosceles triangles. Hence \( \angle OBP = \angle OPB \), and \( \angle OPA = \angle OAP \). It follows then that \( \angle AOP = 180^\circ - 2\angle OPA \), and similarly, \( \angle BOP = 180^\circ - 2\angle OPB \). But

\[ \angle AOB = 360^\circ - \angle AOP - \angle BOP, \]

so

\[ \angle AOB = 360^\circ - (180^\circ - 2\angle OPA) - (180^\circ - 2\angle OPB) = 2\angle APB. \]

We will leave the proof of when the center lies outside the angle as an exercise.

In particular, as a corollary, we get any two angles subtended by the same chord are equal.

Or, \( \angle AQB = \angle APB \), in the picture.

**Book IV** ends with the construction of the pentadecagon that was outlined in the previous chapter.

**Books V and VI** encompass most of the work of Eudoxus concerning the equality of ratios, and the similarity of triangles.

**Books VII, VIII and IX** constitute the chapters on number theory (or higher arithmetic, or plain arithmetic to the Greeks), and we will explore some of these ideas in detail. Some of them are natural from the Greek point of view, but yet different from the modern point of view. For example, today one says that a whole number \( a \) is a divisor (or factor) of another whole number \( b \) if the ratio \( \frac{b}{a} \) is a whole number. Thus, 2 is a divisor of 6, but not of 5. But to a Greek a more natural view of a divisor would be that the smaller number measured the larger number in the sense that it went into it a whole number of times—as discussed in the section on Pythagoras. In fact, Euclid defines:

A number is a part of a number, the less of the greater, when it measures the greater.

As \( A \) measures \( B \) in the picture

\[ \begin{array}{c}
\text{A} \\
\hline
\text{B}
\end{array} \]

The reciprocal notion to being a divisor is that of being a multiple, namely \( a \) is a divisor of \( b \) is synonymous to \( b \) being a multiple of \( a \). In Euclid’s words:
The greater number is a multiple of the less when it is measured by the less.

Euclid defines prime as a number which is measured by a unit alone. One should never consider 1 a prime number, and it is not clear that Euclid ever did.

Proposition 1 of Book VII then hints at what today is known as the Euclidean Algorithm—although to the Greeks the process was known as Antanairesis:

Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before until a unit is left, the original numbers will be prime to one another.

A reasonable model is that where one has two pieces of cloth (or two strings, or two ropes). One is attempting to measure just using each other. What unit of measurement can one use? One natural first step is to lay the small one on the long one, and to keep doing this until what is left over is too small.

We would refer today to two numbers prime to one another as relatively prime. Two numbers are called relatively prime if their only common divisor is 1. Thus 6 and 35 are relatively prime, but 6 and 4 are not since 2 divides both.

Proposition 2 goes directly to the point of the Euclidean Algorithm:

Given two numbers not prime to one another, to find their greatest common measure.

Today we use the term greatest common divisor rather than measure. Using mainly geometric language, then Euclid gives the basic of the algorithm. We will discuss it from a more modern point of view. Rather than repeated subtraction, we talk of division.

The Division Algorithm:

input: Two positive integers: $D$ & $d$.
output: Two nonnegative integers: $q$ & $r$, quotient and remainder.
requirement: $D = qd + r$, $0 \leq r < d$.

This process is nowadays known as long division. The numbers $D$ & $d$ are read in order, $D$ is the dividend, $d$ is the divisor, then $D$ is, uniquely, a multiple of $d$ plus a remainder which is nonnegative, but less than $d$.

When the remainder is 0 we say $d$ divides $D$. Again,

$d$ divides $D$ if $d$ measures $D$.

from the Greek point of view. Easily, as Euclid claims, if $d$ measures $D_1$ and $D_2$, it
certainly measures $D_1 + D_2$. Also if $d$ measures $D$, then it certainly measures any multiple of $D$. Of course, one can consider this nothing but repeated subtraction since from the model point of view, that is really what is being done. For example, if we let $D = 2133$ and $d = 138$. Then since $2133 = 15 \times 138 + 63$, we have $q = 15$ and $r = 63$.

If we get no remainder, we have found the their largest common measure—the smaller segment. The larger piece of cloth is so many copies of the smaller one, and one has succeeded in measuring the pieces of cloth. But suppose one does get a remainder, what does one do now? Start by observing that

**any piece of cloth that measures $D$ and $d$ will also measure the remainder $r$ since $r = D - qd$,**

or saying it in yet another way,

**if $n$ is a common divisor of both $D$ and $d$, then $n$ divides $r$.

One then take this smaller piece of cloth, $r$, the remainder, and use it to compare it with the smaller of the two pieces we started with, $d$, the divisor, and proceed to get yet a smaller remainder (possibly 0) by doing the division algorithm on our new pair: $d$ and $r$. Suppose first that indeed we get no remainder (in other words, a remainder of 0). Then we can argue that $r$ is the largest measurement that will measure both $D$ and $d$. First $r$ measures $d$ since we get no remainder, second $r$ also measures $D$ since $D = qd + r$, a multiple of $r$. Moreover, we have already argued that anything that measures $D$ and $d$ also measures $r$, so $r$ is the largest common measurement for $D$ and $d$.

Suppose instead that when we divided $d$ by $r$ we did get a nonzero remainder, let us refer to it by $r_3$ (as we will label: $D = r_0$, $d = r_1$, and $r = r_2$ in the example below). Moreover, any piece of cloth that will measure both $D$ and $d$, will clearly also measure $r$, so it will measure the new remainder, $r_3$, also, by repeating the argument. We then try to measure $r$ by $r_3$, and suppose we do succeed. Then $r_3$ measures $r$ and $d$, so it also measures $d$ and $D$, and we get that $r_3$ is the largest common measurement.

But what if we got yet another remainder? We continue with the same strategy. We are getting smaller and smaller remainders all the time, so will the enterprise end? Numerically, we are dealing with integers, and since the remainders are getting smaller, and they are all nonnegative,

**the process does end,**

and eventually we will have a remainder 0. The last remainder before we hit zero is the largest piece of cloth that will measure both of our originals, or in numerical terms, it is the greatest common divisor of our original numbers. As in the picture, we start with $r_0$ and $r_1$ (which we called $D$ and $d$ before). By attempting to measure $r_0$ using $r_1$, we see
that \( r_1 \) fits three times into \( r_0 \) while leaving a remainder of \( r_2 \). Now we attempt to measure \( r_1 \) using \( r_2 \) and we obtain a remainder of \( r_3 \). Again, we now try to measure \( r_2 \) using \( r_3 \) and we get a quotient of 1 and a remainder of \( r_4 \). Finally, \( r_4 \) fits exactly twice into \( r_3 \), and we can see in the picture it fits evenly into each of the preceding \( r \)'s: 3 times, 5 times and 18 times, respectively.

By now we have informally argued for:

The Euclidean Algorithm

**input:** two positive integers: \( D \) & \( d \)

**output:** their greatest common divisor, (g.c.d., from now on), \( \delta \):

**description:** Label \( D = r_0 \) and \( d = r_1 \). Apply the division algorithm to them to obtain \( q_1 \) and \( r_2 \), if \( r_2 = 0 \), then \( r_1 \) divides \( r_0 \), and then their g.c.d., \( \delta \), is \( d = r_1 \). Otherwise let \( r_2 \) be the new divisor and apply the division algorithm to the pair \( r_1 \& r_2 \), to obtain \( q_2 \) and \( r_3 \). If \( r_3 = 0 \), then \( \delta = r_2 \). Otherwise let \( r_3 \) be the new divisor and apply the division algorithm to \( r_2 \) and \( r_3 \), to obtain \( q_3 \) and \( r_4 \). Eventually, since \( r_0 > r_1 > r_2 > r_3 > \cdots \), we must obtain a remainder of 0, and then the previous remainder is the g.c.d. of \( D \) and \( d \). In symbols, \( r_0 = q_1 r_1 + r_2 \), where \( 0 \leq r_2 < r_1 \), if \( r_2 = 0 \), \( \delta = r_1 \), and we are done. If not, let \( r_1 = q_2 r_2 + r_3 \) where \( 0 \leq r_3 < r_2 \). If \( r_3 = 0 \), \( \delta = r_2 \), if not, let \( r_2 = q_3 r_3 + r_4 \), etcetera.

Observe the important remark made in passing above that the g.c.d. of two numbers is exactly that, the greatest of all common divisors, but that actually it obtains this property because every other common divisor divides it, hence the g.c.d. is the larger one.

**Example.** Let \( D = 59,059 \) and \( d = 12,508 \), then

so the algorithm is finally over and \( \delta = 59 \) is the g.c.d. since that is the last nonzero remainder. Hence, \( D \) and \( d \) have only two common divisors 1 and 59 (since 59 is prime).

<table>
<thead>
<tr>
<th>Stage</th>
<th>Dividend</th>
<th>Remainder</th>
<th>Quotient</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>59059</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>12508</td>
<td>9027</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>9027</td>
<td>3481</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3481</td>
<td>2065</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>2065</td>
<td>1416</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1416</td>
<td>649</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>649</td>
<td>118</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>118</td>
<td>59</td>
<td>5</td>
</tr>
<tr>
<td>8</td>
<td>59</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

One possible way to modernize the algorithm is to put in the form of a table.

A few centuries later, Indian mathematicians would provide a significant addition to this algorithm. We will study that in a future chapter.

**Proposition 29** claims that

Any prime number is prime to any number which it does not measure.
Today we would say if \( p \) is a prime, then for any \( n \), either \( p \) divides \( n \), or \( p \) and \( n \) are relatively prime.

Euclid could argue that every number has a prime factor,

**Proposition 31. Any composite number is measured by some prime number.**

His argument is still very much ours: take a number \( m \). Then it is either prime or it isn't. If it be prime, we are done. If not, then it has some factor \( n \neq 1 \). If \( n \) is prime, we are done. If not, \( n \) has a factor \( q \), and if \( q \) is prime, we are done since any factor of \( n \) is also a factor of \( m \). If not, …. The fact that we have positive integers forces the inevitability of stopping.

Moreover, one could perhaps state that Euclid was aware of the theorem referred to today as the Fundamental Theorem of Arithmetic, namely that every positive integer can be factored into primes in only one way, however, this theorem as such is not stated in the Elements.

The reciprocal notion to being a divisor is that of being a multiple, namely \( a \) is a divisor of \( b \) is synonymous to \( b \) being a multiple of \( a \). To Euclid, as before the greatest common divisor would be the largest segment that would measure two given segments while the least common multiple was the smallest one that could be measured by two given ones:

**Proposition 34. Given two numbers to find the least number which they measure.**

We finish our discussion on Euclidean number theory with a proof of Euclid's theorem that there is an infinity of primes. Of course, his statement is more cautious (Book IX, Proposition 20):

**Theorem. Euclid.** Prime numbers are more than any assigned multitude of prime numbers.

*Proof.* Suppose that \( a, b, c, d, \ldots, z \) are primes. We claim there is some prime not on that list. Consider \((abc\ldots z)+1 = m\). Then some prime has to divide \( m \), but none of the given primes could since all of them divide \((abc\ldots z)\), and hence if any of them also measured \( m \), then it would measure the difference, which is 1, and this is impossible.

The proof is often misunderstood as claiming more than it does. What the proof is claiming is that if we take a collection of primes, and multiply them and then add 1 to the product, then this new number will either be prime or will be divisible only by primes that were not members of our original collection, and actually both of these possibilities occur often.

We will exhibit with a table by using the first so many consecutive primes. To simplify
notation, and in honor of Euclid, we will let $E_n$ denote the product of the first $n$ primes plus 1.

<table>
<thead>
<tr>
<th>Stage</th>
<th>Prime</th>
<th>$E_n$</th>
<th>Factorization</th>
<th>Stage</th>
<th>Prime</th>
<th>$E_n$</th>
<th>Factorization</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>3</td>
<td>Prime</td>
<td>2</td>
<td>3</td>
<td>7</td>
<td>Prime</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>31</td>
<td>Prime</td>
<td>4</td>
<td>7</td>
<td>211</td>
<td>Prime</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
<td>231</td>
<td>Prime</td>
<td>6</td>
<td>13</td>
<td>30031</td>
<td>59×509</td>
</tr>
<tr>
<td>7</td>
<td>17</td>
<td>51051</td>
<td>$19\times277$</td>
<td>8</td>
<td>19</td>
<td>969969</td>
<td>347×27953</td>
</tr>
<tr>
<td>9</td>
<td>23</td>
<td>223092871</td>
<td>$317\times703763$</td>
<td>10</td>
<td>29</td>
<td>6469693231</td>
<td>331×571×34231</td>
</tr>
<tr>
<td>11</td>
<td>31</td>
<td>200560490131</td>
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<td>12</td>
<td>37</td>
<td>7420738134811</td>
<td>181×676421×60611</td>
</tr>
</tbody>
</table>

Note that at stage 7, we have 19 being a divisor of $2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 + 1$, the next prime, so the conditions are precise.

A thoughtful reader, aware of Greek views, would question this proof since the multiplication of many numbers would be something that a Greek would find nonsensical. How to avoid this? One obvious way (and actually the way Euclid thought about it) was to consider the smallest length that was measured by all of the lengths $a$, $b$, $c$, ..., $z$, in other words, the least common multiple of these numbers, and then we always stay in measurements that are linear so dimensions would not plague us. Hence instead of taking $(abc\cdots z) + 1$, we would take the least common multiple of all the numbers with a unit added to it. Then it is clear that none of the measurement we started with could measure this new length. Which proof may seem more natural is a question of perspective.

We finish our discussion of Euclid by briefly describing what is contained in the remaining books, some of the deeper and more difficult material. Book X deals with the theory of incommensurables, while Books XI, XII and XIII deal with the geometry of space, and it is in these books that claims about the volume of the cylinder, the pyramid, the cone and other solids, as well as the classification of the regular (or Platonic) solids are discussed.

Eratosthenes

A close friend and correspondent of Archimedes, Eratosthenes (276 BC-194 BC), wrote a major book in Geography, and in mathematical circles, is known for two contributions: his sieve and the measurement of the Earth. We will start with the latter.

Alexandrine scientists did not think the Earth was flat, on the contrary they believed it to be a sphere, and they could offer arguments for their beliefs. First was the fact that the sun and the moon were round, and not flat since the moon could be seen with the naked eye to be a solid. But even more convincingly, when ships approached the shore one could see the mast before one could see the rest of the ship indicating there was definite curvature to the surface of the Earth.
Knowing this, Eratosthenes set out to measure the circumference. He used geographical facts he was acquainted with: the cities of Syene (present day Aswan) and Alexandria were approximately on the same meridian, so the occurrence of noon would be simultaneous in both places. He also knew the distance between the cities.

He observed that at midday on a certain day (summer solstice to be precise), the sun shone brightly at the bottom of a deep well in Syene, thus he deduced that on that day at that time, the rays of the sun were perpendicular in Syene. He also supposed that since the Sun was so far from the Earth one could assume that the rays of the sun are parallel at all points of the Earth.

Hence if he could measure the angle \( \alpha \) of the rays in Alexandria, he could estimate the circumference of the Earth. He accomplished the measurement of that angle by planting a vertical pole at Alexandria and using the shadow at noon.

As it turned out the angle in Alexandria was 7.2° which is \( \frac{1}{50} \) of 360°.

Hence the distance from Alexandria to Syene which was 5000 stades (a unit of linear measurement) is also \( \frac{1}{50} \) of the circumference of the Earth, and thus he estimated the circumference of the Earth to measure 250,000 stades which he adjusted to 252,000 to make it divisible by 60.

Although we do not know what the stade used by Eratosthenes measured exactly, under one possibility his calculations amount to 39,690 km as compared to the actual measurement of 40,075 km, which means his calculation was more than 99% accurate. In any case, his technique was immaculate.

The sieve of Eratosthenes is a way for finding primes. Of course every one knows 2, 3, 5,
7, 11, 13, 17 and 19 are primes, but how about 1111111111? Is this number prime? A fundamental computation, with very modern repercussions, is to be able to decide whether a number is a prime or not.

We will use the sieve to discover the primes that are less than or equal to 50. The process is very simple. Start with the numbers from 1 to 50 in a table, and ignore 1 since 1 is not a prime. The first number that is not ignored is 2. Highlight it, so 2 is a prime and then cross out every number when one counts by twos. Hence 4, 6, 8, 10, 12, etcetera, get crossed out.

The first number that has not been underlined is 3. Hence 3 is a prime. Highlight it and proceed to cross out every number that is encountered when counting by 3's.

Since 5 has not been underlined, 5 is a prime, highlight it and proceed to count by 5's.

Whatever numbers remain that have not been crossed out have to be primes. Why is it then that after running just through 2, 3, 5 and 7, we know that, for example, 47 is a prime.

The reason for this is simple. Especially if we think Greek. To us, we think of factorization as purely a numerical quest, but to the geometrically minded Greek, factoring amounted to rearranging rectangles.

Thus they could see the two non-trivial factorizations of 12: \(2 \times 6\) and \(3 \times 4\) in very graphical terms.

Let \(x\) be a real number. We define \(\lfloor x \rfloor\) (it reads: the floor of \(x\)) to be the largest integer less than or equal to \(x\). Thus \(\lfloor \pi \rfloor = 3\) while \(\lfloor e \rfloor = 2\). Often, \(\lfloor x \rfloor\) is written \(\lceil x \rceil\). Dually, one lets \(\lceil x \rceil\) (the ceiling of \(x\)) be the smallest integer greater than or equal to \(x\). Thus, \(\lceil \pi \rceil = 4\). These two functions occur often
when one considers rounding off numbers.

A rationale for Eratosthenes' sieve is achieved in the statement:

**Theorem.** Let \( n \) be composite. Then \( n \) has a prime divisor \( k \) with 
\[
1 < k \leq \left\lfloor \sqrt{n} \right\rfloor.
\]

**Proof.** Suppose \( n \) is composite, then \( n = km \) for some \( k \) and \( m \) greater than 1. Without loss of generality, \( k \leq m \). But then \( k^2 \leq km = n \), hence \( k \leq \sqrt{n} \), and since \( k \) is an integer, 
\[
k \leq \left\lfloor \sqrt{n} \right\rfloor.
\]
If \( k \) is prime we are done, if not, take any prime divisor of \( k \). Geometrically, think of all the shapes of \( n \), and take one with the smallest length bigger than 1, then that rectangle has to be larger than the square on the smaller side. 

Since \( \left\lfloor \sqrt{50} \right\rfloor = 7 \), we have indeed tested all necessary primes for \( n \leq 50 \). Notice that the same primes would have been sufficient for testing all the way to \( n = 100 \), since 7 is still the largest prime \( \leq \sqrt{100} \).

Geometrically, for example, the nontrivial factorizations of 12 are given by the two rectangular arrays above, and one side always has to be smaller than \( \left\lfloor \sqrt{12} \right\rfloor = 3 \). We end the sieve discussion by finding the primes up to 800.

In order to find all the primes \( \leq 800 \), we need to run the sieve through all the primes \( p \) that satisfy \( p \leq \sqrt{800} \approx 28 \). In other words, we need to run it through 2, 3, 5, and 7, and 5 more primes: 11, 13, 17, 19 and 23.

We do not need to go further than 23 since the next prime is 29 and \( 29^2 = 841 \)—or, equivalently, \( \sqrt{800} = 28 \). In the table in the next page, we have highlighted the primes. Thus 499 is a prime, and 797 is the largest prime in our group.

Often, the complete sieve would be extremely long to use with the purpose of finding whether a given number is a prime or not. For example, suppose we wanted to know whether \( 2047 = 2^{11} - 1 \) is a prime. Then we can execute the sieve but just for that number. What we do is keep testing primes to see whether they divide it or not, and we keep testing until we have reached 43 since \( \left\lfloor \sqrt{2047} \right\rfloor = 45 \). Thus, either 2047 is prime or it has a prime divisor \( < 43 \). Indeed, \( 2047 = 23 \times 89 \). Take 10001, \( \left\lfloor \sqrt{10001} \right\rfloor = 100 \), so either it is prime or it has a prime divisor below 97, and actually \( 10001 = 73 \times 137 \). The sieve is most exhausting in cases like the last one where the two factors are close to the square root of the number. For example, \( 11,663 = 107 \times 109 \) would require a maximum effort.
Archimedes

Archimedes (287 BC-212 BC) is almost universally considered the greatest scientific mind of antiquity. Although he was born and died in Sicily, he was educated in Alexandria and kept close ties with that city until his death—in particular, through correspondence with Eratosthenes. He left behind impeccably written manuscripts on a variety of deep topics in mathematics, mechanics, physics and engineering. We will discuss one at length and dabble lightly on some of the others.

We start with the work of Archimedes associated with the famous Eureka story. The story goes that King Hieron (who was his relative) asked Archimedes to determine whether a crown that had been made for him had used the correct amounts of gold and silver that the artisan claimed, or whether more silver and less gold had been used. Naturally, the crown was to be preserved, so melting and separating was out of the question. The weight of the crown was known, but not its volume.

If he could find the volume, he would have enough information to determine the amount of each metal (in modern terms, he would have two linear equations on two unknowns). The legend goes that while taking a bath, Archimedes noticed that the amount of water displaced while entering the tub was related to the volume that was being submerged in the water, and this gave him a clue as to how to find the volume of the crown without destroying it. He was so excited by this discovery that he ran through the city—some say, naked—yelling Eureka, which means I have found it.

On Floating Bodies is the work in which he presented his hydrostatic principles we discussed above. One of them being, in his own words:

If a body which is lighter than a fluid is placed in the fluid, it will be immersed to such an extent that a volume of fluid which is equal to the volume of the body immersed has the same weight as the whole body.

The principle used to solve the problem of the crown (if indeed it is a true story) could have been:

If a body is placed in a fluid which is lighter than itself, it will fall to the bottom. In the fluid the body will be lighter by an amount which is the weight of the fluid which has the same volume as the body itself.

Hence, the volume of the crown was the same as the volume of the water displaced by the crown when submerged in the water.

Next we will discuss one of his favorite subjects—levers, as his famous dictum asserts:

Give me where to stand, and I will move the earth.

In fact when defending his native city of Syracuse from Roman attack, during the first Punic War, Archimedes used levers and pulleys in a multitude of ways, and his
reputation was such that the Roman army was intimidated by any movement behind the walls of the city. Alas, Syracuse did fall to the Romans, and Archimedes was killed while drawing in the sand by an impatient Roman soldier, waiting to escort him to his general.

On the Equilibrium of Planes or Centers of Gravity (Parts I and II) is concerned more with mechanics and statics than with mathematics, yet we will mention two elementary, but interesting facts from this work. He discusses the Law of the Lever which he used so well not only in a physical manner, but also as a guide to mathematical reasoning as we will see further below. The Law of the Lever in his own words:

Commensurable magnitudes are in equilibrium when they are reciprocally proportional to the distances at which they are suspended.

Which means that in the picture:

\[
\frac{W_1}{d_1} = \frac{W_2}{d_2},
\]

or equivalently, to us,

\[W_1d_1 = W_2d_2.\]

As he was interested in more complicated problems including the center of gravity of a parabolic segment, he had to discuss the center of gravity of a triangle.

The center of gravity of a flat object of uniform density is that point at which it can be balanced. For a triangle the answer is easy. A median of a triangle is a line joining the midpoint of a side to its opposite vertex. As it turns out the three medians of a triangle intersect, and that point is the center of gravity of the triangle.

Theorem. Center of Gravity. The three medians in a triangle intersect.

Proof. Consider an arbitrary triangle \( \triangle ABC \), and let \( D \) be the midpoint of side \( AB \), and \( E \), the mid point of side \( AC \). Let \( X \) be the intersection of \( DC \) and \( EB \). Then \( \triangle ADE \) is similar to \( \triangle ABC \) by SAS. So \( DE = \frac{1}{2} BC \) and \( DE \) is parallel to \( BC \). This forces \( \triangle XED \) to be similar to \( \triangle XBC \), and so \( DX = \frac{1}{2} XC \), and that determines the location of the point \( X \) independent of the median \( BE \), and so the point \( X \) is in all medians.

More than two thousand years later, the great Euler would prove that the three perpendiculars of a triangle intersect in what is now called the Euler point, or orthic center.

The next work that we discuss is one of Archimedes’ best known.
Measurement of the Circle was concerned with the computation of the area of a circle. In it, he proves rigorously that the area of a circle is the same as the area of the triangle with base the circumference and height the radius.

Then he sets out to estimate the circumference, more accurately, the ratio of the circumference to the diameter, by using the perimeter of inscribed and circumscribed polygons around a circle.

Since the eighteenth century, we have referred to this ratio by the letter \( \pi \), and we will follow Archimedes reasoning, but we will adopt modern ways and let the diameter of the circle be one unit in length, and thus, its circumference is exactly \( \pi \).

Start by inscribing and circumscribing hexagons on our circle. Since the shortest distance between any two points is the straight line, the perimeter of the inscribed hexagon is less than \( \pi \)—the side of the polygon being less in the length than the respective arc.

Less obvious, yet true, is the fact that the perimeter of the circumscribed polygon is more than the circumference.

In order to assert this less obvious fact, he stated a clear intuitive assumption:

That of all lines that are not straight, if, lying in one plane, they have the same extremities, two are unequal whenever both are concave in the same direction and moreover one of them is wholly included between the other and the straight line with the same extremities with it, or is partly included by and partly coincides with the other; and that the line which is included is the lesser.

In modern words, given two points, and two curves that do not change concavity going between them, if one of them is contained in the other, then it is shorter, thus in the picture, the inside curve is shorter:

Back to the hexagons, suppose we let \( p_1 \) and \( q_1 \) denote the perimeters of the inscribed and circumscribed hexagons respectively. Then we have, \( p_1 < \pi < q_1 \),
and since the radius of the circle is the side of the hexagon, we have easily that $p_1 = 3$. Since the interior angle of a hexagon is $120^\circ$, he could use the triangle obtained at a corner of the circumscribing hexagon to accurately get $q_1 = \frac{6}{\sqrt{3}}$, and we have our first approximation:

$$3 = p_1 < \pi < q_1 \approx 3.4641.$$

Then by halving sides, he built dodecagons, and thus improving both sides of the approximation by estimating $p_2$ and $q_2$. And so on, going through polygons with 24, 48 and finally 96 sides, to end with his famous final estimate:

$$3 \frac{10}{71} < \pi < 3 \frac{1}{7},$$

or in decimal notation:

$$3.1408 < \pi < 3.1428.$$

Although much work went into this estimate—imagine 96 sides—we only get 2-digit accuracy, and indeed Apollonius is reported to have achieved more accuracy with a different estimate.

Quadrature of the Parabola concerns the measurement of the area of a parabolic segment, and he gave the elegant answer:

It is manifest that the area of any segment which is comprehended by a straight line and an orthotome (his word for parabola) is greater by one-third than the triangle which has the same base as the segment and equal height.

Let us understand what he is stating. We start with a parabolic segment:

And we want to compute the area. Archimedes claims it is one-third more than the triangle with the same base and height.
But what is the height of this segment? It is the distance between the base and the tangent which is parallel to the base.

Thus he claims the area of the parabolic segment is \( \frac{4}{3} \) the area of the triangle.

**On Spirals** is the work in which he introduced the spiral that bears his name.

In modern notation, we define **Archimedes' spiral** by the polar equation \( r = a\theta \) where \( a \) is some constant. But, with echoes from **Hippias**, we could define it as follows. Suppose we have a ray, and at the end of the ray, a bug starts moving with uniform velocity simultaneously as the ray starts rotating with uniform velocity anchored at its end. What is the path of the bug? It is one of Archimedes' spirals.

As usual, he was pleased by the nice pattern he noticed about these spirals, namely the area made by one full cycle of the spiral (together with the ray from the beginning to the end) is exactly one-third the area of the circle with radius at the end point and center at the beginning point. In the picture, the shaded area is one-third the area of the circle.

In the **Book of Lemmas**, Archimedes gives a variety of results some of which we look at in the exercises, and it is there that he gives his simple trisection of any angle.

Suppose \( \angle AOB = \beta \) is to be trisected. Then with any radius \( r \) one chooses, draw a circle with center \( O \). Let the circle intersect \( OA \) at \( Q \). Here comes the step that can not be done with straightedge and compass alone. Find the point \( X \) such that
when the line from \( X \) to \( Q \) is drawn, and \( P \) is the point where it intersects the circle, then \( XP \) has length \( r \). Then we claim \( \alpha = \angle PXO \) is one-third of \( \angle AOB = \beta \). The proof is very easy. Since \( PX = PO \), \( \alpha = \angle POX \). Also, since \( OP = OQ \), \( \angle QPO = \angle QQP \), but \( \angle QPO = 2\alpha \) since it is an exterior angle to \( \triangle OPX \). But then since \( \beta \) is an exterior angle to the triangle \( \triangle XOQ \), \( \beta = \angle QXO + \angle OQX = \alpha + 2\alpha = 3\alpha \).

Archimedes must have smiled at the simplicity of this construction.

7. **The Sand-Reckoner** is more interesting philosophically and astronomically than mathematically. His purpose was to count the number of grains of sand in the whole universe, and although hampered by not the best notation for numbers, he argues, based on Aristarchus’ astronomy, that the number of grains is bounded by \( 10^{63} \). Aristarchus was one of the best Greek astronomers, and among his suggestions was the fact that the Earth went around the sun, instead of the other way around.

And finally we arrive at one of his proudest achievements:

8. **On the Sphere and Cylinder** where he computed the volume of a sphere. We have seen above that Eudoxus proved that the volume of the cone is one-third the volume of the circumscribed cylinder. Before Archimedes the volume of the sphere was not known, and he was particularly pleased with the answer he got. Consider a sphere, the circumscribing cylinder, and inside the cylinder the inscribed cone. Then the ratio of the volumes of the cone:sphere:cylinder is 1:2:3. Thus the cone is one-third the cylinder, or one-half the sphere, while the sphere is two-thirds the cylinder.

Equivalently, the cone and the sphere combined make up the cylinder. How he arrived at this wonderful answer (he requested be engraved on his tombstone), and how he proved it is what we delve in next.

What is surprising about the argument is that we use another cylinder and another cone to do the reasoning.

We start with a circle with center \( O \) and diameter \( AB \). At \( B \) erect the perpendicular \( BD \) of length \( AB \), and join \( AD \). Let \( C \) be the point of intersection of \( AD \) with the circle. Observe that \( \angle DAB = 45^\circ \) and hence \( OA=OC \). Extend the diameter \( AB \) to
the point \( P \) where \( PA \) is the same as \( AB \). Note that if we rotate the picture on the axis \( PB \), we will get a sphere, a cone and a cylinder (but not the circumscribing cylinder and inscribed cone).

Take an arbitrary point \( X \) on the diameter \( AB \), and draw the vertical at \( X \). And let \( Z \) be the point where the perpendicular intersects the line \( AD \), and let \( Y \) be the point in the perpendicular that intersects the circle. Clearly, the distance between \( A \) and \( X \) is the same as the distance between \( X \) and \( Z \). Let \( z \) denote this distance, and let \( y \) denote the distance from \( X \) to \( Y \). By the construction of the geometric mean, \( y \) is the geometric mean of \( z \) and \( 2r - z \) (we are letting \( r \) denote the radius \( AO \)), that is,

\[
y = \sqrt{z(2r - z)}.
\]

If we take a slice at \( X \) from each of the three solids obtained from rotating the picture around \( PB \), we have three circles, one with radius \( z \), one with radius \( y \) and the last with radius \( 2r \).

We are going to balance the three circles as the picture indicates, the two small ones at \( P \), and the large one at \( X \). Is there equilibrium?

The ratio of the distances is \( \frac{z}{2r} \). The reciprocal of the ratio of the weights is \( \frac{y^2 + z^2}{(2r)^2} \). Since \( y^2 = 2rz - z^2 \), we have equilibrium for any point \( X \).

If we think of each of the solids as comprised of the slices, then we will stack all the slices of the cone and the sphere at \( P \), but the slices of the cylinder are stacked each at \( X \) for all the various points \( X \). Hence we get the following equilibrium—see picture on the right.

But since the center of gravity of a cylinder is at its center, we get the equivalent equilibrium of solids—see picture below.
This last claim then states that the volume of the cylinder with radius $BD$ and height $AB$ is twice the volume of our sphere (which has radius $OA$) and the cone with radius $BD$ and height $AB$ since the distance $AP$ is twice the distance $AO$.

To simplify matters, we will let $S$ denote the volume of the sphere, $CY$ denote the volume of the cylinder with radius $BD$ and height $AB$ and $CO$ the volume of the cone with the same radius and height. Thus what we have shown is that $S + CO = \frac{1}{2} CY$. But we know from Eudoxus, that $CO = \frac{1}{3} CY$, thus $S = \frac{1}{6} CY$. But $CY$ is 4 times the volume of our circumscribing cylinder since its radius is just $AO$. If we let $C$ denote the volume of the circumscribing cylinder, then $C = \frac{1}{4} CY$, and thus we arrive at

$$S = \frac{1}{3} C$$

which is what we need. We already knew that the inscribed cone has one-third of the volume of the circumscribing cylinder.

There is also the famous Archimedes' Cattle Problem, which involves four kinds of cows and four kinds of bulls (hence 8 unknowns) satisfying 7 linear equations. In addition, there is a requirement that 2 of the kinds of bulls add up to a square number while the remaining 2 add up to a triangular number. The problem with just the first 7 linear equations is bad enough where 8 digit solutions are required. But if we add the last two requirements, the total number of bovine is 5,916,837,175,686.

Finally, when Pappus is discussed in the next chapter, we will look into the Archimedean or semiregular solids, which are similar to the Platonic solids in that all faces are regular polygons, but not all faces have to have the same number of sides. Pappus attributes all 13 of these polyhedra to Archimedes. We end this wonderful chapter in the history of mathematics with a discussion of the younger rival of Archimedes:

**Apollonius**

The major contribution of Apollonius (265 BC-170 BC) is the book *Conics*, which as expected is dedicated to the study of the conic curves. The most important family of curves (after the straight line and the circle) is that of the *conic sections*. Although they are older than Apollonius, it is his name that is closely associated with them, and he is the person that names them the *ellipse*, the *parabola* and the *hyperbola*.
It is also Apollonius that showed the connections between them and the cuts of a cone—
namely that when a cone is cut by a plane the trace left on the surface of the cone by the
plane is one of the three curves.

The reason he called the curves by
their names depended on the angle
of the plane as compared with the
angle of the cone. Namely, a
hyperbola is obtained (third cone
on the picture) if the angle of the
plane is larger than the cone angle
(hyper means more than, as in our
word hype), a parabola is obtained when the angle is exactly what is needed (same root
as the word parable), and finally an ellipse is called that because the angle falls short of
what is needed (same root as the word ellipsis).

In this section we will summarize the basic knowledge about these wonderful curves, that
have been admired throughout the centuries, and that many centuries after Alexandria's
glory will come back to play a major role in physics and mechanics.

From the powerful hindsight of the twentieth century we understand that the Conics
represent arbitrary quadratics on \(x\) and \(y\). More precisely, any equation of the form
\[
ax^2 + bxy + cy^2 + dx + ey + f = 0
\]
represents a conic as long as not all \(a\), \(b\) and \(c\) are zero. And we are told in analytic
geometry that the type of conic depends as follows:

<table>
<thead>
<tr>
<th>Type of Conic</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ellipse</td>
<td>(4ac - b^2 &gt; 0)</td>
</tr>
<tr>
<td>Parabola</td>
<td>(4ac - b^2 = 0)</td>
</tr>
<tr>
<td>Hyperbola</td>
<td>(4ac - b^2 &lt; 0)</td>
</tr>
</tbody>
</table>

But most of us are often exposed to other definitions of the conics:

An ellipse is usually defined as the set of points whose
total distance (the sum of the two distances) to two fixed
points (each called a focus) is
fixed. A clear extension of the
circle when the two foci
become one, the center of the
circle.

The parabola is usually given in terms of a point called the
focus and a line, the directrix, and the parabola is the set of
points whose distance to the line is the same as their distance
to the point. But one can also think of a parabola as an
extreme ellipse where one of the foci went to infinity.
Finally the **hyperbola** is similarly defined to the ellipse as the set of points that when their distance to two fixed points (the **foci**, of course) are taken, then the difference of those two distances is always a fixed number.

But as we mentioned above, one of Apollonius major contributions was to connect these definitions, which were older with the sections of the cone. How does one do that? What makes a conic section a conic in the other sense of the word?

This is the question that Apollonius was interested in. And indeed he proved that when a plane intersects a cone, one always gets one of the three conics. **His arguments are very difficult and laborious.**

From the modern algebraic point of view, the intersection of any plane Rather, we will give a nineteenth century argument applied to the cylinder instead of the cone, but which will clearly elucidate the idea.

First we need to review a basic fact: Consider the tangents from a point outside a circle to the circle. **There are two of them and they are of the same length.**

What is the three-dimensional analog of this fact? If a point outside a sphere is taken, how many tangents are there from that point to the sphere? We should realize that **there are many tangents from the point to the sphere, but they are all of the same length.**

Keep this in mind as we proceed to give the argument that when a plane cuts a cylinder, the curve that is traced on the surface of the cylinder is an ellipse.

The obvious question is if it is to be an ellipse where are the foci?

The answer is easy: Think of a balloon that fits exactly around the cylinder and drop it gently until it touches the plane, and mark the point of tangency. We do the same from the bottom. We claim these **two points of tangency are**
our foci.

To prove this we need to argue that any point on the surface of the cylinder where it intersects the plane has a fixed distance to the two points. Consider the fifth cylinder in our picture. If we take any point on our alleged ellipse, and measure its distance to the (alleged) focus stemming from the sphere above the plane, we get that since both the point in the ellipse and the focus are in the plane, and the plane and the sphere are tangent, the line drawn from the point to the focus is a tangent line from the point to the sphere. As such its length is the same as any other tangent from the point to the sphere. But there is another tangent line from our point on the ellipse to the sphere and that one is along the surface of the cylinder since the sphere and the cylinder are tangent. That tangent line goes from the point to the point of the equator of the sphere that is vertically above our given point. Hence the distance from the point to the focus is the same as the distance from that point to the equator above it.

Similarly, if we switch to the focus obtained from the sphere below, we get that the distance from the point to that other focus is the distance from the point to the equator below it. Hence the sum of the two distances is nothing but the distance between the two equators. That distance does not depend on the point we chose on the ellipse, hence we indeed have an ellipse!

The argument for the cone is on the same vein, except the spheres have changing diameters so as to fit snugly against the cone at all times.
Chapter 7
Greek Twilight

As we ended the last chapter, the Romans were about to conquer Cartage, and within 150 years of Archimedes' death, the whole Mediterranean world had fallen to Roman armies. While very able in engineering and law, the Romans did not contribute as meaningfully to mathematics, and Alexandria would remain the major center of mathematical activity throughout the Roman Republic and Empire. However, the concentration of genius that occurred during the first century of Alexandria will not occur again. Hence, the mathematicians in this extended period—which covers roughly 500 years—will come interspersed through time. Yet even Greek twilight is far brighter than the mathematical darkness that will encompass Western Europe for the next 700 years. Alexandrian mathematics ends with the death of Hypatia in the year 415 when she is murdered by a mob that would not tolerate her pagan beliefs. We will consider 4 mathematicians of this late Alexandrine period: Heron, Ptolemy, Diophantus and Pappus.

Heron

Heron was a major scientist and mathematician with an extensive bibliography. He is said to have founded the first College of Technology in Alexandria. One of his major works, *Metrica*, was not found until 1896. Not much is known about Heron—not even exactly when he lived except that he definitely followed Archimedes and definitely preceded Pappus. We will place him about 105 BC, but scholars have also placed him to have lived some time during the first two centuries of our era.

Some authors have judged Heron harshly as not having the mental rigor that was exemplified by Eudoxus and Archimedes. They would blame him for not knowing the difference between a good approximation and the answer. Yet there is a subtle philosophical perspective that is stressed in this criticism—namely a Platonic perspective that would definitely commit to the existence of, for example $\sqrt{2}$, while in a different perspective all one can ever do is approximate it, never realize it, hence a good approximation is all one will ever achieve, and hence is as good as the answer.

We are going to look at two contributions of Heron. The first one is traditionally called Heron's formula, but it has been argued that it is due to Archimedes. In any case it is a wonderful formula, and certainly Heron, if not the originator, is a propagator of it.
We all know that the area of a triangle is one half of the base times the height. But if surveying was our business, calculating the height may well take time and care. It would be more advantageous to have a formula that produces the area from the lengths of the sides. Heron’s formula is such a formula. The derivation below is not Heron’s; it rather uses elementary but powerful algebraic techniques that were not available to him. Take any triangle with vertices \( A \), \( B \) and \( C \) and sides \( a \), \( b \) and \( c \) respectively. By dropping the perpendicular from \( A \) to \( BC \) we get the lengths \( x \), \( y \) and \( z \) which satisfy:

\[
\begin{align*}
    x + y &= a, \\
    x^2 + z^2 &= c^2, \\
    y^2 + z^2 &= b^2.
\end{align*}
\]

Then the area \( A \) of the triangle is given by \( A = \frac{az}{2} \). We proceed to eliminate unknowns from the equations in attempting to have just the lengths of the sides in the expression for the area—namely, we just want \( a \), \( b \) and \( c \) in an expression for \( A \).

From \( (3) \), we have that \( y = \sqrt{b^2 - z^2} \) (note that the negative sign in front of the square root would address the oblique triangle case). Substituting in the other two equations, we have that \( x + \sqrt{b^2 - z^2} = a \) and \( x^2 + z^2 = c^2 \), and the \( y \) has been eliminated.

From \( (1) \), we obtain, \( x = a - \sqrt{b^2 - z^2} \), and substituting in \( x^2 + z^2 = c^2 \), we have

\[
a^2 - 2a\sqrt{b^2 - z^2} + b^2 - z^2 + z^2 = c^2,
\]

and so \( 2a\sqrt{b^2 - z^2} = a^2 + b^2 - c^2 \), and so \( b^2 - z^2 = \left(\frac{a^2 + b^2 - c^2}{4a^2}\right)^2 \). Hence,

\[
z^2 = \frac{4a^2b^2 - (a^2 + b^2 - c^2)^2}{4a^2}.
\]

Since \( A^2 = \frac{a^2z^2}{4} \), we obtain \( 16A^2 = 4a^2b^2 - \left( a^2 + b^2 - c^2 \right)^2 \). It is time to pursue a symmetric expression on the 3 letters, and since the right hand factors easily, we have

\[
16A^2 = \left(2ab + \left(a^2 + b^2 - c^2\right)\right)\left(2ab - \left(a^2 + b^2 - c^2\right)\right),
\]

which can easily be rearranged into

\[
16A^2 = \left(\left(a + b\right)^2 - c^2\right)\left(-\left(a - b\right)^2 + c^2\right) = (a + b + c)(a + b - c)(a - b + c)(a - b - c)
\]

which is symmetric in the three letters. If we let \( s \) denote the semiperimeter of the triangle, \( s = \frac{a + b + c}{2} \), then we get that

\[
a + b + c = 2s, \ a + b + c = 2s, \ a + b + c = 2s, \ a - b + c = 2s - 2b,
\]
Thus
\[ 16A^2 = 2s(2(s-b))(2(s-c))(2(s-a)) \]
and we get
\[ A^2 = s(s-b)(s-c)(s-a) \]
or equivalently,
\[ A = \sqrt{s(s-b)(s-c)(s-a)}. \]

Either of the last two expressions is called **Heron’s formula**.

As observed before, note that the symmetry of the three sides of the triangle is reflected algebraically in the formula since regardless how one permutes \( a, b \) and \( c \), \( s \) remains unchanged, and so does \( A \) as it should be from the geometry.

The oblique triangle case is just as easily handled.

The other contribution of Heron that we consider has to do with the following geometric problem.

**Given two points** \( A \) **and** \( B \) **not on a line, what is the shortest path from** \( A \) **to** \( B \) **that touches the line?**

In other words, of all possible paths from \( A \) to the line and from the line to \( B \), which is the shortest? The answer turns out to be both beautiful and simple. First find the point \( C \) that is the **reflection** of the point \( B \) on the line. Equivalently, the line \( BC \) is perpendicular to the given line, and the distance from \( B \) to the line (in other words, its distance to \( O \) ) is the same as the distance from \( C \) to the line.

Then the shortest path is the one through \( P \) where \( P \) is the point of intersection of the line \( AC \) with the original line.

The reason for this path being the shortest one is simple. Consider any other path and let \( Q \) be the point where the path intersects the line. Then the distance from \( Q \) to \( B \) is the same as the distance from \( Q \) to \( C \) since the triangles \( \triangle QOC \) and \( \triangle QOB \) are congruent right triangles (\( OC \approx OB \) and \( OQ \approx OQ \)), but then the distance from \( A \) to \( Q \) and from \( Q \) to \( B \) is longer than the path through \( P \) since \( A \) to \( P \) and \( P \) to \( C \) is a straight line.

Note that the path through \( P \) is exactly the one that makes the angle of reflection equal to the angle of incidence: \( \angle APR = \angle BPO \).
since the triangle $\triangle QBC$ is isosceles and $QO$ is the perpendicular bisector to the base. This important equality of angles is often referred to as the Law of Reflection, and light obeys it as it reflects off a flat mirror.

The Greeks were definitely interested in Optics and several authors wrote treatises on the subject—including Euclid and Ptolemy. Archimedes, in the fanciest of legends, is reputed to have used huge mirrors to burn Roman ships during the siege of Syracuse. While the technology to build such a mirror may not have been available, the mathematical knowledge was certainly within his grasp.

It is even perhaps possible that the Greeks got interested in tangent lines to the conics because of an interest in mirrors and light reflecting off them. As we mentioned above:

- **light bounces off a flat mirror by taking the shortest possible path from one point to the other,**

but what happens if the mirror is not flat? How does the ray of light bounce off? This is where the tangent line comes in. The ray bounces off the tangent line by the law of reflection—in other words, the angle of incidence with the tangent line to the curve at the point of contact with the mirror is the same as the angle of reflection with the tangent. Or equivalently, it goes from one point to the other by the shortest possible path from one to the other through the tangent line.

For an example, we will work out what happens in an elliptical mirror. In our argument we will use the following intuitive fact: we know the ellipse contains all points whose distance to the foci is a given constant. Suppose we take a point outside the ellipse? What is the total distance from that point to the two foci? Clearly, since we are walking two legs of a triangle, it is greater than from the ellipse itself. Consider now an ellipse with foci $A$ and $B$, and consider a point $P$ on the ellipse. How do we find the tangent line to the ellipse at the point $P$?

Among all possible lines through $P$, which one is the tangent?

By Heron's result and the intuitive fact mentioned above, the answer is easy: since the shortest path from $A$ to $B$ using the tangent line is the path through $P$, we must have $\angle APX = \angle BPY$,

which means that when a beam of light is lit at one focus in an elliptical mirror, it shines at the other focus.
Ptolemy (c.170 AD) has an indelible name in the history of astronomy. His masterpiece in the subject, which we know mainly by its Arab name of the *Almagest* (or Great Work), was the reference book on the subject for over 12 centuries. We hear much, and deservedly so, about the Copernican revolution in astronomy, however, Ptolemy's humility in his claims should be pointed out.

- First, although Aristarchus (from Athens) had proposed a heliocentric system, Ptolemy adopted a geocentric one because this approach was prevalent in his time (mainly due to Aristotelian influence), and he observed that this prevalent influence was one of the reasons for the geocentric view.
- Second, he was committed to circles partly because of Platonic influence, and hence he used *epicycles*, or *epicycles of epicycles*, etcetera. (up to 27 different types)—which will be clarified briefly. Actually, Copernicus himself believed in the circular path of the planets around the sun. This is not a great aberration since although we know today the paths to be elliptical; these are ellipses that resemble circles.
- Finally, Ptolemy emphasized that all he had in his book was a system that worked—and indeed it did for over a thousand years, and no implication of truth was ever made. He indeed was much wiser (and humbler) than many of his followers who insisted on his views as being without the possibility of error.

We will start our discussion of Ptolemy by describing briefly what an *epicycle* is—we will not use them afterwards at all. In order to justify the movement of the astral bodies, but using only circles, epicycles were created. The basic epicycle functions as follows: consider the path of a point going around a circle with uniform velocity, where simultaneously the center of the circle is going around another circle with uniform velocity. There were many types of basic epicycles depending on the radii and relative velocities of the centers. We have given examples of two. If that was not enough we could have a point going around a circle whose center is going around a circle whose center is going around a circle, etcetera.

Mathematically, Ptolemy's greatest contribution was a *first-rate trigonometric table*. More accurately, he left behind a *table of chords*. As any other astronomer, Ptolemy
was interested in solving triangles, which means that if we are given 2 sides and an angle, or two angles and a side, or 3 sides, of a triangle we must be able to solve for the other angles and sides. And as is today, the trigonometric functions are invaluable for this purpose. Actually, the Greeks did not consider our present day functions; rather they had only one function called the chord of an angle. Namely, given an angle \( \alpha \), chord \( \alpha \) is the length of the chord subtended by \( \alpha \) in a given circle.

Ptolemy used circles of radius 60 (reminiscences of Babylon) and used the hexagesimal notation for writing his decimal numbers, thus 3;15,30 indicated \( 3 + \frac{15}{60} + \frac{30}{3600} \) where we use the semicolon to indicate the decimal point.

For example, then, chord 60\(^\circ\)=60 since the chord is the radius. On the other hand, by using the Pythagorean theorem, chord 90\(^\circ\)=60\(\sqrt{2} \). Ptolemy would use the old Babylonian approximation of 1;24,51,10 for \( \sqrt{2} \) which is correct to 6 decimal places.

Let us compute chord 72\(^\circ\), which is precisely the length of the side of the pentagon. From the Athenian construction, we know \( BC = AB = \sqrt{60^2 + 30^2} \), and thus \( CO = 30\sqrt{5} - 30 \), and finally, we get \( AC = \text{chord} 72^\circ = 30\sqrt{10 - 2\sqrt{5}} \), which Ptolemy wrote in the form 70;32,3.

Thus Ptolemy set out to compute a table of chords. What tools did he use? One of the tools was easily gotten from Thales' Theorem,

\[
\text{chord}(180^\circ - \alpha) = \sqrt{120^2 - (\text{chord} \alpha)^2},
\]

But there were other tools he used extensively, and some were based on what is often called Ptolemy's Theorem. The main tool in proving this theorem is Proposition 20 of Book III of Euclid, which was proven in the previous chapter:

any two angles subtended by the same chord are equal.

Now, we have

**Theorem. Ptolemy.** Let the quadrilateral ABCD be cyclic, in other words, inscribed in a circle. Then

\[
\]

**Proof.** Find the point \( E \) so that \( \angle EAD = \angle BAC \). Then the triangles \( \triangle BAC \) and \( \triangle EAD \) are similar since \( \angle EAD = \angle BAC \) and \( \angle BDA = \angle ACB \) since they subtend the same chord \( AB \). Thus,
\[ \frac{ED}{BC} = \frac{AD}{AC} \], or equivalently, \( ED \times AC = AD \times BC \). Concurrently, \( \Delta BAE \) is similar to \( \Delta CAD \) since \( \angle CAD = \angle BAE \) and \( \angle ABE = \angle ACD \), and thus \( \frac{BE}{CD} = \frac{AB}{AC} \), or
\[ BE \times AC = AB \times CD. \]
Since \( BE + ED = BD \), by adding the two equations we obtain the theorem.

The theorem was the crucial step to one of the two weapons that Ptolemy used the most.

If one knows the chord of two angles, does one know the chord of the sum of the two angles?

And if one knows the chord of an angle, does one know the chord of half that angle?

To both of these questions we can answer YES, and both solutions can be gotten from Ptolemy's theorem (which of course is much older than Ptolemy).

First we can use Ptolemy's theorem to compute the chord \((90^\circ - \alpha)\) given that we know the chord \(\alpha\). So in the picture we know \(AB\), and, of course we know \(\alpha\) (\(=120\)). Also, by Thales' Theorem, \(BD\) is known and so are \(CD = AC = 60\sqrt{2}\). Hence by Ptolemy's theorem, since
\[ AD \times BC + AB \times CD = BD \times AC, \]
we can know \(BC\), the chord \((90^\circ - \alpha)\).

Suppose \(\alpha\) and \(\beta\) are given, and suppose their chords are given.

How do we find chord \((\alpha + \beta)\) ?

By the previous remark, we know the chords of \(90^\circ - \alpha\) and \(90^\circ - \beta\), and since the complementary angle is \(\alpha + \beta\), we can see that in the picture, we seek to compute the length \(BC\) \((= \text{chord}(\alpha + \beta))\) while we know \(AB\), \(CD\), \(AD\). By Thales' theorem we also know \(AC\) and \(BD\), and thus the only unknown quantity in Ptolemy's Theorem is \(BC\).
To obtain the chord of half an angle, we reason as follows. We know $\mathbf{AB}$ (chord$\alpha$) and $\mathbf{OC} = \mathbf{OB} = \mathbf{OA} = 60$. We also know that where the chord $\mathbf{AB}$ intersects the radius $\mathbf{OC}$ is the midpoint of the chord $\mathbf{AB}$, and we also know they intersect at a 90° angle. Let $\mathbf{D}$ be this point of intersection. Since we know $\mathbf{AD}$ and $\mathbf{OA}$, we know $\mathbf{OD}$, and since we know $\mathbf{OC}$ and $\mathbf{OD}$, we know $\mathbf{DC}$, but then we can know $\mathbf{AC}$, and that is the chord we desire.

Ptolemy used these two steps over and over to build his table. For example, since he had chords for 60° and 72°, he could find 12°, and then 6°, and 3°, etcetera.

But what does one really need in order to solve a triangle? For example, suppose that in the given right triangle we are given the hypotenuse $c$ and the angle $\delta$. **What do we do in order to find the side $a$?**

Putting the triangle in a circle, it becomes clear that, by similarity of triangles, $\frac{c}{60}$ is the same as $\frac{2a}{\text{chord}(2\delta)}$. Or equivalently,

$$a = \frac{\text{chord}(2\delta)}{2} \times \frac{c}{60},$$

and thus **half the chord of twice the angle** became an important function—in fact, more important than the chord, and thus the Hindus (and the Arabs) built tables not of chords, but of this new expression, which we know as the **sine function**, a name it derived in the West through mistranslation.

**Diophantus**

Many consider **Diophantus** (c.250 AD) one of the original algebraists, although it is the Arabs that give us both the word **algebra** and a more refined art for it. In fact, Diophantus uses a symbol for the unknown, and symbols for its powers, which he considers all the way to the sixth power—the consideration of any power above three represented a more sophisticated view than those in the Greek beginnings, but remember, by now, more than 500 years have passed since Euclid.
He also introduces equations, and their elementary manipulations. His notation is all very different from ours, yet the ingredients are there. Most of our modern algebraic notation comes from much later, the fifteenth and sixteenth centuries, and its mainly Western European in origin.

Diophantus' masterpiece is called the *Arithmetica*, a book not on arithmetic, but on what we would today call *number theory*, although primes and divisibility are not discussed anywhere. The Arithmetica will be translated into French in the seventeenth century, and through it influence mathematics considerably. In particular, *Fermat*, the best French mathematician of that century is going to read it and comment on it considerably.

The *Arithmetica* consists of a list of problems that are to be solved with rational numbers. As a matter of fact, the words *diophantine equations* have come to mean equations that are to be solved in integers, but it is easy to go from equations to be solved in rationals to equations that are to be solved in integers. However, the techniques may differ considerably.

We will look at two problems in the *Arithmetica*. They both emphasize the use of algebra as an art where one thinks carefully about technique. This attitude is rarely stressed nowadays, yet it should. Since he worked with only one unknown, he had to be careful about what he chose to be that unknown. Both of the next two examples illustrate this cleverness, where we would normally jump at two unknowns, he made do with one.

One problem would be easily handled today, but perhaps not as elegantly as Diophantus:

**Suppose two numbers are given where their sum is 20 and their product is 96. Find the numbers.**

Let $x$ be the amount by which the larger number exceeds 10. Thus, the larger number is $10 + x$, and so the smaller number is $10 - x$. But then their product is $100 - x^2$ which is supposed to be 96, and hence, $x = 2$, which gives the solution 12 and 8.

He would not allow irrational numbers, and, so, for example, if the product had been 97 instead, he would have announced no solution to the problem, since $x^2 = 3$ does not have rational solutions.

A modern problem in the same vein:

**If Rita gives Tony $30, then Tony would have twice as much money as Rita would. On the other hand, if Tony gives Rita $50, then Rita would have 3 times as much money as Tony would. How much money does each have at present?**

Let $x$ be the amount of money Rita has left if she gives Tony the $30. But then, Tony would have $2x$. On the other hand, if Tony gives the $50 instead, then he would have $2x - 30 - 50 = 2x - 80$ while Rita would have $x + 30 + 50 = x + 80$, and we know that the amount Rita has is three times the amount that Tony has. Thus, $x + 80 = 3(2x - 80)$,
which translates into $5x = 320$, or $x = 64$, and so Rita started with $94$, while Tony started with $2x - 30 = 98$.

We look at another problem in the Arithmetica:

*To break 13 into two squares, both of which are greater than 6.*

Notice this is an indeterminate problem in that it does not have a unique solution, and Diophantus is not particularly interested in finding all solutions, just one. However, that solution has to consist of rational numbers. In modern notation, we want $a$ and $b$ such that

\[ a^2 + b^2 = 13, \quad a^2 > 6, \quad b^2 > 6. \]

In modern terms, Diophantus is interested in finding a rational point in the circle $x^2 + y^2 = 13$ that is near the $y = x$-line. Although we will adopt a modern approach to his quest, many authors feel that except for the difference in notation (and our level of comfort with irrationals), the ideas were Diophantus’.

The point $(3, 2)$ is a rational point in the circle. Any line with rational slope going through this point will intersect the circle at another rational point. Since we want to end near the $45\degree$-line, we should aim at the point $(\sqrt{6.5}, \sqrt{6.5})$, so the slope should be a rational number close to $\frac{\sqrt{6.5} - 2}{\sqrt{6.5} - 3} \approx -1.219$. Thus we can try slope $-\frac{6}{5}$. The line going through $(3, 2)$ with slope $-\frac{6}{5}$ is the line $6x + 5y = 28$, and searching for the intersection of this line with the circle, we obtain, by substitution for $y = \frac{28 - 6x}{5}$, the following quadratic, $25x^2 + 784 - 336x + 36x^2 = 325$, which simplifies to $61x^2 - 336x + 459 = 0$, and readily solves into $x = \frac{168 \pm 15}{61}$, and so we have $x = \frac{153}{61}$ (as the other possible $x$), and so $y = \frac{158}{61}$, and the problem has been solved since $x^2 > 6$.

But, perhaps the oldest objects of interest in number theory are Pythagorean triples, of which the ancient Mesopotamians left behind a table as we saw before. Indeed the search for triples of positive integers $a$, $b$ and $c$ such that

\[ a^2 + b^2 = c^2, \]

for whatever reason, is old.

When we discussed the Mesopotamians, we mentioned a procedure for producing triples. Let $\alpha$ and $\beta$ be relatively prime positive integers, with $\alpha > \beta$. Also one may assume (exactly) one of them is even. Let $a = \alpha^2 - \beta^2$, $b = 2\alpha\beta$ and $c = \alpha^2 + \beta^2$,

then $a$, $b$ and $c$ is a Pythagorean triple. Moreover, any triple can be obtained this way. Following Diophantus, we adopt a slightly different view. Rather than looking for positive integer solutions to the equation $a^2 + b^2 = c^2$, one could divide by $c^2$ and
switch the point of view and consider rational solutions to the equation \( x^2 + y^2 = 1 \). One can then see that such solutions correspond to Pythagorean triples and vice versa, any Pythagorean triple \( a, b \) and \( c \) corresponds to the rational solution \( x = \frac{a}{c} \) and \( y = \frac{b}{c} \). For example, the triple 3, 4 and 5 would correspond to the rational solution \( x = \frac{3}{5} \) and \( y = \frac{4}{5} \).

This is similar to the approach in the previous problem from the Arithmetica.

Geometrically what are we doing? We are looking for rational points on the unit circle—more accurately, for nonnegative rational points on the unit circle. Consider the point \((-1,0)\), which is on the circle. If we draw any line with rational slope going through this point, the line will intersect the circle at another rational point. This is because once a quadratic equation with rational coefficients has one rational root, then it must have another—since the sum of the roots is the coefficient of \( x \), or because the product of the roots is the constant term. Conversely, the line going through any rational point in the circle and the point \((-1,0)\) will have rational coefficients.

Thus we consider an equation of the form \( y = mx + m \) (where \( m \) is rational). This represents an arbitrary line through the point \((-1,0)\) and we proceed to find its intersection with the unit circle. In fact we can take \( m \) so that \( 0 < m < 1 \) so that our intersection point is in the first quadrant—we might as well take \( x \) and \( y \) to be positive.

The cases \( m = 0 \) and \( m = 1 \) are not interesting, giving as intersection points \((1,0)\) and \((0,1)\) respectively.

But then we have \( x^2 + (mx + m)^2 = 1 \), and so \( (m^2 + 1)x^2 + 2m^2x + (m^2 - 1) = 0 \). Solving for \( x \), we get \( x = \frac{-2m^2 \pm \sqrt{4m^4 - 4m^4 + 4}}{2(m^2 + 1)} = \frac{-2m^2 \pm 2}{2(m^2 + 1)} = \frac{-m^2 \pm 1}{m^2 + 1} \), and so besides the already known \( x = -1 \), we obtain \( x = \frac{1 - m^2}{1 + m^2} \), and, then \( y = \frac{2m}{1 + m^2} \).

Now we can let \( m = \frac{\beta}{\alpha} \) where \( 0 < \beta < \alpha \) are positive integers (remember \( 0 < m < 1 \)), and since we can take a reduced fraction for \( m \), we can also assume that \( \alpha \) and \( \beta \) have no divisors in common (they are relatively prime). Substituting into the expressions for \( x \) and \( y \) above, we obtain

\[
x = \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} \quad \text{and} \quad y = \frac{2\alpha\beta}{\alpha^2 + \beta^2}.
\]

and thus, since \( c \) is the denominator, \( c = \alpha^2 + \beta^2 \), and then \( a = \alpha^2 - \beta^2 \) and \( b = 2\alpha\beta \).
This is a general procedure that will succeed with all quadratic equations, and hence we have a theorem that goes back, at least tacitly, to Diophantus:

**Theorem.** If a quadratic equation on \( x \) and \( y \) with rational coefficients has a rational solution, then it has infinitely many.

As mentioned above, the Arithmetica was indeed a source of inspiration to Fermat, and through him to many others.

### Pappus

As Diophantus was interested in arithmetic and algebra, Pappus' (c. 300 AD) interest was geometrical. As Diophantus influenced Fermat, Pappus had influence on Fermat's contemporary, Descartes. And it is even possible that Pappus' problems were part of the motivation for the creation by Descartes of **Cartesian geometry**. It is not surprising that Descartes would be influenced by Pappus since in the **Collection**, which is considered Pappus' major contribution to mathematical literature, there are many interesting geometrical problems—something that had not occurred often in the literature of the 200 years prior to Pappus. The Collection, as its name indicates, is a collection of mathematical facts due to many authors, including Pappus.

One of the problems that Descartes is going to find interesting has to do with **kissing circles**. The problem that Pappus considered is the following. Take the Shoemaker's Knife that consists of three tangential semicircles. And start building circles that are tangential to the three previous circles. Let \( d_i \) denote the diameter of the first circle, and let \( h_i \) the height of the center of this first circle.
Similarly, let \( d_2 \) denote the diameter of the second circle, and let \( h_2 \) be the height of the center of the second circle, let \( d_3 \) and \( h_3 \) denote the diameter and the height of the center of the third circle, etcetera. Then a wonderful thing happens:
\[
h_1 = d_1, \quad h_2 = 2d_2, \quad h_3 = 3d_3, \quad h_4 = 4d_4, \ldots
\]

Thus, for example, in our illustration, we start with two semicircles of radius 3 and 2, so the large semicircle has radius 5. Then the diameter of the first circle is \( \frac{10}{3} \), and so is the height of its center, while the second circle has diameter \( \frac{60}{17} \), but the height of its center is \( \frac{120}{31} \), and continuing, we get \( d_3 = \frac{70}{17} \) and \( h_3 = \frac{60}{17} \), \( d_4 = \frac{60}{17} \) while \( h_4 = \frac{240}{79} \), and finally, in our diagram, \( h_5 = \frac{40}{79} \) and \( d_5 = \frac{120}{79} \). Pappus' proof of this fact is ingenious and interesting, but, alas, beyond the scope of this book. It suffices to remark that in that proof, as throughout the Collection, Pappus uses conics very comfortably and powerfully, and solves many of the classical constructions as intersections of conics.

One could not discuss Pappus and not mention the interesting theorem that bears his name:

Take any two lines, and any three points in each of them. Let us label the points \( A, B \) and \( C \) in one line and \( A, B \) and \( C \) in the other. Consider the point \( C \) of intersection of the line \( AB \) and the line \( AC \), the point \( B \) which is the intersection of \( AC \) and \( BC \), and the point \( A \) where \( BC \) and \( BC \) intersect.

Then \( A, B \) and \( C \) are collinear.

One of the nice features of Pappus' theorem is its generality. Observe that it does not matter how we label the points, and under any circumstances, the theorem is still true.

In the Collection, Pappus has a wonderful discussion on bees and the shape of their beehive. To quote him:

...the vessels which we call honey-combs,
with cells all equal, similar and contiguous to one another, and hexagonal in form. And they have contrived this by
virtue of a certain geometrical forethought we may infer in this way. They would necessarily think that the figures must be such as to be contiguous to one another, that is to say, to have their sides common, in order that no foreign matter could enter the interstices between them and so defile the purity of their produce. Now only three rectilinear figures would satisfy the condition, I mean regular figures which are equilateral and equiangular; for the bees would have none of the figures which are not uniform.... There being then three figures capable by themselves of exactly filling up the space about the same point, the bees by reason of their instinctive wisdom chose for the construction of the honeycomb the figure which has the most angles, because they conceived that it would contain more honey than either of the two others. Bees, then, know just this fact which is of service to themselves, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material used in constructing the different figures.

Finally, it is Pappus who discusses the Archimedean or semi regular solids.

A convex solid is called Archimedean if all of its faces are regular polygons, but they do not all have the same number of sides. However it is required that every corner have the same polygons coming into it.

Pappus lists 13 such polyhedra, giving credit to Archimedes for their discovery. We examine how to produce some of them easily from the Platonic solids.

We will exemplify the procedure using the cube. Starting with an arbitrary cube, we can use a plane to
cut any of its corners, as the picture illustrates. By being careful about what planes we choose, we can produce a semi regular solid.

One way that does it is to take the plane that goes through the midpoints of the three edges coming into a corner, and we do this at each of the corners we obtain the **cuboctahedron** which is made up of 8 triangles (one for each corner of the cube) and 6 squares (one for each of the faces of the original cube), and every corner is made of 2 squares and 2 triangles in an alternating fashion.

But there is another way to cut the corners of the hexahedron so as to form a semi regular solid. Namely, cut the corner off so that each face becomes a regular hexagon. This is the **truncated cube**, which consists of 6 octagons (one for each face) and 8 triangles (one for each original corner), and each new corner has two octagons and a triangle at it.

In general, we can truncate each of the regular polyhedron to obtain a semi regular solid, and we will leave these for the exercises.
It is true that a mathematician, who is not somewhat of a poet, will never be a perfect mathematician—\textit{Weierstrass}.

We now travel to the East to keep up with mathematical tradition. As we mentioned in one of the disclaimers in the preface—we simply do not have the time to pay proper tribute to all the great cultures that contribute to our subject. This is indubitably true in this and the next chapter where we look at two great civilizations that enhanced in very meaningful ways all areas of mathematics.

Both the \textbf{Hindus} and the \textbf{Muslims} made major contributions in mathematics that had tremendous influence on the development of mathematics in Western Europe. As Western Europe sank into intellectual darkness for many centuries, mathematics flourished in the East. Our numeral notation is still justifiably referred to as \textbf{Hindu-Arabic}. It is possible that it developed as a series of continuous improvements on the original Babylonian notation.

However, there were some very meaningful quantum leaps, one of them being the \textit{creation of the numeral 0 by the Indus}. Another was their \textbf{healthy, aggressive manipulation of symbols}—including negative numbers, rid of any intrinsic geometric meaning, which enabled future generations to abstract and gain power, and thus aid in the development of our present day notation.

One would expect an ancient history such as that of India to be very difficult to summarize in a few paragraphs, or in a table, and indeed that is the case. The civilization in the Indus and Ganges Valleys dates back to about 6,000 years ago. Furthermore, there has been a large variety of different people entering the subcontinent from both Asia and Europe—including Alexander and other Greeks later on.

\begin{center}
\textbf{Indian History Highlights}
\end{center}

\begin{tabular}{lc}
\textbf{Date} & \textbf{Event} \\
4,000 BC & Civilization develops in the Indus Valley. \\
1,750 BC & End of Indus civilization. \\
1,500 BC & Aryans move into Northern India and flourish. \\
1,000 BC & The Rig-Veda, a collection of hymns, is composed and passed down orally.
\end{tabular}
566 BC  Gautama the Buddha is born in the foothill of the Himalayas.
540 BC  Gandhara is born. He is the founder of Jainism.
327 BC  Alexander enters India.
305 BC  Chandragupta Maurya extends his empire from the Ganges to Afghanistan.
260 BC  Ashoka, grandson of Chandragupta converts to Buddhism.
185 BC- Several rulers around the land.
320-500 The Guptas rule from the Arabian Sea to the Bay of Bengal.
985 Indian Mathematicians use modern numeral system.
997-1,030 In Northern India, many attacks from the Ghaznavid Empire of Persia.
1,014 Chola king Rajendra I opens commercial routes from the Arabs to the Chinese.
1,180 Decline of Chola kingdom.
1,321 The Tughluq Muslim dynasty is founded
1,336 Hindu capital of Vijayanagar is founded.
1,411 Ahmadabad founded by Ahmad Shah.
1,526 Moghul rule begins.

Mathematics, too, has an ancient tradition in India—as far back as 3000 BC. By 800 BC they have the following statement of the Pythagorean Theorem:

The rope stretched along the length of the diagonal of a rectangle makes an area which the vertical and horizontal sides make together

About 1,000 BC, some of the Vedas were composed. In them, one finds fascination with arithmetic, geometry and patterns. One contemporary author refers to the continued additions of the digits of a number until only one digit is obtained as the **Vedic transformation**. If we do this, then the multiplication table is transformed as below—where many patterns are readily accessible:

\[
\begin{array}{cccccccccc}
\times & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
2 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\
3 & 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\
4 & 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 \\
5 & 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\
6 & 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 \\
7 & 7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 \\
8 & 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \\
9 & 9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 \\
\end{array}
\]

By 600 BC, in a medicinal treaty, there is already mention that a total of 63 different combinations can be made out of the six flavors possible: bitter, sour, saltish, astringent, sweet and hot. In modern days, we would state this as

\[
\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 6 + 15 + 20 + 15 + 6 + 1 = 63 .
\]
Although this mathematical tradition has continued into the present day—as verified by the fact that most would consider the Indian mathematician Ramanujan among the greatest of the twentieth century—we speak of the period 400 AD to 1200 AD as the Classical Period of Indian mathematics.

We know that by the beginning of that era, they had the numeral 0 as well as great arithmetical algorithmic capability—an ability that will lead naturally to the development of the rudiments of algebra by them as well as by the Arabs. In passing, the word algorithm comes from the mispronunciation of an Arab name—see the next chapter.

Some authors would consider the culture to have a passion for high and low numbers, and they had words to describe them such as paduma, which stood for $10^{29}$. In one piece of literature from the fourth century, Buddha gives the number $108,470,495,616,000$ as an answer, and in another religious writing, the answer $2^{96}$ is offered for the totality of human beings since creation.

We will exemplify by using one of their algorithms for multiplication. The method was called vertically and crosswise and we will use it to multiply $1234 \times 567$. They would do it as follows:

We clearly see that this algorithm is not very far from similar algorithms we would use today, that there is a solid understanding of the base and its powers behind the notation, and what position means is well understood. Yet the technique is also not too distant from something more abstract, namely the multiplication of polynomials or convolution as it is often referred to:

$$ (x^3 + 2x^2 + 3x + 4)(5x^2 + 6x + 7) = 5x^5 + 16x^4 + 34x^3 + 52x^2 + 45x + 28. $$

We will discuss three, of the many, mathematicians from the classical period: Aryabhata, Brahmagupta and Bhaskara II.

**Aryabhata**

All three of the Indian mathematicians we will discuss excelled in astronomy. Aryabhata was born in 476 and died in 550. While he lived in Kusumapura, near the city of Palalipultra, in Northern India, he wrote, at age 23, a much-admired astronomical work, the Aryabhatiya. At that time, Palalipultra was the capital of the Gupta Empire.
As most of the mathematical works of the time, the Aryabhatiya is composed in verse, and consists of 118 verses. It summarizes the Hindu mathematics up to that time, and its mathematical component contains 33 verses. The remaining verses reflect astronomical knowledge that show thorough understanding of the nature of eclipses, the size of the earth and the shape of the orbits of the planets, which he believed to be ellipses! In fact, he gives 62,832 miles as the circumference of the earth.

In the second chapter, where most of the mathematics occurs, the first verse is:

Half of the circumference multiplied by half of the diameter is the area of a circle.

Later on he states:

Add four to one hundred, multiply by eight and then add sixty-two thousand. The result is approximately the circumference of a circle of diameter twenty thousand. By this rule the relation of the circumference to diameter is given.

This gives the value \( \pi = \frac{62832}{20000} = 3.1416 \) exactly. Aryabhata himself occasionally used \( \sqrt{10} \approx 3.1622 \) for \( \pi \). One should observe the lack of symbols in the previous paragraph—just words.

Another major contribution of Aryabhata is the occurrence of half-chords of angles, or as we have seen before when we discussed Ptolemy, sines of angles. Again, some authors disagree on whether the idea of chord was obtained from the Greeks or not. In any case, the very convenient switch to the half-chord is a meaningful contribution of the Indus. Aryabhata gave a table of sines (24 values) by using some original trigonometric identities. He also introduced the function versine, which equals \( 1 - \cos \).

Although Aryabhata did not use what eventually became our numerals, some authors believe that he was aware of them, of the place-value system and of the numeral zero. He may also have contributed to their further development by having invented a system for representing numbers based on the 33 consonants of his alphabet. His system allowed representation of numbers as large as \( 18^{18} \).

Finally, the Aryabhatiya contains some of the earliest work on the solutions of the linear diophantine equation \( ax + by = c \), and the use of the pulverizer, a topic which will be further developed when Brahmagupta is discussed below. This type of analysis was needed in trying to find a common unit to measure the period of the planets.

The Aryabhatiya, as well as all the other works from the region, has suffered a great deal by mistranslation. We already saw the mistranslation into sine. In some versions of the work, wrong expressions for the volume of the pyramid and the sphere are given. Most authors feel that as capable and knowledgeable a mathematician as Aryabhata was, he was not capable of these errors, and rather the most likely explanation is that of yet another mistranslation.
Brahmagupta (c. 628) wrote an important book, the Siddhanta, a book that through the Arabs will make Western Europe aware of Indian mathematical (and astronomical) traditions and techniques. He was part of the great astronomical tradition of the city of Ujjain. In this work, Brahmagupta exhibits a deep understanding of the number system, and he specifically gives properties of it such as:

When zero is added to a number or subtracted from a number, the number remains unchanged; and a number multiplied by zero becomes zero.

He also discusses positive and negative numbers, which are referred to as fortunes and debts:

A fortune subtracted from zero is a debt.
The product or quotient of two fortunes is a fortune.
The product or quotient of two debts is a fortune.
The product or quotient of a debt and a fortune is a debt.

He even attempts to extend the arithmetic to include division by zero. He wrongfully states that

Zero divided by zero is zero.

In the Siddhanta, algorithms for multiplication are given which are direct ascendants of our present school-days algorithm. We exemplify by multiplying 315 by 426.

Multiply 315 by the left member in each row maintaining the position.

<table>
<thead>
<tr>
<th></th>
<th>4</th>
<th>1</th>
<th>2</th>
<th>6</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Add the columns:

<table>
<thead>
<tr>
<th></th>
<th>4</th>
<th>1</th>
<th>2</th>
<th>6</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

and the answer is read at the bottom row.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>3</th>
<th>4</th>
<th>1</th>
<th>9</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

One of the chapters of the Siddhanta is called Kuttaka, which means pulverizer. This referred to the Euclidean algorithm and other algebraic techniques associated with it. It was so quaintly called because it crushed problems, it solved them.

Of equal importance to finding the g.c.d. of two numbers is the fact that one can also use the algorithm to solve the crucial equation: \( \delta = xD + yd \) where \( \delta \) is the greatest common divisor of \( D \) and \( d \). This equation is usually referred to as writing the g.c.d. as a linear combination of the two original numbers.
We revisit an example from **Chapter 6** before we visit an actual example due to Brahmagupta. In that chapter we considered the situation when \( D = 59059 \) and \( d = 12508 \). We found then that \( \delta = 59 \). We actually revisit the table for that example but we add three columns to the table:

<table>
<thead>
<tr>
<th>Stage</th>
<th>Dividend</th>
<th>Divisor</th>
<th>Quotient</th>
<th>x’s</th>
<th>y’s</th>
<th>( x_iD + y_id )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>59059</td>
<td>12508</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>59059</td>
</tr>
<tr>
<td>1</td>
<td>12508</td>
<td>9027</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>12508</td>
</tr>
<tr>
<td>2</td>
<td>9027</td>
<td>3481</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>9027</td>
</tr>
<tr>
<td>3</td>
<td>3481</td>
<td>2065</td>
<td>1</td>
<td>3</td>
<td>-4</td>
<td>2065</td>
</tr>
<tr>
<td>4</td>
<td>2065</td>
<td>1416</td>
<td>0</td>
<td>3</td>
<td>-14</td>
<td>1416</td>
</tr>
<tr>
<td>5</td>
<td>1416</td>
<td>649</td>
<td>2</td>
<td>7</td>
<td>-33</td>
<td>649</td>
</tr>
<tr>
<td>6</td>
<td>649</td>
<td>118</td>
<td>2</td>
<td>7</td>
<td>-33</td>
<td>649</td>
</tr>
<tr>
<td>7</td>
<td>118</td>
<td>59</td>
<td>5</td>
<td>18</td>
<td>55</td>
<td>59</td>
</tr>
<tr>
<td>8</td>
<td>59</td>
<td>0</td>
<td>2</td>
<td>97</td>
<td>-458</td>
<td>59</td>
</tr>
</tbody>
</table>

For the \( x \)'s and \( y \)'s one simply starts \( x_0 = 1, \ x_1 = 0 \) and \( y_0 = 0, \ y_1 = 1 \). Then from then on

\[
x_{n+2} = x_n + q_{n+1}x_{n+1}
\]

and

\[
y_{n+2} = y_n + q_{n+1}y_{n+1}.
\]

Note this is a recursive definition. Observe that the recursive relation satisfied by the \( x \)'s and the \( y \)'s is identical to the recursive relation satisfied by the \( r \)'s:

\[
x_n = q_{n+1}x_{n+1} + x_{n+2}
\]

(and similarly for the \( y \)'s). These recursions are nothing more than good accounting for going backwards in the algorithm to recover the \( x \) and the \( y \) we desire. It is the last (unnecessary) column where we see how the algorithm works. At all times (namely, in each row),

\[
r_i = x_iD + y_id
\]

and since \( r_n \) is the g.c.d. \( \delta \) of the two inputs \( r_0 = D \) and \( r_1 = d \), a solution to \( \delta = xD + yd \) has been obtained. In our example above,

\[
59 = (97)59059 - (458)12508.
\]

In general, Indian mathematicians were fond of indeterminate equations—problems that have many solutions, similar to the ones we encountered when we studied Diophantus, and, in fact, they were excellent at solving various types of Diophantine equations. But more than just a theoretical tool, they needed the Euclidean algorithm in order to do astronomical calculations.

For example, without giving the astronomical details, Brahmagupta needed to solve the following Diophantine equation:

\[
4567x - 10,000y = 2166.
\]

How do we solve such an equation? We use the Kuttaka, the pulverizer, the Euclidean algorithm with the added columns.

First we proceed to find numbers \( m \) and \( n \) such that

\[
10000m + 4567n = \delta
\]

where \( \delta \) stands for their g.c.d. Doing our table, we get that
10000 × 501 + 4567 × (−1097) = 1.

Equivalently:

4567×(−1097) + 10000×501 = 1.

If we multiply this equation by 2,166, we get

4567(−1097 × 2166) + 10000(501 × 2166) = 4567(−2376102) + 10000(1085166) = 2166.

But, we also have to take care of the sign for 10,000,

4567(−2376102) − 10000(−1085166) = 2166,

and thus, we have that

(x, y) = (−2376102, −1085166)

is a solution to our equation

4567x − 10,000y = 2166.

But, Brahmagupta needed positive whole numbers that satisfy our equation. How do we find a positive solution to it? Let us see what we are doing geometrically (and taking a giant leap into the future): What we have is a line, and we are looking for the points that lie on that line but which have coordinates that are whole numbers.

We have found one such point, how do we get to the next point on the line? We use the slope of the line:

\[ \text{rise} = \frac{4567}{10000}, \]

which in our case is \( \frac{4567}{10000} \), so we add 10,000 to the run (x-coordinate) while we add 4,567 to the rise (y-coordinate). So we have that

\[ (x, y) = (−2376102, −1085166) + (10000, 4567) \]

is also a solution to our equation, but then we can do it again, and readily we see that any point of the form

\[ (x, y) = (−2376102, −1085166) + k(10000, 4567) \]

is a solution.

Naturally, it is trivial to verify this fact algebraically: if we substitute

\[ x = −2376102 + 10000k, \quad y = −1085166 + 4567k \]

into \( 4567x − 10,000y = 2166 \), we easily see that we have a new solution—the terms with the \( k \)'s canceled out.

Now in order to reach a positive solution, and the smallest one is usually what is
preferred, we must find the smallest $k$ that will make $x = -2376102 + 10000k$, $y = -1085166 + 4567k$ both positive—in geometric terms, we are computing the first point we encounter as we enter the first quadrant of the plane.

For $x$, we must have $k \geq \frac{2376102}{10000} \approx 237.6$, and so $k \geq 238$, and for $y$, we must have $k \geq \frac{1085166}{4567} \approx 237.6$, so again $k \geq 238$ (this is a common coincidence) and so if we let $k = 238$, we arrive at the solution:

$$x = 3898, \quad y = 1780.$$ 

Observe that there are infinitely many positive solutions since we can keep adding to our solution to get bigger and bigger ones—we are dealing with indeterminate equations.

Kuttaka can be pushed even farther to try to approximate some irrationals or to solve even more complicated equations. Brahmagupta did this, perhaps under Greek influence, perhaps not. But we give an example by applying the Euclidean algorithm to irrational numbers! In more modern times, this theme is called continued fractions.

Looking into both the past and the future, suppose we will exemplify by applying this extended Euclidean algorithm to the numbers $\frac{1 + \sqrt{5}}{2}$ and $1$.

But we have to be careful with what we call the quotient—once we decide on the quotient; the remainder is easy since it is always

$$\text{Remainder} = \text{Dividend} - \text{Quotient} \times \text{Divisor}.$$ 

Since we want the quotient to be an integer, it seems correct to say that when we are dividing $D$ by $d$, the quotient should be $\left\lfloor \frac{D}{d} \right\rfloor$, the floor of $\frac{D}{d}$. Note that this is what we have done in the past, in any case. Once we have made this pseudo-correction we can apply the Euclidean algorithm to $\frac{1 + \sqrt{5}}{2}$ and $1$.

However, there is another small change in the algorithm. The $x$'s and $y$'s, as we usually define them, do not play a meaningful role, instead there are two new quantities $a$ and $b$ which have the same starting condition as the $x$'s and $y$'s, but their recursion is slightly different (just the negative sign has been changed):

$$a_{i+2} = a_i + q_{i+1}a_{i+1}, \quad b_{i+2} = b_i + q_{i+1}b_{i+1}.$$ 

We do a part of the table:
Some observations:

- The algorithm does not stop—namely, this is an on-going, never-ending process. The algorithm does stop if and only if the ratio of the two quantities we started with is rational, in other words, the quantities are commensurable.

- The last column on the table, whose entries correspond to the fractions $\frac{b}{a}$, is giving better and better, rational approximations to the number we started with,

$$\frac{1 + \sqrt{5}}{2} \approx 1.61803398875.$$

- The quotients were all computed to be 1 in this situation.

Brahmagupta’s name is also well known for his theorem concerning the area of **cyclic quadrilaterals.** A quadrilateral is **cyclic** if it can be inscribed in a circle. Then

**Brahmagupta’s Theorem.** Half the sum of the sides set down four times and severally lessened by the sides, being multiplied together; the square root of the product is the area.
In more modern terminology, if we let \( a, b, c \) and \( d \) be the lengths of the sides, then the area of the quadrilateral \( A \) is given by
\[
A^2 = (s-a)(s-b)(s-c)(s-d),
\]
where \( s \) is, as usual, the semi perimeter. Note that by letting one of the sides be zero we obtain Heron’s formula. The proof of the theorem is laborious so it will not be given.

It can be proven that a quadrilateral with given side lengths will maximize its area when it is cyclic.

**Bhaskara II**

Bhaskara II (c. 1150 AD) is also known as Bhaskaracharya for Bhaskara, the Teacher. Another Indian astronomer-mathematician, he continues the tradition of our former mathematicians of the area, as he read them and was influenced by them.

Bhaskara wrote several books on mathematics, all highly regarded for their clarity of exposition. One of them was the Lilavati, named after his daughter to whom it was dedicated. He dedicated the book to her in an effort to console her for a misfortune that resulted in her remaining unmarried!

The Lilavati is an arithmetic manual showing a thorough understanding of the subject. We give one of its many word problems:

> From a swarm of bees, a number equal to the square root of half the total number of bees flew out to the lotus flowers. Soon after, eight ninths of the total swarm went to the same place. A male bee enticed by the fragrance of the lotus flew into it. But when it was inside the night fell, the lotus closed and the bee was caught inside. To its buzz, its consort responded anxiously from outside. O my beloved! How many bees are there in the swarm?

In modern notation, if we let \( x \) be the number of bees in the swarm, then we have the equation \( \sqrt{\frac{x}{2} + \frac{8}{9}x + 2} = x \), which clears into \( 2x^2 - 153x + 648 = 0 \) with roots \( 72 \) and \( \frac{2}{7} \), Bhaskaracharya giving appropriately 72 as the only reasonable answer.

In another of his treatises, Bhaskaracharya follows some previous work of Brahmagupta and discusses the Diophantine equation:

\[ X^2 = 67Y^2 + 1. \]

This type of equation has a long history, going back to at least the Greeks, however, in modern times; it has been referred to as an example of (unjustifiably-named) Pell’s
Equation. In the eighteenth century, the method of continued fractions—discussed in the previous section—was used to solve such an equation. But Bhaskaracharya had an even more general procedure, which we examine briefly: he considered the more general set of equations

\[ X^2 = 67Y^2 + f \]

where \( f \) was an arbitrary integer, even possibly negative. Bhaskaracharya makes the following 3 wise observations—the first one being the most important:

<table>
<thead>
<tr>
<th>Observation 1: Combination</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ((x,y)) is a solution to (X^2 = 67Y^2 + f) and ((a,b)) is a solution to (X^2 = 67Y^2 + g), then ((xa + 67yb, xb + ya)) is a solution to (X^2 = 67Y^2 + fg).</td>
</tr>
</tbody>
</table>

This is easily verified algebraically. We will refer to the step as a combination of solutions. The puzzled reader should realize that if we wrote \(x + y\sqrt{67}\) for \((x,y)\), then the third line expression is obtained by algebraic multiplication of the other two.

<table>
<thead>
<tr>
<th>Observation 2: Obvious solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>For any (c), ((c,1)) is a solution to (X^2 = 67Y^2 + (c^2 - 67)).</td>
</tr>
</tbody>
</table>

and

<table>
<thead>
<tr>
<th>Observation 3: Canceling</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ((x,y)) is a solution to (X^2 = 67Y^2 + f) and for some positive integer (k), (k</td>
</tr>
</tbody>
</table>

So we now proceed to solve \(X^2 = 67Y^2 + 1\), the original equation by solving others with various constant terms.

The process is schematized as follows:

<table>
<thead>
<tr>
<th>Step</th>
<th>Solution</th>
<th>Constant Term</th>
<th>Procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(8,1)</td>
<td>-3</td>
<td>A place to Start</td>
</tr>
<tr>
<td>2</td>
<td>(8c + 67,8 + c)</td>
<td>-3(c^2 - 67)</td>
<td>2, and then 1</td>
</tr>
<tr>
<td>3</td>
<td>(123,15)</td>
<td>54</td>
<td>Let c = 7—see below.</td>
</tr>
<tr>
<td>4</td>
<td>(41,5)</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>(41d + 335,5d + 41)</td>
<td>6(d^2 - 67)</td>
<td>2, 1 as in step #2</td>
</tr>
<tr>
<td>6</td>
<td>(540,66)</td>
<td>-6(42)</td>
<td>Let d = 5 as in step #3.</td>
</tr>
</tbody>
</table>
And we get the solution \( x = 48842 \) , \( y = 5967 \) to the equation \( X^2 = 67Y^2 + 1 \).

The only step which requires justification is the one at step \#3. How do we find \( c = 7 \) ?

What we want to accomplish is the cancellation of the \(-3\) in step \#1, and we want to use the third observation. Hence we want \( 8 + c \) , \( 8c + 67 \) and \( c^2 - 67 \) to be multiples of \( 3 \)—which will make \(-3(c^2 - 67)\) a multiple of \( 9 \). Fortunately, if we satisfy the first one, the other two will be automatic.

Suppose \( 8 + c \) is a multiple of \( 3 \). Then \( c = 3t - 8 \) for some integer \( t \). Thus,

\[
8c + 67 = 24t - 64 + 67 = 24t + 3 = 3(8t + 1)
\]

and

\[
c^2 - 67 = 9t^2 - 48t + 64 - 67 = 3(3t^2 - 16t - 1) .
\]

Now in order to shorten the steps, it would be best to make \( |c^2 - 67| \) as small as possible, and from both of these—making \( 8 + c \) a multiple of \( 3 \) and \( |c^2 - 67| \) as small as possible, we get \( c = 7 \).

He was particularly proud of having solved the more difficult equation

\( X^2 = 61Y^2 + 1 \).

We show the solution in the table, but we do not give justifications for each step. In order to be more historically accurate we have stayed with only positive constant terms. Note the symmetry in the last column of the table.
Bhaskara II, in another one of his writings, again influenced by the great Brahmagupta, discusses some of the subtle questions involving the arithmetical operations with 0. Some centuries later, division by 0, and the consequent problems of vanishing quotients, is certainly stimulus for the development of the theory of limits and its connections to Calculus. He and others had anticipated many of these difficulties.
Chapter 9
Islam

Algebra is generous, she often gives more than is asked of her—D’Alembert

Mohammed, the prophet died in 632 AD. By the year 750, his influence, through both political and religious conversion, had spread from China to Spain. Although the movement was led by people from the Arabian Peninsula, it soon involved a multitude of nations, languages, races and ethnicities.

After the fall of Alexandria, and Northern India, the Muslims came in contact with the mathematical legacy of Greece, and India. For several crucial centuries, under the leadership of enlightened Caliphs, in their magnificent capitals, first Damascus, then Baghdad, and other cities such as Cairo, Samarkand and Córdova, spreading in three continents, the Arabs and other Muslims were the holders, disbursers and originators of most of the intellectual activity of the period. Baghdad became for a while the new Alexandria with a wonderful school: the House of Wisdom built in the early part of the ninth century by the Caliph Al-Mamum.

Vast energy was spent in the translation and additional commentary of most of the works that we have discussed during the Greek chapters—and lest we forget, Ptolemy’s work is still known by the name given by the Arab conquerors of Alexandria. But the Arabs will not only translate and annotate, but also synthesize, clarify, and make crucial additions, as we will see below. They affected all branches of mathematics and science in their pursuits, from geometry to trigonometry to astronomy to geography to arithmetic to number theory to the creation of a more systematic and powerful algebra.

In the early part of the thirteenth century, the Mongols, led by Genghis Khan, ruthlessly invaded Persia and other parts of the Muslim world. But even before the Mongols then, extended contact between the Muslim world and the Chinese civilization had occurred. To cite just one example, the technology of paper came to Western Europe via this route, Chinese-Arab-Spanish (and Sicilian) soon after the Arabs reached China in 751.
## Highlights of Arab and Islamic History

<table>
<thead>
<tr>
<th>Date</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>570</td>
<td>Prophet Mohammed is born in Mecca.</td>
</tr>
<tr>
<td>622</td>
<td>The Hegira: Mohammed flees Mecca for Medina—start of the Muslim calendar.</td>
</tr>
<tr>
<td>632</td>
<td>Death of Prophet Mohammed—Sunnis and Shiites argument develops.</td>
</tr>
<tr>
<td>642</td>
<td>Arabs conquer Persia and Syria led by Caliph Omar.</td>
</tr>
<tr>
<td>661-750</td>
<td>The Omayyad dynasty rules, Damascus is their capital.</td>
</tr>
<tr>
<td>711</td>
<td>Most of Spain is conquered by armies from North Africa. First Muslim state in India is established by Omayyads.</td>
</tr>
<tr>
<td>732</td>
<td>The franks stop Muslim invasion by winning the battle of Poitiers.</td>
</tr>
<tr>
<td>751</td>
<td>Islam reaches China.</td>
</tr>
<tr>
<td>762</td>
<td>Baghdad becomes capital of the Abbasid caliphate.</td>
</tr>
<tr>
<td>786-809</td>
<td>Harun Al-Rashid rules a united empire as the fifth Abbasid caliph.</td>
</tr>
<tr>
<td>813-833</td>
<td>Caliph Al-Mamum founds House of Wisdom.</td>
</tr>
<tr>
<td>992</td>
<td>The Ghaznavids form an independent state in Persia, and lead attacks against India.</td>
</tr>
<tr>
<td>1,071</td>
<td>Muslim Seljuk Turks conquer Persia and defeat Byzantine army.</td>
</tr>
<tr>
<td>1,085</td>
<td>Christians capture Toledo and its library.</td>
</tr>
<tr>
<td>1,096</td>
<td>First Crusade starts.</td>
</tr>
<tr>
<td>1,167</td>
<td>Birth of Genghis Khan in Mongolia.</td>
</tr>
<tr>
<td>1,187</td>
<td>Saladin captures Jerusalem.</td>
</tr>
<tr>
<td>1,219</td>
<td>Mongols attack Persia and Turkey.</td>
</tr>
<tr>
<td>1,400</td>
<td>Damascus is devastated by the Mongols led by Tamurlane.</td>
</tr>
</tbody>
</table>

From the Western European point of view, other Arab intellectual centers also play a very meaningful role in the history of our subject. Some would consider the fall of Toledo, and its library, with its immense holding of texts, at the end of the eleventh century as pivotal in the resurgence of mathematical activity in Europe. Many translators came to Toledo and started the slow translation process of the Greek and Arab texts into vernacular European languages.

Islamic mathematics last roughly from 750 to 1,450. In that period they finished the development of the decimal system, together with the algorithms for most of the arithmetical operations: addition, subtraction, multiplication, division and extraction of roots. On the last topic, they not only improved the extraction of square and cube roots, they also developed an algorithm for the extraction of fifth roots!

Islamic mathematicians also contributed to the development of geometry and to the numerical solution of equations of higher degree than two. They began the consistent use of all six of our current trigonometric functions, going beyond the Hindu’s sine and cosine. Al-Tusi from the 13th century would make trigonometry a mathematical science rather than a branch of astronomy, and he would clearly state laws such as the **Law of Sines**:
In number theory, the Pythagorean and Diophantine tradition of perfect and amicable numbers was continued. A number is **perfect** if it equals the sum of its proper divisors, for example $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$ are both perfect. Islamic mathematicians of the 10th century (al-Baghdadi) would correctly claim that

*He who affirms that there is only one perfect number in each power of 10 is wrong; there is no perfect number between ten thousand and one hundred thousand. He who affirms that all perfect numbers end with the figure 6 or 8 are right.*

This was in response to some false assertions made by Alexandrian mathematicians of the first century.

Two numbers are said to be **amicable** if the sum of the divisors of each equals the other number. From Pythagorean times, 220 and 284 had been known to be amicable since the divisors of 284 add up to 220, $1 + 2 + 4 + 71 + 142 = 220$, and, vice versa, the divisors of 220 add up to 284: $1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$.

From the 9th century, Thabit would claim that

For $n > 1$, let $p_n = 3 \times 2^n - 1$ and $q_n = 9 \times 2^{2n-1} - 1$. If $p_{n-1}$, $p_n$ and $q_n$ are prime numbers, then $a = 2^n p_{n-1} p_n$ and $b = 2^n q_n$ are amicable numbers.

Thabit produced a new pair by using his claim when $n = 4$, so $p_4 = 23$, $p_4 = 47$ and $q_4 = 1151$, so $a = 17296$ and $b = 18416$.

They devoted great energy to both astronomy and geography. The *Almagest* would be studied and annotated by many authors during many centuries. Of particular interest was the calculation of the direction of Mecca as a function of terrestrial latitude and longitude since pilgrims attempted to visit the holy city from all over the world.

And in medicine, the physicians from Islam were superior to any other as exemplified by Ibn Sina, who is known in the West as Avicenna. He is considered to have written one of the most important books in the history of medicine, *The Canon of Medicine*. But he also wrote on mathematics, other sciences, astronomy and music.

As a tenet of religious faith, humanistic representations of God are forbidden, hence, Muslims decorated their mosques with extraordinary abstract tilings, and rarely have such tessellations of the plane been more elaborate or
beautiful. Naturally, such considerations lead to serious mathematical activity. For example, in one of their most famous buildings, the Alhambra in Granada (Spain), one can find all possible different symmetry tessellations of the plane.

They had several multiplication and division algorithms, and notably the two of them we will discuss had geometric tilings in their background.

One of their most popular multiplication algorithms was called the *gelosia* or *lattice* method. The method came to Western Europe via al-Khwarizmi's book (see below), and very soon in Italy, this method was extremely popular, and most early texts have examples of it. The example we give below is actually from a Florentine text from 1430. It involves the multiplication of $456,789 \times 987,654$.

\[
\begin{array}{ccccccc}
4 & 5 & 6 & 7 & 8 & 9 \\
4 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
5 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
4 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
9 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
4 & 8 & 3 & 0 & 0 & 6 \\
\end{array}
\]

and we readily read the answer $451,149,483,006$.

We could use this example to illustrate another arithmetical contribution of the times, the method of *casting-out nines* which was taught in school as late as the 1950’s. It was an effective way to test whether an error had occurred during a multiplication. The key idea was the Vedic transformation we mentioned in the previous chapter. Recall that this involved the consecutive addition of a number until a single digit remained. The rule was simple. If one is to multiply two numbers, then the Vedic transformation of their product is the same as the product of their Vedic transformation, itself transformed. One would do the following: take the numbers we are multiplying, $456,789 \times 987,654$, and write.
their Vedic transformations in the top and the bottom of $\frac{3}{9}$, and we get $\frac{3}{9}$ since the Vedic transformation of 456,789 is $456789 \mapsto 39 \mapsto 12 \mapsto 3$ and similarly for the other factor. Multiply these two numbers (and apply the Vedic transformation if necessary) and write the result on the right $\frac{3}{9}$, and finally, do the Vedic transformation to the product of the two numbers and write it on the left, the two numbers should be the same: $\frac{3}{9}$ which they are since $451,149,483,006 \mapsto 45 \mapsto 9$, which makes us happy! In modern times we would refer to this process as doing arithmetic modulo 9.

We now give an example of division. The method is called the galleon method, and it is a form of long division. Once again, the Arab influence on arithmetic is exemplified by the fact that the first printed occurrence of long division in the Americas occurred in a Spanish colony, New Spain (or Mexico) in the 16th century. The reason for the name galleon is from the shape of the finished table that resembles a ship or galleon.

The table represents the division $65,284 \div 594$ with the result 109 with remainder 538.
Again, the importance of the ability to perform both multiplication and division quickly and efficiently cannot be overstressed. There is the tale of the fifteenth century German merchant who asked a German scholar where he should send his child for higher learning, and the scholar replied that if all he needed was for the child to perform addition and subtraction, then German universities would suffice. But that if the merchant wanted the child to do multiplication and division, he best send his child to Italy.

We will discuss three Islamic mathematicians, one from the early period, one from the middle and one from the late: Abu Ja’far Muhammad ibn Musa al-Khwarizmi, Omar Khayyam and Jamshid al-Kashi.

---

Al-Khwarizmi

Details of al-Khwarizmi’s life are scarce. We know he lived from about 780 until 850. Some authors are confident he was born in Baghdad, but some believe he was originally from Central Asia, near the Aral Sea, in the outreaches of Persia. His mathematical contributions were made while at the House of Wisdom in Baghdad, and his works were dedicated to the Caliph al-Mamum.

He was momentous in establishing the decimal tradition in the Islamic world, and thus also in Europe. His lasting influence is still felt in two very common words in the mathematical lexicon.

First, he wrote a book on arithmetic called The Book of Addition and Subtraction According to the Hindu Calculation. When this book came to Western Europe, his name is going to be associated with the idea of knowing how to do arithmetic. With an Italian pronunciation, his name became alcorismi, from which our word algorithm comes from.

Second, another of his books is called in the original Arab: Hisab al-jabr wa’l-muqabala, which means The Book of Restoring and Balancing. It is this title that gave us the word algebra, and, naturally, this book is pivotal in the history of the subject.

Not as symbolic as present-day algebra, this early algebra uses mostly words. What we usually denote by $x$, it was called root or thing. A constant in an equation was number, what we call $x^2$ was referred to as mal.

Al-Khwarizmi considers mainly linear and quadratic equations and he identifies six types (three more than Euclid as he allows 0 as a coefficient):

<table>
<thead>
<tr>
<th>squares equal to roots</th>
<th>squares equal to numbers</th>
<th>roots equal to numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 = 3x$</td>
<td>$x^2 = 9$</td>
<td>$3x = 15$</td>
</tr>
<tr>
<td>squares and roots equal to numbers</td>
<td>squares and numbers equal roots</td>
<td>roots and numbers equal to squares</td>
</tr>
<tr>
<td>$x^2 + 10x = 39$</td>
<td>$x^2 + 21 = 10x$</td>
<td>$3x + 4 = x^2$</td>
</tr>
</tbody>
</table>
Restoring and balancing refer to transforming an equation into one of these 6 forms. For example, \( x^2 = 40x - 4x^2 \) is \textit{al-jabr} into \( 5x^2 = 40x \) since we removed a subtraction; while the equation \( 50 + 3x + x^2 = 29 + 10x \) is \textit{al-muqabala} first into \( 21 + 3x + x^2 = 10x \) and then \textit{al-muqabala} again into \( 21 + x^2 = 7x \).

One of the problems from al-Khwarizmi's book is:

\[ \text{Solve mal and 10 root equals 39.} \]

His solution is given in

\[ \text{... a mal and 10 roots are equal to 39 units. The question therefore in this type of equation is about as follows: what is the mal which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. Now the roots in the problem before us are 10. Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3. The number three therefore represents one root of this square, which itself, of course is 9. Nine therefore gives the square.} \]

This paragraph represents syncopated algebra where common words, as opposed to symbols, are used to express the algebraic manipulations. The symbolic algebra will not be born until the Renaissance in Western Europe. The main contributions will be Italian and German as we will see in that chapter.

Al-Khwarizmi then gives a geometric justification for the argument by completing the square:

\[
\begin{array}{ccc}
& \frac{5x}{2} & \frac{5x}{2} \\
\frac{5x}{2} & x^2 & \frac{5x}{2} \\
\frac{5x}{2} & \frac{5x}{2} & \frac{25}{4} & \frac{25}{4} \\
\end{array}
\]

Naturally, nowadays, we would simply proceed by

\[ x^2 + 10x = 39 \quad (x + 5)^2 = 39 + 25 = 64 \quad x + 5 = 8 \quad x = 3. \]

The negative root \( x = -13 \) is ignored.

Al-Khwarizmi further discusses (all in words, naturally) the multiplication of such expression such as \((2x + 3)(4x + 5)\).

One of the roles mathematics played in Islam was in the computation of inheritances. There were specific laws that dealt with inheritance. Some of the guidelines were as follows:
• If a wife died, the husband inherited one-fourth of the state, and the rest is divided among the children.
• A son receives twice what a daughter receives.
• A stranger cannot receive more than one-third of the state without approval of the other heirs.
• If a share is left to a stranger, this is taken off the top—before any distribution is made.

So another problem we encounter in al-Khwarizmi’s Algebra is:
A woman dies leaving a husband, a son and three daughters. She also leaves a bequest consisting of \( \frac{1}{8} + \frac{1}{7} \) of her estate to a stranger. Calculate the shares of her estate that go to each of her beneficiaries.

The stranger then receives, \( \frac{1}{8} + \frac{1}{7} = \frac{15}{56} \). Leaving \( \frac{41}{56} \) of the estate to be shared by husband and children. The husband receives then one-fourth of \( \frac{41}{56} \), which equals \( \frac{41}{224} \). This leaves \( \frac{123}{224} \) of the estate. We divide this into 5 equal parts—giving the denominator 1120, since it has to be shared in the ratio 2:1:1:1.

Final allocations are then:

<table>
<thead>
<tr>
<th></th>
<th>Strangers</th>
<th>Husband</th>
<th>Son</th>
<th>Each Daughter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>300</td>
<td>205</td>
<td>246</td>
<td>123</td>
</tr>
<tr>
<td></td>
<td>1120</td>
<td>1120</td>
<td>1120</td>
<td>1120</td>
</tr>
</tbody>
</table>

Omar Khayyam (c.1048-1131) is widely known in world literature for his poetic masterpiece, the Rubaiyat. Not as well known is the fact that he was an excellent mathematician. He lived his life in what was then Persia during very turbulent times. He served the Seljuk Turk ruler of the region at the observatory in Esfahan for more than 18 years, which was one of his most productive periods. It was at the observatory that Khayyam participated in a reform of the calendar. At one time he suggested that instead of every fourth year being a leap year, it should be eight years out of every 33 that ought to be leap years, and thus the length of the years would be 365.2424 which is closer to the astronomical year (365.2422) than the Julian year of 365.25, but also than the presently used Gregorian, 365.2425. He accurately measured the length of the year to be 365.24219858156 days. Naturally, there is no necessarily straightforward way to select 8 years out of 33 of them.

Khayyam had a truly wide spread scope of interests and talents as evidenced by both poetry and mathematics. But even within mathematics, he studied the foundations of Euclidean geometry, in particular the fifth postulate of Euclid, as well as the nature of the ratio of two numbers, including Eudoxus’ definition given in Euclid. Khayyam’s name is closely associated with what was perhaps the first general study of
cubic equations—a very popular topic in the early fifteen hundreds in Italy. Partly in the
desire for the duplication of the cube, and, partly out of general intellectual curiosity,
Omar Khayyam set out to classify the types of cubic equations that there were. Many
centuries later, Newton would be interested in cubic equations in two variables, and in
the late 20th century, cubic equations would be used in the solution of Fermat's last
theorem.

We have seen that the quadratic equation: \( x^2 + ax + b = 0 \) appeared very early in history,
and was completely solved possibly as far back as 4,000 years ago. We looked at how al-
Khwarizmi had given complete solutions for all 6 types of quadratic equations. The cubic
emerges early in history too; for example the duplication of the cube is nothing else but a
solution to the equation \( x^3 = 2 \).

As with the quadratic, we, in modern times, think of only one type of cubic equation:
\( x^3 + ax^2 + bx + c = 0 \).

But that is because we freely consider negative and positive numbers and zero as
equivalent. Moreover, we completely ignore the nature of the roots, whether they are real
or not. Khayyam naturally concentrated on the ones that had real solutions. Thus,
\( x^3 + 2x - 4 = 0 \) and \( x^3 - 2x + 4 = 0 \) seem of similar type to our eyes, but to Omar
Khayyam, they seem as of two different types:

\[
\begin{align*}
x^3 - 2x + 4 &= 0 & x^3 + 4 &= 2x & \text{Cube and numbers equal sides.} \\
x^3 + 2x - 4 &= 0 & x^3 + 2x &= 4 & \text{Cube and sides equal numbers.}
\end{align*}
\]

The last column indicates how Omar Khayyam would refer to the type of equation. In all,
he classified 25 different types of cubic equations, and he gave methods for solving all of
the ones that had had real (mainly positive) solutions.

For eleven different types, he gave methods that involved only the Euclidean tools,
straightedge and compass, but for others, he used conics to get the solution. He would be
proven correct more than seven centuries later when he claimed that straightedge and
compass alone were not sufficient to solve all cubics.

We end our discussion of Omar Khayyam with an example of one of his solutions:

\[
\begin{align*}
\text{Cube and sides equal numbers.} & \quad x^3 + px = q \quad \text{where } p \text{ and } q \text{ are} \\
\text{positive numbers} & \quad \text{Take } a \text{ so that it is the side whose square equals} \\
& \quad \text{the number of roots.} \\
& \quad a = \sqrt{p} \\
\text{Let } h \text{ be the side of a rectangular parallelepiped} & \quad h = \frac{q}{p} \\
\text{whose base is } a^2 \text{ and whose volume is } q. 
\end{align*}
\]
Take a parabola whose vertex is $B$, axis $BZ$ and parameter $a$, and place $h$ perpendicular to $BZ$ at $B$.

Consider the parabola $ay = x^2$, and find $h$ in the $x$-axis.

<table>
<thead>
<tr>
<th>On $h$ as diameter describe a semicircle and let it cut the parabola at $D$.</th>
<th>Find the intersection $D$ of the parabola with the circle $\left( x - \frac{h}{2} \right)^2 + y^2 = \left( \frac{h}{2} \right)^2$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>From $D$, drop $DE$ perpendicular to $h$ and the ordinate $DZ$ perpendicular to $BZ$.</td>
<td>Find the coordinates of $D$.</td>
</tr>
<tr>
<td>Then $DZ = BE$ and $BE$ solves our equation</td>
<td>The $x$-coordinate of $D$ is a solution.</td>
</tr>
</tbody>
</table>

The modern proof that the construction works is immediate: the equation of the semicircle is $x^2 - hx + y^2 = 0$, or equivalently, $y^2 = x(h - x)$, or also $a^2y^2 = a^2x(h - x)$. From the parabola, we know that $x^2 = ay$ so $x^4 = a^2y^2$. Equating we get $x^4 = a^2x(h - x)$, and canceling $x$, $x^3 = a^2(h - x) = p(h - x)$, thus $x^3 + px = ph - q$ and we have solved the equation.

Omar Khayyam in his construction respects at all times the homogeneous nature of the equation $x^3 + px = q$ in that $p$ is an area and $q$ is a volume, so the geometric spirit of the Greeks lives on. By the way, in order to build $h$, he needed a segment of length 1, and then he gave a relatively straightforward Euclidean construction of $h$.

---

**Al-Kashi**

As one may say that the art of Omar Khayyam was geometric algebra, one should state that the art of Jamshid al-Kashi was numerical algebra—he was a number-cruncher. Appropriately, he wrote a book called *Calculator’s Key*. We do not know when al-Kashi was born, but we know that he was active from 1406-1429, when he passed away. He lived most of his productive years in Samarkand, in Central Asia, in what was then Persia. There he was the leading astronomer and mathematician of a large group of scientists assembled there by the ruler-scientist Ulugh Beg, a grandson of the Mongol conqueror, Tamurlane.

Al-Kashi improved on the algorithm for taking square roots that the Indus had developed, but more remarkably he gave a precise algorithm to compute the *fifth root of a number*. However, in our modern days of calculators, the fifth root algorithm is beyond our
manual dexterity, and instead we will delve into an algorithm for taking square roots, which as mentioned above also was of al-Kashi’s concern.

We view the algorithm in the concrete by looking at an example. As all similar root algorithms, it computes the square root one digit at a time. Clearly, the algorithm can be extended to pursue decimal digits (as many as we desire) as we will exemplify below.

<table>
<thead>
<tr>
<th>Power</th>
<th>Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>100</td>
<td>10,000</td>
</tr>
<tr>
<td>1,000</td>
<td>1,000,000</td>
</tr>
</tbody>
</table>

Before we proceed it would be convenient to have a table of the digits and their squares. We also need to observe what the powers of 10 have for their squares

Suppose now we want to compute $\sqrt{605,061,699}$. The first step is to realize that we are taking the square root of 605 million plus. This will give us the square root of 605 thousands plus, so we need to start by taking the square root of 605. Clearly, the square root of 605 is a two-digit number: $10a + b$ where $a$ and $b$ are digits. Thus when we square we have $(10a + b)^2 = 100a^2 + 20ab + b^2$, and so $a^2 \leq 6$, and so $a = 2$.

That is exactly the starting point of the algorithm. Actually, more accurately, the algorithm starts by separating the number whose square root we are taking into blocks of two digits starting on the right (since the square of 10 is 100).

Thus we have found $a$, the first digit for the answer for the square root of 605. Subtract its square and the remainder together with next block, namely, 20,5 will be used to estimate $b$, the second digit. To do this, we double what we have obtained so far for our square root (only a 2 so far), and separate the last digit of our remainder (since remember the term is $20ab$), and

Ask how many times does our doubled result goes into our remainder: $20 \div 4 = 5$, and so we attach the 5 behind the 4, and multiply it by 5, obtaining 225.
which is too big since our remainder is only 205, so our estimate \( b = 5 \) was too high, and here is one of the crucial ingredients to our algorithm, the **need to erase at times**. Thus, before this algorithm could be immensely popular, technological improvements such as paper and pencil, had to allow erasure. So we change our estimate to 4

\[
\begin{array}{ccc}
6,05,06,16,99 & 2 \\
4 & 44 \\
205 & 4 \\
176 & 176 \\
29 &
\end{array}
\]

so our next digit is 4, we enter it, and double our result. We also bring the next block next to our remainder, and again separate the last digit

\[
\begin{array}{ccc}
6,05,06,16,99 & 24 \\
4 & 24 & 48 \\
205 & 4 \\
176 & 176 & 290,6 \\
\end{array}
\]

Again we ask \( 290 \div 48 = 6 \), and we can estimate the next digit to be 6,

\[
\begin{array}{ccc}
6,05,06,16,99 & 24 \\
4 & 24 & 486 \\
205 & 4 & 6 \\
176 & 176 & 2916 \\
290,6 &
\end{array}
\]

but, as before, it is too high, so we have to change down again

\[
\begin{array}{ccc}
6,05,06,16,99 & 24 \\
4 & 24 & 485 \\
205 & 4 & 5 \\
176 & 176 & 245 \\
2906 & & 291 \\
2425 &
\end{array}
\]

Thus we enter our new digit 5, double our result, and bring the next group of two digits, and separate the last digit:
and estimating \(4811 \div 490 = 9\), and testing

\[
\begin{array}{c|c|c|c|c}
6,05,06,16,99 & 245 & \\
4 & 24 & 485 & 490 \\
205 & 4 & 5 & 9 \\
176 & 176 & 2425 & 44181 \\
2906 & 2425 & \\
48116 & 44181 & \\
3935 & \\
\end{array}
\]

so no erasure is necessary this time. Continuing,

\[
\begin{array}{c|c|c|c|c|c}
6,05,06,16,99 & 2459 & \\
4 & 24 & 485 & 4909 & 4918 \\
205 & 4 & 5 & 9 \\
176 & 176 & 2425 & 44181 \\
2906 & 2425 & \\
48116 & 44181 & \\
3935 & 9, 9 \\
\end{array}
\]

and estimating \(39357 \div 4918 = 8\), and testing

\[
\begin{array}{c|c|c|c|c|c|c}
6,05,06,16,99 & 24598 & \\
4 & 24 & 485 & 4909 & 49188 \\
205 & 4 & 5 & 9 \\
176 & 176 & 2425 & 44181 & 393504 \\
2906 & 2425 & \\
48116 & 44181 & \\
393599 & \\
393504 & \\
95 & \\
\end{array}
\]

and we are done; our answer is 24,598 (we have a remainder).
Now we show how to compute decimal digits by adding blocks of 2 zeroes for each
decimal digit we wanted.

We will proceed to do the first three decimals, without any dialog

<table>
<thead>
<tr>
<th>6,05,06,16,99,00,00,00</th>
<th>24598.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>24</td>
</tr>
<tr>
<td>205</td>
<td>4909</td>
</tr>
<tr>
<td>176</td>
<td>49188</td>
</tr>
<tr>
<td>2906</td>
<td>24598</td>
</tr>
<tr>
<td>48116</td>
<td>491860</td>
</tr>
<tr>
<td>393599</td>
<td>4919600</td>
</tr>
<tr>
<td>9500,0</td>
<td>49196001</td>
</tr>
<tr>
<td>950000,0</td>
<td></td>
</tr>
<tr>
<td>95000000</td>
<td></td>
</tr>
<tr>
<td>45803999</td>
<td></td>
</tr>
</tbody>
</table>

So the answer to three decimals is 24,598.001.

Al-Kashi may have been one of the first ones to have used decimal notation for digits to
the right of the decimal point. This notation will not be used in the West for another
century. As a matter of fact, Fibonacci, who introduced Hindu-Arabic numerals to Italy
in the thirteenth century, always used hexagesimal notation to express numbers.

Al-Kashi used both decimal and hexagesimal notation to give an approximation of $2\pi$
which is correct to 16 decimal places. Namely,

$$2\pi = 6; 16, 59, 28, 1, 34, 51, 46, 14, 50,$$

or, decimally,

$$2\pi = 6.2831853071795865.$$

He accomplished his approximation by using the half-angle formula 28 times, and being
very careful with the error estimates. This is tantamount to inscribing and circumscribing
polygons with 805,306,368 sides in a circle! But, as we mentioned above, he used
numbers to do the estimation.

This represented a considerable improvement over the two digits of Archimedes. Also, it
was considerably better than the approximation given in the fifth century AD by the
Chinese mathematician Zu Chongzhi who gave the fraction $\frac{355}{113}$ as an approximation.
This is accurate to 6 digits. However, when Calculus was developed, much more powerful methods of approximation will be developed, methods that will supersede al-Kashi’s approximation by tens of digits. Many distinguished mathematicians participate in this hunt—including Newton. In recent times, the first two billion digits of $\pi$ have been computed using both mathematics and computers.
Chapter 10
China

As the prerogative of Natural Science is to cultivate a taste for observation, so that of Mathematics is, almost from the starting point, to stimulate the faculty of invention—Sylvester

We continue our journey to the East by visiting the Chinese culture, an ancient and inventive civilization. As we see from the table of dates below, the age of the Chinese civilization rivals that of Egypt and Mesopotamia.

<table>
<thead>
<tr>
<th>Date</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>5000 BC</td>
<td>Rice is being cultivated on the Eastern Coast of China.</td>
</tr>
<tr>
<td>1300 BC</td>
<td>The Shang Dynasty has several capitals, among them Anyang. New system of writing and measurement and a new calendar are introduced.</td>
</tr>
<tr>
<td>1050 BC</td>
<td>The Western Zhou Dynasty takes over. Period of prosperity for the first 200 years.</td>
</tr>
<tr>
<td>770 BC</td>
<td>Central authority collapses. Former fiefdoms become rival states-Warring States period starts in the fifth century.</td>
</tr>
<tr>
<td>480 BC</td>
<td>Confucius, the Great Philosopher dies.</td>
</tr>
<tr>
<td>221 BC</td>
<td>The Qin emerge victorious from the Warring States period, and their leader Shi Huangdi becomes emperor of all of China. The Great Wall is finished.</td>
</tr>
<tr>
<td>206 BC</td>
<td>After the death of Shi Huangdi, the Qin are overthrown, Liu Bang becomes the first Han emperor, and the capital is Chang'an.</td>
</tr>
</tbody>
</table>
We will discuss mathematics in China from antiquity to what is considered its zenith of the ancient period, 1279 AD.

Since the emperor, Shi Huangdi, ordered the burning of many books in 211 BC, many records prior to that time have been lost. We do know that by 500 BC they were acquainted with the Pythagorean Theorem, called the **Gougu Theorem**. In fact, possibly the oldest extant written diagram illustrating the Pythagorean Theorem is Chinese from this period in time. In modern notation, we see that

$$c^2 = (a - b)^2 + 4(ab / 2) = a^2 - 2ab + b^2 + 2ab = a^2 + b^2.$$

These and other mathematics were recorded in such classics as **The Circular Paths of Heaven**, or **Nine Chapters on the Mathematical Arts**.

Also prior to the Warring States period, they already had a **magic square**, which according to legend, the emperor saw in the shell of a river turtle. The **Lo shu**, as the diagram is called, resembles the figure, and it is equivalent to the

---

136 BC  Confucianism is declared the state religion.
124 BC  An imperial university is established in the capital.
65 AD  Buddhism enters China from India.
105 AD  Paper and porcelain are invented.
220 AD  The Han dynasty ends and China is divided into three kingdoms.
280 AD  The Jin dynasty unifies China.
316 AD  Chang’an falls to the barbarians. China is divided, north and south for more than 250 years.
581 AD  The Sui dynasty unifies China.
610 AD  The Grand Canal linking Chang’an to the Yangtze is finished.
618 AD  The Tang dynasty begins and in 626 adopts Buddhism. It lasts until 907 AD.
868 AD  The Diamond Sutra, the oldest printed book still in existence is produced by wood blocks.
960 AD  The Sung dynasty reunifies China.
1000 AD  Gunpowder is perfected.
1090 AD  Water clock built in the capital.
1120 AD  Playing cards are invented.
1279 AD  Kublai Khan, grandson of Genghis Khan becomes ruler of all China.
1368 AD  Mongols driven out of China, Zhu Yuanzhan founds Ming Dynasty.
1644 AD  Qing Dynasty starts. It will last until 1911.
4 9 2

magic square: 3 5 7. This is a 3×3 magic square, which is a square arrangement of the numbers 1,2,..,9 = 3^2 so that every row and every column has the same sum. Often it also required that the two diagonals have that same sum, in this case 15. Later on, in the thirteenth century, the mathematician Yang Hui would list magic squares of every size up to 10×10. We give the 7×7 example:

Observe that the central nine positions, have also a constant row, column and diagonal sum of 75, and that the central 25 positions also have a constant row, column and diagonal sum of 125, while each row, column and diagonal of the whole square adds to 175. There was also a balance in the construction built around the number 50.

But, perhaps, the most essential feature of ancient Chinese mathematics is their use of counting rods. Sets of these rods—some were bamboo sticks, some were made of bone, some were just scratches on clay—have been discovered that are more than 2,500 years old. These rods were very useful not just for expressing numbers, but also for doing all the arithmetical operations as well as taking roots, and later on for solving polynomial equations!

First, let us address the writing of numbers. The Chinese always used base 10, and they had position in their notation, but in a unique fashion. They had two sets of digits! The symbols were similar, except some were vertical and some were horizontal, and they resembled the rods.

The writing of numbers would be accomplished by using the digits called even above for the even powers of 10, while the odd digits would be used for the odd powers of 10. In the words of Master Sun, a mathematician of the fifth century AD,

Units are vertical, tens are horizontal,
Hundreds stand, thousands lie down;
Thus thousands and tens look the same,
Ten thousands and hundreds look alike.
Hence 362 would be written \( \underline{||| \underline{|} | |} \), while 623 would be written \( \underline{\underline{|} || |} \).

A 0 was not needed since, for example, 201, represented by \( \underline{||} \) \( \underline{|} \) could not be confused with \( \underline{\underline{|}} \) which stood for 21. Usually, a small space left in between was all that was needed. However, 2001 could be confused with 21, but that was unlikely from context. Finally, in the thirteenth century, the Chinese adopted a 0.

Effective algorithms for arithmetic were developed early, possibly as far back as the fifth century BC. One of the advantages of the counting rod digits was that they could easily be transformed, making the carry-over of digits much smoother.

We exemplify with an addition, and observe that they started the addition on the left, accentuating the fact that changing digits was easily done with the rods. We do the addition \( 273 + 968 \) in both notations. In this example, as in all that follow, we will indicate different steps by a thick boundary,

**Chinese Addition**

<table>
<thead>
<tr>
<th></th>
<th>Chinese Addition</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \underline{\underline{</td>
<td>}</td>
<td></td>
</tr>
<tr>
<td>( \underline{\underline{</td>
<td>}</td>
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<td>( \underline{\underline{</td>
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<td>( \underline{\underline{</td>
<td>}</td>
<td></td>
</tr>
<tr>
<td>( \underline{\underline{</td>
<td>}</td>
<td></td>
</tr>
</tbody>
</table>

Subtraction was accomplished in a similar fashion. Note that the notation leads to rectangular arrays very naturally, and matrices would not be totally foreign to the ancient Chinese. In fact, later on, certainly by the third century, they would indeed develop methods to solve systems of linear equations in a method not dissimilar to our modern day matrix methods, and by the thirteenth they would be able to solve systems of polynomial equations. We will see an example below.
Continuing with the arithmetic review. We will multiply $258 \times 376$ to exemplify their methods. We will do it in our modern notation to make it easier for the reader. But we should keep in mind that the counting rods of the Chinese were very easy to rearrange, and hence in their notation the algorithms would be easy to implement. They used a table with three rows, and we have more rows at times to make it easier to learn the algorithm. These middle steps would more naturally be performed in our heads.

### Chinese Multiplication

<table>
<thead>
<tr>
<th>Explanation</th>
<th>2 5 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplier—Top Row</td>
<td></td>
</tr>
<tr>
<td>Multiplicand—Bottom Row</td>
<td>3 7 6</td>
</tr>
<tr>
<td>Top Row</td>
<td>2 5 8</td>
</tr>
<tr>
<td>Bottom Row, lined up so that last digit of bottom coincides with first digit of top.</td>
<td>3 7 6</td>
</tr>
<tr>
<td>Middle Step, the first digit of the multiplier is used to multiply by.</td>
<td>6</td>
</tr>
<tr>
<td>Middle step, just continuing the process.</td>
<td>7 4</td>
</tr>
<tr>
<td>True Middle Row, 2 has been multiplied by the bottom row.</td>
<td>7 5 2</td>
</tr>
<tr>
<td>Erase the first digit, since it has been multiplied.</td>
<td>5 8</td>
</tr>
<tr>
<td>Bottom row is again lined up so that last digit is with first digit.</td>
<td>3 7 6</td>
</tr>
<tr>
<td>Middle Row from Above.</td>
<td>7 5 2</td>
</tr>
<tr>
<td>Middle Step: Multiplying the 5 times the 3, and adding to previous middle row.</td>
<td>9 0 2</td>
</tr>
<tr>
<td>Middle Step: Multiplying the 5 times the 7, etcetera.</td>
<td>9 3 7</td>
</tr>
<tr>
<td>Second Middle Row.</td>
<td>9 4 0 0</td>
</tr>
<tr>
<td>Erase the second digit, since it has been multiplied</td>
<td></td>
</tr>
<tr>
<td>Bottom Row, lined up</td>
<td>3 7 6</td>
</tr>
<tr>
<td>Middle Row from Above.</td>
<td>9 4 0 0</td>
</tr>
<tr>
<td>Same process as above.</td>
<td>9 6 4 0</td>
</tr>
<tr>
<td>Same process as above.</td>
<td>9 6 9 6</td>
</tr>
<tr>
<td>Final Middle Row.</td>
<td>9 7 0 0 8</td>
</tr>
<tr>
<td>It is the Answer.</td>
<td></td>
</tr>
</tbody>
</table>

The Chinese table would be even briefer than the one below when we have kept just the minimal information:

<table>
<thead>
<tr>
<th>Top Row</th>
<th>2 5 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bottom Row, lined up</td>
<td>3 7 6</td>
</tr>
<tr>
<td>Erase the first digit, since it has been multiplied.</td>
<td>5 8</td>
</tr>
<tr>
<td>Bottom row is again lined up so that last digit is with first digit.</td>
<td>3 7 6</td>
</tr>
<tr>
<td>Middle Row from Above.</td>
<td>7 5 2</td>
</tr>
<tr>
<td>Erase the second digit, since it has been multiplied</td>
<td></td>
</tr>
<tr>
<td>Bottom Row, lined up</td>
<td>3 7 6</td>
</tr>
<tr>
<td>Middle Row from Above.</td>
<td>9 4 0 0</td>
</tr>
<tr>
<td>Final Middle Row. It is the Answer.</td>
<td>9 7 0 0 8</td>
</tr>
</tbody>
</table>
Division was done by a method similar to the galleon method we mentioned in the previous chapter. Except, repeated subtraction was used much more as part of the algorithm—since, again, the rods could easily be rearranged, and repeated subtraction was easy to perform.

We will perform \(106838 \div 456\),

### Chinese Division

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Quotient</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0 6 8 3 8 4 5 6</td>
<td>First Digit of Quotient Dividend Row First Divisor</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5 6</td>
<td>First Remainder</td>
</tr>
<tr>
<td>1</td>
<td>0 6 8 3 8 4 5 6</td>
<td>Second Dividend Second Divisor</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5 6</td>
<td>Second Digit of Quotient</td>
</tr>
<tr>
<td>1</td>
<td>0 6 8 3 8 4 5 6</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3 9</td>
<td>Second Remainder</td>
</tr>
<tr>
<td>2</td>
<td>5 6 5</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0 6 8 3 8 4 5 6</td>
<td>Final Remainder</td>
</tr>
<tr>
<td>1</td>
<td>3 9</td>
<td>Third Dividend Third Divisor Third Digit of Quotient</td>
</tr>
<tr>
<td>2</td>
<td>5 6 5</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0 6 8 3 8 4 5 6</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3 9</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5 6 5</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0 6 8 3 8 4 5 6</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

And the answer is \(234 \frac{134}{456}\).

It is appropriate to mention that the Chinese handled fractions extremely well, and
performed arithmetic on them. The idea of doing arithmetic with the counting rods was extended to include the extraction of roots.

It is through that extraction of roots that, by the fourteen century, the Chinese had developed what, we, in the West, referred to as Pascal’s Triangle (and which we will study in a future chapter). In fact, this triangle is appropriately referred to as Yang Hui’s Triangle in China, and its presentation is the familiar triangular form.

In the fifteenth century, the abacus was widely adopted, and even superior algorithms for arithmetic would be developed based on it.

We mentioned above that they developed methods to solve linear systems. We give an example from The Nine Chapters of the Arithmetic Art, which dates from the first century AD. The method was called of rectangular arrays. We should remark a Dou is a unit of volume. The problem is the following:

Three bundles of top-grade ears of rice, together with two bundles of medium grade, and one bundle of low-grade ears of rice make 39 dou of rice. Also two bundles of top-grade ears of rice with three bundles of medium grade and one bundle of low-grade ears of rice make 34 dou of rice. Finally, one bundle of top-grade ears of rice, and two bundles of medium grade with three bundles of grade ears of rice make 26 dou of rice. How many dou are there in a bundle of top-grade, of medium grade, and of low-grade ears of rice?

The Chinese would write the system vertically and from right to left:

$$
\begin{array}{c}
1 & 2 & 3 \\
2 & 3 & 2 \\
3 & 1 & 1 \\
\end{array}
$$

Today, we would write this problem as a system of three equations on three unknowns:

$$
3x + 2y + z = 39
$$

$$
2x + 3y + z = 34
$$

$$
x + 2y + 3z = 26
$$

and we would use the matrix representation:

$$
\begin{bmatrix}
3 & 2 & 1 \\
2 & 3 & 1 \\
1 & 2 & 3 \\
\end{bmatrix}
$$

Keeping our modern day notation, we will follow the Chinese steps:

Multiply the second row by 3,

$$
\begin{bmatrix}
3 & 2 & 1 \\
6 & 9 & 3 \\
1 & 2 & 3 \\
\end{bmatrix}
$$

Subtract twice the first row from the second row:

$$
\begin{bmatrix}
3 & 2 & 1 \\
0 & 5 & 1 \\
1 & 2 & 3 \\
\end{bmatrix}
$$
Multiply the third row by 3,
\[
\begin{pmatrix}
3 & 2 & 1 & 39 \\
0 & 5 & 1 & 24 \\
3 & 6 & 9 & 78
\end{pmatrix}
\]
Subtract the first row from the third,
\[
\begin{pmatrix}
3 & 2 & 1 & 39 \\
0 & 5 & 1 & 24 \\
0 & 4 & 8 & 39
\end{pmatrix}
\]
Multiply the third row by 5,
\[
\begin{pmatrix}
3 & 2 & 1 & 39 \\
0 & 5 & 1 & 24 \\
0 & 20 & 40 & 195
\end{pmatrix}
\]
Subtract four times the second row from the third row:
\[
\begin{pmatrix}
3 & 2 & 1 & 39 \\
0 & 5 & 1 & 24 \\
0 & 0 & 36 & 99
\end{pmatrix}
\]
Now we can say, \(36z = 99\), so \(z = \frac{99}{36} = 2\frac{1}{4}\). Using the middle row, \(y = 4\frac{1}{4}\), and finally, \(x = 9\frac{1}{4}\).

Note that there is very little difference between the ancient method and our modern day method, except their age—there is more than fifteen centuries between them.

We give a later example of an indeterminate linear system, but diophantine in nature. The example is from the book, (Zhang Qiujian) Mathematical Manual, which dates from the sixth century. Beware, that there are several books by that title.

\textit{One rooster is worth five copper cash; one hen is worth three copper cash; three young chicks are worth one copper cash. Buying 100 fowls with 100 cash, how many roosters, hens and chicks?}

If we let \(R\) stand for the number of roosters, \(H\) for the number of hens and \(C\) for the number of chickens, then the conditions of the problem easily translate to the following two equations:

\[
\begin{align*}
R + H + C &= 100 \\
5R + 3H + \frac{1}{3}C &= 100
\end{align*}
\]

Or in matrix notation this becomes \(\begin{pmatrix} 1 & 1 & 1 \\ 5 & 3 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} R \\ H \\ C \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \end{pmatrix} \). Although the method of solution was not included, the book gives the three positive solutions, although today we would probably list all 4, including the one with one of the unknowns taking the value 0:

\[
\begin{array}{ccc}
R & H & C \\
12 & 4 & 84 \\
8 & 11 & 81 \\
4 & 18 & 78 \\
0 & 25 & 75 \\
\end{array}
\]

We will discuss some of the geometric contributions of the Chinese as we discuss some of their many mathematicians. Naturally, since we are writing in English, we will use Western symbols and spellings. But we will keep the Chinese tradition of writing the last name first. We will discuss three names, Liu Hui, Zu
Chongzhi, and Qin Jiushao.

Chinese mathematics had an especially productive period during the thirteenth century, and although we discuss only one mathematician of that period, Qin Jiushao, there are several others of the same excellent quality. Later on, communication with the West would widen. For example, in 1582, with Jesuit collaboration, Euclid would be translated into Chinese. Although we stop in the thirteenth century, without a doubt, the outstanding mathematical tradition has continued in China until our present day.

**Liu Hui**

Liu Hui (c. 250 AD) set out to improve on the ratio of the circumference to the diameter that was listed in the Nine Chapters on the Mathematical Art, and he indeed wrote the last Commentary on that ancient work. It is this work that we will concentrate on, although he also wrote Sea Island Mathematical Manual.

In the original Nine Chapters, the area of a circle is listed as half the circumference multiplied by half the diameter. Note this is correct since half the circumference is $\pi r$ multiplied by half the diameter, $r$. But Liu Hui pointed out that the value for the ratio of the circumference to the diameter, our $\pi$, that was used, $\pi = 3$, was not correct. He developed a method called circle division, and he made several nice observations.

He pointed out that $\pi = 3$ gives the exact value not for the circle but for the regular dodecagon inscribed in the circle, and so it gives a lower estimate. In fact, since the angle at the center is $30^\circ$, the height of the triangle in the picture on the right is $\frac{1}{2}$, so its area is $\frac{1}{4}$, and thus we have $3$ for the total area.

Liu Hui made the nice observation that if one knows the side of the $2n$-gon, then one can find the area of the $4n$-gon.

Suppose we let $s_{2n}$ denote the side of the $2n$-gon, and we let $A_{2n}$ be the area of the $2n$-gon. So, in the picture below, $FB = s_{2n}$. Let $BD$ be the side of the $4n$-gon. We are interested in $A_{4n}$, the area of the $4n$-gon. But, as the Nine Chapters claims, we have then

$$A_{4n} = 2n \left( \frac{(FB)(OA)}{2} + \frac{(FB)(AD)}{2} \right) = 2n \left( \frac{s_{2n}r}{2} \right) = nrs_{2n}.$$

From the same picture we can also observe that if we let $A$ denote the area of the circle, then

$$A_{4n} < A < A_{4n} + (A_{4n} - A_{2n}) \text{ for any } n.$$
The first inequality is obvious. The second one:

\[ A = 4n \times \text{sector } \text{OBD} = 4n \times (\Delta \text{OBD} + \text{sliver } \text{BD}). \]

But this sliver \( \text{BD} \) is less than \( \Delta \text{BCD} \), which equals \( \Delta \text{BDA} \). And so

\[ A < 4n \times (\Delta \text{OBD} + \Delta \text{BDA}) = 4n \times \Delta \text{OBD} + 4n \times \Delta \text{BDA}. \]

But,

\[ A_{4n} - A_{2n} = 2n \times \Delta \text{FDB} = 4n \times \Delta \text{BDA}. \]

Finally, the last key observation is that from \( s_{2n} \), we can calculate \( s_{4n} \) by the Gougu (Pythagorean) Theorem. Again, referring to the same picture, in a computation reminiscent of Ptolemy,

\[ \text{OB} = r, \ \text{AB} = \frac{s_{2n}}{2}, \ \text{AO} = \sqrt{r^2 - \left(\frac{s_{2n}}{2}\right)^2}, \ \text{AD} = r - \text{AO}, \]

and so we get,

\[ s_{4n} = \text{BD} = \sqrt{(\text{AD})^2 + (\text{AB})^2} = \sqrt{r^2 - 2\sqrt{r^2 - \left(\frac{s_{2n}}{2}\right)^2} + \left(\frac{s_{2n}}{2}\right)^2} = \sqrt{2r^2 - 2\sqrt{r^2 - \left(\frac{s_{2n}}{2}\right)^2}.} \]

Finally, in a similar, yet independent, fashion to Archimedes, Liu Hui started with the area of the hexagon, and by doubling the number of sides, he went as far as the side of the 96-gon, and thus the area of the 192-gon. All the time fully aware that all he has are better and better approximations to the true value of the ratio, not the true ratio, in other words, in full realization of the idea of limit.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 2n )</th>
<th>( 4n )</th>
<th>( s_{2n} )</th>
<th>( A_{4n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>12</td>
<td>( \frac{\sqrt{6} - \sqrt{2}}{2} )</td>
<td>3 ( (\sqrt{6} - \sqrt{2}) )</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>24</td>
<td>( \sqrt{2} - \sqrt{2 + \sqrt{3}} )</td>
<td>12 ( \sqrt{2} - \sqrt{2 + \sqrt{3}} )</td>
</tr>
<tr>
<td>12</td>
<td>24</td>
<td>48</td>
<td>( \sqrt{2 - \sqrt{2 + \sqrt{3}}} )</td>
<td>24 ( \sqrt{2 - \sqrt{2 + \sqrt{3}}} )</td>
</tr>
<tr>
<td>24</td>
<td>48</td>
<td>96</td>
<td>( \sqrt{2 - \sqrt{2 + 2 + \sqrt{3}}} )</td>
<td>48 ( \sqrt{2 - \sqrt{2 + 2 + \sqrt{3}}} )</td>
</tr>
<tr>
<td>48</td>
<td>96</td>
<td>192</td>
<td>( \sqrt{2 - \sqrt{2 + 2 + \sqrt{3}}} )</td>
<td>48 ( \sqrt{2 - \sqrt{2 + 2 + \sqrt{3}}} )</td>
</tr>
</tbody>
</table>

and for the value of \( A_{192} \) he gives the fraction \( 3.14 \frac{64}{625} \), which is equivalent to \( \pi = 3.141024 \).
Both Zu Chongzhi (429-500 AD) and his son Zu Geng were first-rate mathematicians. We have encountered the name of the father in the previous chapter when we mentioned his excellent approximation of $\pi$, $\frac{355}{113}$, $\pi = 3.141592920$.

As his predecessor, he was fully aware that this was not exact, he said, **Close ratio: diameter 113, circumference 355.**

We do not know exactly how he arrived at this wonderful fraction, but it probably involved some geometric division of the circle, similar work to that done by Liu Hui.

When he was 33 years old, he introduced an improvement to the Chinese calendar. Unfortunately, due to political considerations in the court, there was reluctance to accept it. It was, eventually, adopted after his death due to the insistence of his son.

The contribution of Zu’s that we concentrate on is his computation of the volume of the sphere.

Consider a sphere of radius $r$. From the top hemisphere, take one quadrant (one fourth), so we have $\frac{1}{8}$ of the sphere, and this sector can be inscribed in a cube with side $r$.

We are going to use two cylinders to cut this cube. In fact, in the picture on the left, think of two cylinders, one on each of the sides of the cube, cutting the cube into four pieces.

With a little intuition, we can see the pieces:

and the last and most important piece which Zu called the part umbrella:
First we will compute the volume of the three small pieces.

In order to do so, we will use a principle that Zu stated as follows:

Matching membranes, same stature,  
then the volume cannot be different.

Meaning that if two objects have the same height (stature), and their corresponding slices have the same area (membrane), then the two objects have the same volume.

Consider then the cube sliced at a point \( P \), as in the picture below. We know that \( OB = AB = BC = r \), the radius of the sphere.

By the Pythagorean Theorem, 
\[ r^2 = h^2 + (AP)^2 \]
where \( h \) denotes the height \( BP \).

But then what are the areas of the shaded regions? \( \square \) = \( (r - AP)^2 \), so together, they amount to 
\[ 2rAP - 2AP^2 + r^2 - 2rAP + AP^2 = h^2. \]

Since the inverted pyramid of height \( r \) with a square base of side \( r \) gives the same area, \( h^2 \), at height \( h \) for any \( h \), the volume of the three pieces is equivalent to \( \frac{1}{3} \) of the volume of the cube. By the way, the ancient Nine Chapters had the volume of the pyramid. Therefore the volume of the part umbrella is \( \frac{4}{3} r^3 \).

Can we compare the volume of the part umbrella to the volume of the sphere? If we take a slice of the part umbrella at any height, we will get a square of variable side depending on the height. Let us refer to the length of the side by \( s \). While if we take a slice of our piece of the sphere, we obtain a quarter of a circle of the same radius as the side of the square obtained in the part umbrella.

Hence the ratio of the slice of the part umbrella to the section of the circle is \( \frac{s^2}{\frac{1}{3} \pi s^2} \), or equivalently, \( \frac{4}{\pi} \). And so 4 times the volume of our segment of the sphere is equal to \( \pi \) times the volume of the part umbrella. But above we saw that the latter equals \( \frac{4}{3} r^3 \). Finally, we need 8 times the volume of our segment for the volume of the sphere and so we arrive at the volume of the sphere is
\[ 2 \times \pi \times \frac{2}{3} r^3 = \frac{4}{3} \pi r^3. \]

An outstanding accomplishment even if it followed 700 years after Archimedes.

Incidentally, Zu’s Principle mentioned above and used to evaluate these volumes is known in the West as Cavalieri’s Principle and we will study it when we reach the Renaissance, approximately a thousand years after Zu lived.

Qin Jiushao

Ever since Master Sun’s Mathematical Manual from the third century, there has been interest in China in what is commonly known today as the Chinese Remainder Theorem. Indeed, in that classic, the following problem occurs:

There are an unknown number of things. Three by three, two remain; five by five, three remain; seven by seven, two remain. How many things?

What it means is that if we count the collection of objects by 3’s, we will have a remainder of 2, while if we count them by 5’s, we will have 3 left over, and finally, if we count them by 7’s, the remains will be 2. How many objects are there? There is an intuitive, temporal way to visualize the question. Consider a clock with only three markers on its face, 0, 1 and 2, and every second (or minute, or whatever unit of time one chooses), the clock moves from one of the marks to the next one, so the clock ticks 0→1→2→0→1→2→0→⋯.

We appeal to our intuition to appreciate that counting a collection of objects by 3’s and examining the remainder is equivalent to counting time in that clock, and seeing what time the clock indicates. For example, suppose we have 25 objects. Then counting by 3’s, we will have exactly 1 object left over. This is the same as asking, suppose 25 units of time go by in a 3-tick clock (always starting at 0). What time is the clock point at? We should be ready to answer 1, since 25 = 8×3 + 1.

We use a modern notation to indicate this. This brilliant notation is due to none other than Gauss. We say \( 25 \equiv 1 \mod 3 \), which reads 25 is congruent to 1, modulo 3.

Suppose that now we have 2 clocks: A 3-tick clock and a 5-tick clock. And they tick at the same time, both starting at 0. Then we
can see how time is measured:

<table>
<thead>
<tr>
<th># of units</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-tick clock shows</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>5-tick clock shows</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

After 15 seconds, both clocks show 0, and the pattern starts again. So, after 12 seconds, for example, the 3—tick clock shows 0 (since there is no remainder when we count 12 by 3’s), and the 5—tick clock shows 2 (since we have a remainder of 2). Conversely, suppose that some one said, we have the remainders 2 and 1 in the 3—tick and 5—tick clock respectively. What time could it be, or equivalently, how many objects could we have started with? Using the table, we can see that 11 is the only time in which those remainders occur, so we either started with 11 objects, or 26 objects, or 41 objects, etcetera.

Note that every possible pair of markings in the clocks occurs exactly once in the table. Why is this true? Trivially, when does the 3-tick clock go back to 0—every multiple of 3. When does the 5-tick clock go back to 0—every multiple of 5. So when will the process restart? At the least common multiple of 3 and 5, 15, and since there are 15 possible pairs, every pair will occur exactly once!

What Master Sun was then asking can be interpreted as follows, suppose we have 3 different clocks, a 3—tick, a 5—tick and a 7—tick, what time is it? Using the notation of Gauss, we are asking for a simultaneous solution to

\[
\begin{align*}
x & \equiv 2 \pmod{3} \\
x & \equiv 3 \pmod{5} \\
x & \equiv 2 \pmod{7}
\end{align*}
\]

We could clumsily build the same type of table as we did for the two clocks, but, rather, we will develop some mathematics to solve the problem. Although Master Sun supplies an answer to his problem, the first time the theory is fully developed is in the book Mathematical Treaty in Nine Sections (1247) by Qin Jiushao.

As we mentioned above, the thirteenth century was a very productive period for Chinese mathematics, and Qin Jiushao is one of several outstanding mathematicians of this period. He developed a general procedure for solving problems such as the one above. He recognized correctly that the key step is being able to solve congruences of the type:

\[ax \equiv 1 \pmod{n}\]

and he called the method for solving such a congruence the method of finding one by the great extension. Indeed, interest in such a congruence may go back as far as the I Ching, Book of Changes, where one is required for divination purposes to divide a number of stalks minus 1 into two groups signifying the Yin and the Yang, the harmonious dual elements of Chinese philosophy. ☯

Fortunately, we already have the means to solve such a congruence. For example, let us consider Qin’s own example (see below),
2970x \equiv 1 \mod 83.

All we have to do then is the extended Euclidean algorithm (Kuttaka) we studied in the Chapter in India. Namely the extension used to write the g.c.d. as a linear combination of the components. Doing our table

Where we do not bother to compute the last column because at the end we will have a number \( y \) such that

\[ 23 \times 2970 + y \times 83 = 1, \]

but since 83 is the same as 0 mod 83, we have that

and we have solved the congruence.

Returning to Master Sun’s Problem:

\[ x \equiv 2 \mod 3 \quad x \equiv 3 \mod 5 \quad x \equiv 2 \mod 7 \]

as the first step we have to solve three congruences of the type listed above. Observe first that 105 is an important number to this problem, since 105 is the least common multiple of all three clocks, 3, 5 and 7. Hence, the best information we can gather about this problem will be in 105-tick clock.

The congruences we have to solve are then:

\[ 35x \equiv 1 \mod 3 \quad 21x \equiv 1 \mod 5 \quad 15x \equiv 1 \mod 7 \]

It is clear where the moduli or clocks, 3, 5 and 7 stem from. Also the 1’s are clear. But what about the coefficients? \( 35 = \frac{105}{3}, 21 = \frac{105}{5} \) and \( 15 = \frac{105}{7} \).

Since \( 35 \equiv 2 \mod 3 \), the first congruence is equivalent to \( 2x \equiv 1 \mod 3 \), and easily we can solve \( x = 2 \)—remember there are only 3 possibilities, since there are only 3 markings in the clock.

Since \( 21 \equiv 1 \mod 5 \), the second congruence is immediately solved, \( x = 1 \).

Similarly, since \( 15 \equiv 1 \mod 7 \), \( x = 1 \) is a solution to the third congruence. To solve the system now, we use the constants of each of them. We take the solution to the first congruence
And since \( 233 \equiv 23 \mod 105 \), our eventual solution is 23 objects. Why this is a solution in general is clear. Modulo 3, the product of the first two elements in the first row is 1, and then you are multiplying by what is needed, so we have that 140 is 2, without any effort. But the other two summands are 0, since the 3 has not been canceled out of the first column, so the sum of all the members of the last column is the same as the first entry in a 3-tick clock. Similarly, modulo 5, the first and the third entries in the last column are 0, and the second one is 3, and a similar claim can be made modulo 7.

We end our discussion of Qin with the solution to a problem he actually gave:

*Three farmers start with the same amount of rice measured in dous (a dry measure for rice). But they go to different markets to sell their crop. In one market they buy in multiples of 83 dou, in another market, in multiples of 110 dou, and in the third market they use multiples of 135 dou. The first farmer has a remainder of 32 dou while the second and third have remainders of 70 and 30 respectively. How much rice did each farmer have initially.*

Our query is equivalent to the three congruences

\[
x \equiv 32 \mod 83; \quad x \equiv 70 \mod 110; \quad \text{and} \quad x \equiv 30 \mod 135
\]

which Qin showed was equivalent to the system

\[
x \equiv 32 \mod 83; \quad x \equiv 70 \mod 110; \quad \text{and} \quad x \equiv 30 \mod 27.
\]

The least common multiple is 246,510. Qin Jiushao then requires the solutions to three separate congruences:

\[
2970x \equiv 1 \mod 83 \quad \text{2241}x \equiv 1 \mod 110 \quad \text{9130}x \equiv 1 \mod 27.
\]

This is the method of finding one by the great extension. By kuttaka (see table), we respectively get 23, 51 and 7.

Recall that the coefficients 2970, 2241 and 9130 come from the l.c.m. 246510 divided by

<table>
<thead>
<tr>
<th>l.c.m./mod</th>
<th>Solution to the Single Congruence</th>
<th>Constant Term in the System</th>
<th>Product of Row</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Congruence</td>
<td>35</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Second Congruence</td>
<td>21</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Third Congruence</td>
<td>15</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Solution is the sum</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
each of the moduli: $2970 = \frac{246510}{83}$ for example. To solve the system now, Qin built a table:

<table>
<thead>
<tr>
<th>mod</th>
<th>lcm=246510</th>
<th>I.c.m./mod</th>
<th>Inverse</th>
<th>Constant Term</th>
<th>Product of Row</th>
</tr>
</thead>
<tbody>
<tr>
<td>83</td>
<td>First Congruence</td>
<td>2970</td>
<td>23</td>
<td>32</td>
<td>2185920</td>
</tr>
<tr>
<td>110</td>
<td>Second Congruence</td>
<td>2241</td>
<td>51</td>
<td>70</td>
<td>8000370</td>
</tr>
<tr>
<td>27</td>
<td>Third Congruence</td>
<td>9130</td>
<td>7</td>
<td>30</td>
<td>1917300</td>
</tr>
</tbody>
</table>

Solution is the sum of last column 12103590

And since $12103590 \equiv 24600 \mod 246510$, we can safely claim that each farmer has grown 24,600 dou of rice.

Why Qin’s method works is clear (as before): modulo 83, the product of the first two elements in the first row is 1, and then you are multiplying by what is needed, so we readily have that 2185920 is 32, without any effort. But the other two summands are 0, since 83 is a factor and has not been canceled out of the first column, so the sum of all the members of the last column is the same as the first entry in a $83 – tick$ clock. Similarly, modulo 110, the first and the third entries in the last column are 0, and the second one is 70, and a similar claim can be made modulo 27.
Chapter 11
The Middle Ages

In mathematics as in other fields, to find oneself lost in wonder at some manifestation is frequently the half of a new discovery—Dirichlet

As we prepare to enter the age of European Hegemony, an age which eventually gives rise to our own present day culture, we enter it at the late Middle Ages. We shall remain in Western Europe for the rest of our journey since we will end it in the second half of the seventeenth century.

However, one of the major ignition sparks that got Western Europe bustling, mathematically speaking, was extensive contact with the Muslim world—as we have seen before. The Muslim mathematicians, who had not only translated many of the Greek works, had, in addition, extensive contact with other foreign cultures and ideas, such as the Hindus and the Chinese. The contact between Islam and the West came in various forms, some peaceful, some not. Some, as the fall of Toledo, mentioned in a previous chapter, was not peaceful, and also played the role of a major injection of classical works to the West. But, much was accomplished through simple commerce too, and, in fact, the first of the two mathematicians of this period that we will discuss profited by being a merchant, and a son of merchants. The two mathematicians of the Middle Ages that we will discuss are: Fibonacci and Oresme.

European History Highlights

<table>
<thead>
<tr>
<th>Date</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>800</td>
<td>Charlemagne crowned first Holy Roman Emperor.</td>
</tr>
<tr>
<td>878</td>
<td>Arabs conquer Sicily—Palermo becomes the capital.</td>
</tr>
<tr>
<td>1000</td>
<td>The Viking Leif Eriksson lands in North America.</td>
</tr>
<tr>
<td>1085</td>
<td>Toledo taken from the Arabs by Alfonso VI.</td>
</tr>
<tr>
<td>1096</td>
<td>First Crusade.</td>
</tr>
<tr>
<td>1119</td>
<td>University of Bologna is founded.</td>
</tr>
<tr>
<td>1150</td>
<td>University of Paris is founded.</td>
</tr>
<tr>
<td>1167</td>
<td>Oxford University is founded.</td>
</tr>
<tr>
<td>1249</td>
<td>King Louis IX of France leads the Seventh Crusade.</td>
</tr>
<tr>
<td>1272</td>
<td>Marco Polo journeys to China. He returns to Italy in 1295.</td>
</tr>
</tbody>
</table>
Leonardo of Pisa, better known as Fibonacci (1175-1250), was a merchant and a son of merchants. He is a major figure in bringing Hindu-Arabic numerals to the West and his book the Liber Abaci (the Book of the Abacus) plays a major role in this adoption. Although the adoption of these numerals is a bit slow at the beginning, it eventually—within 100 years—catches fire, and constitutes a tremendous improvement, not just in the business and accounting world, but also in all forms of mathematical activity.

Fibonacci is also famous for the sequence named after him. The sequence originates from a problem in the Liber Abaci.

How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on?

Let us count the pairs with the aid of pictures:
The first month we start with a young rabbit pair

which after one month will grow into

At the third month, we have the adult pair we had, but also a small pair; their offspring since every pair of mature rabbits will give birth to a new pair every consequent month.

By the fourth month, we have but also , since the young pair has grown by now.
Hence by the fifth month, we have as usual, and but also.

Things are beginning to sizzle by the sixth month in which we have, and

In the following month, the seventh, we are beginning to have serious problems: and

Remember, the idea is that every time we have a mature pair one month, the following month will have not only that mature pair but also its offspring, which in turn after one month will grow into a mature pair.

The eighth month finds us with a lot of rabbits:
And the moment has come to start counting in the abstract.

Let \( Y_1, Y_2, Y_3, Y_4, \ldots \) denote the number of young pairs at month #1, month #2, month #3, etcetera. Hence \( Y_1 = 1, \ Y_2 = 0, \ Y_3 = 1, \ Y_4 = 1, \ \ldots \) and let \( O_1, O_2, O_3, O_4, \ldots \) denote the number of old pairs (at the same times), so \( O_1 = 0, \ O_2 = 1, \ O_3 = 1, \ O_4 = 2, \ \ldots \). Also, if we let \( F_n \) denote the total number of pairs at \( n \) months, then \( F_n = Y_n + O_n \).

By the reproductive rules, we have that for every month \( n \),
\[
O_{n+1} = O_n + Y_n = F_n \quad \text{and} \quad Y_{n+1} = O_n.
\]

Hence,
\[
F_{n+1} = O_{n+1} + Y_{n+1} = O_n + Y_n + O_n = F_n + F_{n-1}.
\]

This last equality, namely,
\[
F_{n+1} = F_n + F_{n-1},
\]
is what is called a recursion, and the Fibonacci sequence: \( F_1, F_2, F_3, \ldots \) is one of the oldest examples of a recursive array, in this case, a one-dimensional array, a sequence.

An array of numbers is said to be recursive if at any stage of the development of the array, the present depends, by some understood relation, from the past. When we look into Pascal's world, we will discuss a very important two-dimensional array.

Once we understand the recursive nature of the sequence we can write as many terms as we wish:

<table>
<thead>
<tr>
<th>Month</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pairs</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
<td>55</td>
<td>89</td>
<td>144</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Month</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pairs</td>
<td>233</td>
<td>377</td>
<td>610</td>
<td>987</td>
<td>1597</td>
<td>2584</td>
<td>4181</td>
<td>6765</td>
<td>10946</td>
<td>17711</td>
<td>28657</td>
<td>46368</td>
</tr>
</tbody>
</table>

Thus, the answer to Fibonacci's original problem was: \( F_{12} = 144 \).

Recursion is a powerful tool for various reasons. Among them are:

- Recursive statements lend themselves to be proven by using recursive and inductive techniques.
- Machines will understand recursive instructions very easily, so that programs for generating such arrays are readily available. For example, it takes very little energy to generate the Fibonacci sequence on any spreadsheet software.
- Recursive thought can be very helpful in tackling problems in general, and we will exemplify that in a future chapter when we study Pascal.
- Recursion seems to be a major force in nature, and many complex organisms follow a simple recursive principle for generation.

For example, the Fibonacci sequence itself occurs in nature as in the number of petals in a flower, or in the arrangement of the cells of a pineapple, and actually although rabbits do not quite breed à la Fibonacci, the ancestry of a drone does obey Fibonacci rules. As it
turns out, the egg of a bee if not fertilized by a male, will automatically produce a male
offspring, while anytime it is fertilized will produce a female offspring. Bluntly put, a male only has one parent, a mother; while a female always has two parents, a mother and a father. Starting with a male
(a drone), let's look at its ancestry:

So that at each level the number of ancestors equals: 1, 1, 2, 3, 5, 8, 13,
etcetera.

Suppose now we wanted to count the complete family of a drone up to a given level. So at level one, we have only 1, at level 2 we have 2, at level 3, 4, at level 4, 7, etceteras. So what we are doing is adding the first consecutive terms of the Fibonacci sequence.

More formally, we are generating a new sequence: \( A_1, A_2, A_3, \ldots \) where \( A_1 = F_1, A_2 = F_1 + F_2, A_3 = F_1 + F_2 + F_3, \ldots \)—which means, for all \( n \), \( A_n = F_1 + F_2 + \ldots + F_n \). This is the definition of the \( A \) sequence, we can always go back to this. This way of generating a new sequence from an old sequence by adding the first terms is very important (we saw it before when we discussed the Pythagoreans) and when we discuss Leibniz, the idea will become very important—but it should be observed that this is the reverse process to taking the difference of a sequence.

\[
\begin{array}{ccccccccccccccc}
& n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
F \text{ sequence} & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 89 & 144 & 233 & 377 & 610 \\
A \text{ sequence} & 1 & 2 & 4 & 7 & 12 & 20 & 33 & 54 & 88 & 143 & 232 & 376 & 609 & 986 & 1596 \\
\end{array}
\]

Let us see if we can relate the new sequence of \( A \)’s to the \( F \)’s. What is the proper claim then: \( A_1 = 1, A_2 = 2, A_4 = 4, A_7 = 7, A_5 = 12 \)? Can we claim that \( A_n = F_{n+2} - 1 \)?

It certainly seems reasonable. Can we prove it? YES How do we prove it? By induction.

It is true for 1: \( A_1 = 1 = F_3 - 1 \). What about if it is true for \( n \), is it true for \( n + 1 \)? Suppose \( A_n = F_{n+2} - 1 \). Then

\[
A_{n+1} = F_1 + F_2 + \ldots + F_n + F_{n+1} = (F_1 + F_2 + \ldots + F_n) + F_{n+1} = A_n + F_{n+1} = F_{n+2} - 1 + F_{n+1} = F_{n+3} - 1.
\]

So if the statement is true for \( n \), it follows that it is true for \( n + 1 \). Since it was true for 1, then it is true for 2, and so it is true for 3, and hence true for 4, etcetera (which means it is true for all \( n \)).
We finish our discussion of Fibonacci with an observation he made that excited him with joy. But before we do we need to remind ourselves that a segment is broken into the golden mean if the small part is to the large part as the large part is to the whole. This was one of the classic constructions from Greek times, and one that had been used extensively in architecture and painting.

Suppose now the whole segment measures 1. Let $x$ denote the large part, so the small part measures $1-x$. To be the golden mean signifies that $\frac{1-x}{x} = \frac{x}{1}$, and thus $x^2 = 1 - x$, and solving for $x$, we get that $x = \frac{-1 + \sqrt{5}}{2} \approx 0.61803398875$. Now Fibonacci looked at the consecutive ratios of his sequence: $1, 1, 2, 3, 5, 8, 13, \ldots$, and he was thrilled to find that these consecutive ratios got closer and closer to the golden mean, as the list of the first twenty values on the right hand side shows.

As we mentioned in a previous section, Fibonacci always used hexadecimal notation for his approximations.

\[\begin{array}{c|c}
1.00000000 & 0.50000000 \\
0.66666667 & 0.60000000 \\
0.62500000 & 0.61538461 \\
0.61904762 & 0.61764706 \\
0.61818182 & 0.61818181 \\
0.61803381 & 0.61803381 \\
0.61803405 & 0.61803405 \\
0.61803399 & 0.61803399 \\
\end{array}\]

In Nicole Oresme (1323-1382), we find a late Middle Ages French clergyman—he eventually became a bishop—who also contributed to mathematics. We are going to look at his proof of the divergence of the harmonic series. But first we look briefly at series in general. One of the outcomes of the new worldviews acquired during the Middle Ages was a much more ready acceptance of the infinite and infinite processes. In particular, infinite series are going to start being evaluated early in the 1400's using an intuitive meaning for limit. The more rigorous ideas of limit and convergence do not get clarified until the late eighteenth or early nineteenth centuries, which is a long time afterwards.

The most fundamental of all infinite series is the geometric series. The Greeks, and Archimedes, in particular, had used it extensively. Given a number $x$, its geometric series consists of the sum of its powers, the powers of $x$:

\[1 + x + x^2 + x^3 + \cdots.\]

Sometimes, one may see it start with $x$: $x + x^2 + x^3 + \cdots$. They are essentially equivalent expressions, since one can be evaluated exactly when the other one can. Perhaps the
oldest incident of the sum of the powers of a number is that of the powers of \( \frac{1}{2} \):

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots
\]

which equals 1. We can argue this fact a couple of ways. First analytically, let us consider the partial sums:

\[
\begin{array}{c}
1 & 3 & 7 & 15 & 31 \\
2 & 4 & 8 & 16 & 32 \\
\end{array}
\]

which we readily see to be of the form:

\[1 - \frac{1}{2^n},\]

and thus, as \( n \) grows without bound we have the limit going to 1.

But we can also visualize the process geometrically since at each stage we are adding one half of what is left over:

If we approach the general geometric series: \( 1 + x + x^2 + x^3 + \cdots \) in the same way by considering the partial sums, we then have that

\[1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}.\]

We can prove this again by induction. For \( n = 1 \), \( 1 + x = \frac{1-x^2}{1-x} \) is well-known. Note we could have started the induction at \( n = 0 \) also. Now suppose it holds for \( n \). We need to show that

\[1 + x + x^2 + \cdots + x^n + x^{n+1} = \frac{1-x^{n+2}}{1-x}.\]

But

\[1 + x + x^2 + \cdots + x^n + x^{n+1} = \left(1 + x + x^2 + \cdots + x^n\right) + x^{n+1} = \]

\[\frac{1-x^{n+1}}{1-x} + x^{n+1} = \frac{1-x^{n+1} + x^{n+1} - x^{n+2}}{1-x} = \frac{1-x^{n+2}}{1-x}.\]

Hence if the powers of \( x \) (and this happens exactly when \( x \) is between \(-1\) and 1) are vanishing as we take them higher and higher, the series will converge to \( \frac{1}{1-x} \), and thus we get the fundamental fact that if \( |x| < 1 \), then the geometric series converges and it converges to \( \frac{1}{1-x} \):

\[1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.\]

There is another way to view this important equation. The method is older, and does not care about convergence; it is purely symbolic, disregarding what \( x \) stands for. It is relevant because it is dominant point of view during much of the 18\textsuperscript{th} Century. Namely,
the equation

\[ 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \]

is true because when we multiply both sides by \(1-x\) we get equality:

\[ (1-x)(1 + x + x^2 + x^3 + \cdots) = 1 \]

After the geometric series, perhaps the most important series is the **harmonic series**:

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots, \]

which is the one Oresme was interested in.

The partial sums of the reciprocals of the positive integers are often called the **harmonic numbers**. The reason for that name is not easily traceable, but probably is old and goes back to Pythagorean connections between numbers and music.

The fact that this series diverges is also important. What do we mean by the divergence of this series? The idea is simple. If we keep adding terms we will get unboundedly big, indefinitely large, **we will exceed any number we wish**. Why does it diverge? After all the terms are getting very small—true, but they are not decreasing fast enough. Here is the reason.

Consider:

\[ \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \cdots \]

and let us group the terms as follows (and this is Oresme's original good idea):

\[ \left( \frac{1}{1} \right), \left( \frac{1}{2} \right), \left( \frac{1}{3} + \frac{1}{4} \right), \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right), \left( \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} \right), \]

\[ \left( \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20} + \frac{1}{21} + \frac{1}{22} + \frac{1}{23} + \frac{1}{24} + \frac{1}{25} + \frac{1}{26} + \frac{1}{27} + \frac{1}{28} + \frac{1}{29} + \frac{1}{30} + \frac{1}{31} + \frac{1}{32} \right) \]

and so on.

The first two groups: 1 and \(\frac{1}{2}\) are easily added. What about the remaining ones? We have:

\[ \frac{1}{3} + \frac{1}{4} > \frac{1}{2} \]

since \(\frac{1}{3} \geq \frac{1}{4}\) and \(\frac{1}{4} + \frac{1}{4} = \frac{1}{2}\);

\[ \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{2} \]

since each term is bigger than \(\frac{1}{8}\) and there are 4 terms;
furthermore
\[ \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} > \frac{1}{2} \]
since as before each term is bigger than \( \frac{1}{16} \) and there are 8 terms. By now the logic is clear: the next group also adds to something bigger than a half—16 terms all bigger than \( \frac{1}{32} \). And without having to write any more, we see inside our heads the 32 terms of the next group all bigger than \( \frac{1}{64} \), and the 64 terms all bigger than \( \frac{1}{128} \), etcetera. Since we are adding an indefinite number of \( \frac{1}{2} \)'s we see we can exceed any number we wish and the series does diverge.

For example, suppose we wanted to exceed 5. Then besides the first 2, we would need 7 groups exceeding \( \frac{1}{2} \), so if we add all the way to the 256\(^{th} \) term, we will exceed 5 for sure. Of course, we are being very generous with our estimates and it may occur sooner, and indeed it does as we can see from our table below.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( H_n )</th>
<th>( n )</th>
<th>( H_n )</th>
<th>( n )</th>
<th>( H_n )</th>
<th>( n )</th>
<th>( H_n )</th>
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</table>
We also give a more formal proof to help the reader get acquainted with formal mathematics reading and writing.

**Theorem.** The harmonic series diverges.

*Proof.* We start by letting $H_n$ denote the partial sum of the first $n$ terms of the harmonic series. Thus, $H_1 = 1$, $H_2 = 1.5$, $H_3 = \frac{7}{4}$, etc., (see table above). We need to show that the $H$'s grow without bound. Clearly, $H_1 \leq H_2 \leq H_3 \leq H_4 \leq \cdots$. So if we can show that some subsequence increases without bound we will be done. As above we will concentrate on the subsequence with indices that are powers of 2:

$$H_1, H_2, H_4, H_8, H_{16}, \ldots$$

We claim by induction that $H_{2^n} \geq 1 + \frac{n}{2}$. This is certainly true for both $n = 0$ and $n = 1$.

Assume it holds for $n$. That is, assume $H_{2^n} \geq 1 + \frac{n}{2}$. We need to prove that it holds for $n + 1$, or equivalently that $H_{2^{n+1}} \geq 1 + \frac{n+1}{2}$. But

$$H_{2^{n+1}} = H_{2^n} + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^{n+1}-1} + \frac{1}{2^{n+1}}.$$ 

There are $2^n$ terms each exceeding $\frac{1}{2^{n+1}}$ hence their sum exceeds $\frac{1}{2}$. Thus we have that

$$H_{2^{n+1}} \geq H_{2^n} + \frac{1}{2} \geq 1 + \frac{n}{2} + \frac{1}{2} = 1 + \frac{n+1}{2},$$

which is what we wanted.

Before we give yet another argument of the divergence of the harmonic series from a few centuries after Oresme, we need an easy fact:

**Lemma.** For any positive integer $n$, $\frac{1}{n-1} + \frac{1}{n+1} > \frac{2}{n}$.

*Proof.* Adding the left hand side, we get $\frac{2n}{n^2-1} > \frac{2n}{n^2}$ since the denominator is smaller, but by canceling the $n$, we get our Lemma.

Note that by the lemma then the sum of three consecutive reciprocals is greater than 3 times the middle reciprocal:

$$\frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1} > \frac{3}{n}.$$ 

For example, if $n = 7$, then $\frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{3}{7}$, since the left-hand side is $\frac{73}{168} > \frac{72}{168} = \frac{3}{7}$. 
A Different Argument for the Divergence of the Harmonic Series

Suppose we assume the series converges and suppose we let $S$ denote the total sum. We know then that

$$S = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \cdots$$

$$= \frac{1}{1} + \frac{1}{2} \left( \frac{1}{3} + \frac{1}{4} \right) + \frac{1}{5} \left( \frac{1}{6} + \frac{1}{7} \right) + \frac{1}{8} \left( \frac{1}{9} + \frac{1}{10} \right) + \frac{1}{11} \left( \frac{1}{12} + \frac{1}{13} \right) + \frac{1}{14} \left( \frac{1}{15} + \frac{1}{16} \right) + \cdots$$

$$> 1 + \frac{3}{3} + \frac{3}{9} + \frac{3}{12} + \frac{3}{15} + \cdots = 1 + \frac{1}{1} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = 1 + S$$

which is certainly nonsensical.

Very elegant indeed!

We know then that if we add the reciprocals of all counting numbers we get an unbounded sum. But what happens if we add only some subcollection of them? For example we could consider adding only the following subcollections:

<table>
<thead>
<tr>
<th>Type of Numbers</th>
<th>Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>the squares</td>
<td>$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots$</td>
</tr>
<tr>
<td>the primes</td>
<td>$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \cdots$</td>
</tr>
<tr>
<td>the triangulars</td>
<td>$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \frac{1}{28} + \cdots$</td>
</tr>
<tr>
<td>the cubes</td>
<td>$\frac{1}{1} + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \frac{1}{216} + \frac{1}{343} + \cdots$</td>
</tr>
<tr>
<td>the fourth-powers</td>
<td>$\frac{1}{1} + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \frac{1}{625} + \frac{1}{1296} + \cdots$</td>
</tr>
</tbody>
</table>

All of these and more have played a meaningful role in the history of mathematics, especially in the 18th Century. The known results are as follow:

<table>
<thead>
<tr>
<th>Type</th>
<th>Convergence Status</th>
<th>Who did it</th>
<th>When</th>
</tr>
</thead>
<tbody>
<tr>
<td>the squares</td>
<td>Converges to $\frac{\pi^2}{6}$</td>
<td>Euler</td>
<td>1700's</td>
</tr>
<tr>
<td>the primes</td>
<td>Diverges</td>
<td>Euler</td>
<td>1700's</td>
</tr>
<tr>
<td>the triangulars</td>
<td>Converges to 2</td>
<td>Leibniz &amp; Others</td>
<td>1600's</td>
</tr>
<tr>
<td>the cubes</td>
<td>Converges to an Unknown Irrational</td>
<td>That it converges is old. To a proven irrational was done by Apéry.</td>
<td>1980's!</td>
</tr>
<tr>
<td>the fourth-powers</td>
<td>Converges to $\frac{\pi^4}{90}$</td>
<td>Euler</td>
<td>1700's</td>
</tr>
</tbody>
</table>
Chapter 12
The Renaissance

The miraculous powers of modern calculation are due to three inventions: the Arabic Notation, Decimal Fractions and Logarithms—Cajori.

In the early stages of the Renaissance, Italy, due to its privileged geographical, economic and political position, naturally adopts a leadership role in all of human intellectual activities, painting, sculpture, literature, science and mathematics. And, in fact, we start the chapter by discussing two Italian mathematicians: Tartaglia and Cardano.

By the time we enter the second half of the sixteenth century; mathematical activity has spread to all corners of Western Europe. And in fact, the four mathematicians of this latter period that we study are all from different countries. In addition, the proliferation of activity forces us to restrict our study to only major influences. E.g., by the late 1500's, decimal notation is developed in the West, yet we just mention it in passing. We will study two major names in the history of physics and astronomy, one Italian and one German, Galileo and Kepler, an eccentric Lord from Scotland, Napier, and a French courtier, Viète.

European History Highlights

<table>
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<tr>
<th>Date</th>
<th>Event</th>
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<tbody>
<tr>
<td>1494</td>
<td>Luca Pacioli’s Summa is published.</td>
</tr>
<tr>
<td>1501</td>
<td>Michelangelo sculpts the David.</td>
</tr>
<tr>
<td>1503</td>
<td>Leonardo Da Vinci paints La Mona Lisa.</td>
</tr>
<tr>
<td>1517</td>
<td>Martin Luther posts his 95 theses in Wittenberg.</td>
</tr>
<tr>
<td>1519</td>
<td>Cortez enters Tenochtitlan. Horses are brought from Spain to North America. Chocolate is brought from Mexico to Spain.</td>
</tr>
<tr>
<td>1532</td>
<td>Pizarro captures Cuzco, the Inca capital.</td>
</tr>
<tr>
<td>1538</td>
<td>Mercator uses the word America for the first time.</td>
</tr>
<tr>
<td>1543</td>
<td>Copernicus proposes the Sun as the center of the system.</td>
</tr>
<tr>
<td>1559</td>
<td>Elizabeth I becomes Queen of England. She will die in 1603.</td>
</tr>
<tr>
<td>1572</td>
<td>St. Bartholomew’s massacre in Paris—Huguenots are killed in the thousands.</td>
</tr>
<tr>
<td>1582</td>
<td>Gregorian Calendar adopted in Papal States, Spain, Portugal, France, the Netherlands and Scandinavia.</td>
</tr>
<tr>
<td>1600</td>
<td>The telescope is invented.</td>
</tr>
<tr>
<td>1616</td>
<td>Cervantes and Shakespeare die.</td>
</tr>
</tbody>
</table>
But there are at least two areas of mathematical activity to which we are being extremely unfair in this ebullient period in history. First, there is a large group of individuals that would by most not be considered mathematicians, yet they were solving hard and interesting geometrical problems—they were the artists of the Renaissance. Among them we encounter Brunelleschi and Da Vinci—the latter even gave up painting for a while to devote himself exclusively to mathematics, and at one time he was all absorbed by the semiregular solids.

The great improvement in painting from the Middle Ages to the Renaissance was the introduction of perspective, of trying to capture realistically a three-dimensional world into a two-dimensional canvas. It is the study of perspective that was most interesting to the artist, but they were also interested in notions of symmetry. Some of the issues studied then, and later, became important again in the second half of the twentieth century when computer graphics were developed.

We just dabble in one very simple issue, but yet, the example illustrates some of the ability and care of the people involved.

The problem to be addressed is simple: we have a classical black & white tile floor, which from a bird's point of view looks like:

But now the painter is standing at the edge of the floor, and wants to represents what he/she sees onto the canvas.

A little reflection and plain experience will answer some questions easily: it looks like a trapezoid. But, how do we break the trapezoid into the tiles? The vertical split is easy because the painter is standing at the center of the edge, the vertical lines are evenly spaced between the two bases.

But how about the horizontal lines? Are they evenly spaced? If we do that we see at once that description is not quite realistic:

The answer once observed is trivial, but enlightened. Since from the bird's point of view, the diagonal from one corner to the other crosses the corners of the tiles,
that relation does not change as we change views, so we have to make the drawing as to respect that, and hence the horizontal lines are placed automatically. Elegant, isn’t it?

The essential feature of the transformation was then \textbf{that lines on the floor stayed lines in the painting}—the transformations preserved lines.

The other group of people that we are simply ignoring is a large group of writers of mathematics, mainly from Italy and Germany, who developed, very slowly, much of our present day notation of arithmetic and elementary algebra. In particular, by 1500, many of the symbols such as $+,-,=,\times,\div$, and $\sqrt{\cdot}$, have become accepted by a large ensemble of professionals. One of the most influential books is Luca Pacioli’s \textbf{Summa Arithmetica}, published in 1494. However, one should not consider the notation universal; it was not so, nor will it become so for many years. Which symbols came to mean what they do was very haphazard, and while some symbols made sense (= because no two things are as equal as two parallel lines), others did not, they just came to mean what they did.

As we mentioned earlier, we will start by discussing two mathematicians from this first half of the sixteenth century—both Italians, friends, rivals and enemies.

\textbf{Tartaglia}

\textbf{Niccolo of Brescia}, better known as \textbf{Tartaglia} (1500-1557), was nicknamed so because he stammered due to an injury he suffered as a child. He was interested in many aspects of mathematical inquiry including a serious analysis of the path of projectiles such as a cannon ball. But his most famous mathematical achievement, and the one we look into, is \textbf{the solution of the cubic equation}.

As we saw before, the cubic had received considerable attention by many authors in the past including \textbf{Omar Khayyam} and \textbf{Pappus}. The considerable difference in the early part of the sixteenth century is that rudimentary algebra has become quite developed and acceptable, and the method that Tartaglia developed is thus strictly algebraic. Many of the techniques he used were available before, but he put them all together, and he did it under pressure since he was amidst a duel for solving cubics. He eventually humiliated his opponent by solving all the equations his opponent gave him while none of his could be resolved by his opponent since they were of a different type—remember there were many types. Even today we have inherently two types—those with three real roots and those with only one real root.

But we will ignore this and attack the general cubic as of only one type. Let us take

$$x^3 + ax^2 + bx + c = 0$$

as our general cubic. As mentioned above, at the time of Tartaglia and Cardano the
notion of a general coefficient such as the $a$, $b$ and $c$ in our expression was not available yet. Hence when they wrote they would solve a specific cubic at a time, and hope the reader would get the gist of it all.

One standard procedure that preceded Tartaglia is the elimination of the quadratic term by a simple linear substitution. Let $x = y - \frac{a}{3}$, then substituting for $x$, we get

$$\left(y - \frac{a}{3}\right)^3 + a \left(y - \frac{a}{3}\right)^2 + b \left(y - \frac{a}{3}\right) + c = 0.$$ 

Expanding we get

$$y^3 - 3\frac{a}{3}y^2 + 3\frac{a^2}{9}y - \frac{a^3}{27} + ay^2 - 2a\frac{a}{3}y + a\frac{a^2}{9} + by - b\frac{a}{3} + c = 0$$

which after simplifying becomes

$$y^3 + \left(b - a\frac{a^2}{3}\right)y + \left(\frac{2a^3}{27} - \frac{ab}{3} + c\right) = 0,$$

and indeed the quadratic term is eliminated.

If we let

$$p = b - a\frac{a^2}{3}, \quad q = \frac{2a^3}{27} - \frac{ab}{3} + c,$$

then the equation reduces to

$$y^3 + py + q = 0.$$

It is here that we can simplify Khayyam's many types to just 3. In order to have a positive solution, not both of $p$ and $q$ can be positive, so we are left with:

- both $p$ and $q$ negative,
- $p$ positive and $q$ negative, and
- $p$ negative and $q$ positive.

And indeed Tartaglia was the first to be able to solve all three types, which again we treat as just one.

Another idea that was not uncommon at the time was to substitute one unknown by two unknowns and then use another substitution of the new unknowns in order to simplify.

The idea that one equation is worth one unknown is a basic ingredient of algebra which is still valid today, and the fact is that you can trade up or down, namely increase the number of unknowns by adding equations, or eliminate unknowns by reducing the number of equations. In this problem we reach our goal by this method.

Let $y = u - v$. Note that since we only added one equation, but two unknowns, we have the right later to require another relation (equation) between our two unknowns.

When we expand into our original cubic, eliminating the $y$, we have
and, wonderfully, if we let
\[ 3uv = p, \]
or equivalently,
\[ v = \frac{p}{3u}, \]
then our cubic equation becomes
\[ u^3 - v^3 + q = 0 \]
and eliminating \( v \), we get
\[ u^3 - \left( \frac{p}{3u} \right)^3 + q = 0 \]
which after simplification transforms to
\[ u^6 + qu^3 - \frac{p^3}{27} = 0. \]
But this equation is a quadratic on \( u^3 \), and hence we can get a solution for it:
\[ u^3 = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2} \]
and so
\[ v^3 = u^3 + q = \frac{q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2} \]
and thus if we take cube roots, we get \( u \) and \( v \) and so
\[ y = u - v = \sqrt[3]{\frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}} - \sqrt[3]{\frac{q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}} \]

and from there we easily get a value for \( x \). And thus we have a formula that solves the original cubic equation. **Caution:** the ± has to be interpreted as follows, if one uses the + in one radical, one must use the + in the other.

There are, unfortunately, many problems with this formula. Some will be pointed out next when we discuss Cardano—who himself pointed out some of the difficulties, but we will also return to this theme when we discuss Viète below.

Let us exemplify the procedure with a perfect example:
\[ x^3 - 3x^2 - 3x - 4 = 0. \]
We start by letting \( x = y + 1 \) since \( a = -3 \).

And we obtain the equation:
\( y^3 - 6y - 9 = 0 \)

hence \( p = -6 \) and \( q = -9 \). We now substitute into the formula

\[
y = \sqrt[3]{-q \pm \sqrt{q^2 - \frac{4p^3}{27}}} - \sqrt[3]{p \pm \sqrt{p^2 - \frac{4q^3}{27}}}
\]

so

\[
y = \sqrt[3]{\frac{9 \pm \sqrt{81 - 32}}{2}} - \sqrt[3]{\frac{-9 \pm \sqrt{81 - 32}}{2}} = \sqrt[3]{\frac{9 \pm 7}{2}} - \sqrt[3]{\frac{-9 \pm 7}{2}}
\]

So should one add or subtract the 7 inside the cube root? As it turns out, it does not matter. If we add, we get \( y = \sqrt[3]{\frac{9 + 7}{2}} - \sqrt[3]{\frac{-9 + 7}{2}} = \sqrt[3]{8} - \sqrt[3]{1} = 3 \) while if we subtract we obtain \( y = \sqrt[3]{\frac{9 - 7}{2}} - \sqrt[3]{\frac{-9 - 7}{2}} = \sqrt[3]{1} - \sqrt[3]{-8} = 3 \). It is this kind of symmetric that would be exploited centuries later to more deeply understand the solution. Thus we get that in either case, \( x = 4 \).

And indeed we can verify readily that 4 is a solution to our equation. At this place in history they are simply interested in finding one solution to the equation and it is not even clear that there are three solutions to any cubic since complex numbers are yet to be discussed; still they are just around the corner.

It was Tartaglia that solved the cubic, but the first writer to expose the solution to the world, and perhaps the first writer to fully understand it is our next character.

**Cardano**

Perhaps, Girolamo Cardano (1501-1576) is the most colorful character in the history of mathematics. He was a doctor by profession, an astrologer (this is not surprising since horoscopes and medicine were closely associated at the time), a mathematician and an inveterate gambler—it is possible that he gambled at least eight hours a day for most of his life. Undoubtedly, it was his gambling interests that led him to write one of the first treatises on probability, the *Liber di Ludo Alae* (Book of the Games of Dice), in which he clearly enunciates that the probability of rolling a 6 with a die is \( \frac{1}{6} \) since there are 6 equally feasible outcomes to rolling a die. It is surprising that no written claims on probability occurred earlier since mankind has been gambling for an extremely long period of time—certainly since pre-history. As soon as he enunciates that simple probability principle, he makes the first error on a subject plagued by errors throughout history. He incorrectly states that the probability of rolling at least one six in two rolls is \( \frac{1}{3} \), instead of the correct estimate: \( \frac{11}{36} \).
But his most famous work is the *Ars Magna* (Magnificent Art), where *Art* refers to *Algebra*. It is here that Cardano exposes the solution of the cubic. It is told by many authors that Cardano deceived Tartaglia into giving him the solution with the promise that he would not divulge it. However, Cardano has his defenders who would argue that Tartaglia did not fully understand the solution, and that it was Cardano who after being given sketches by Tartaglia, could fully explain it. And as a matter of fact, Cardano was careful to give Tartaglia credit for the cubic, and also gave credit to his student Ferrari for the solution of the *quartic* (fourth degree) equation which is included in the book. We do not at present dwell on the solution of the quartic—which follows similar techniques to those already presented for the cubic. But we should remark that in order to solve a quartic, a cubic is developed which when solved helps solve the original quartic.

But Cardano also pointed out problems with the methods outlined above, and one of his examples is the cubic:

\[ x^3 + 6x = 20. \]

Since the quadratic term is already missing, we have \( p = 6 \) and \( q = -20 \). Thus, substituting into the formula, we get

\[
x = \sqrt[3]{\frac{20 + \sqrt{400 + 32}}{2}} - \sqrt[3]{\frac{-20 + \sqrt{400 + 32}}{2}},
\]

which simplifies to

\[
x = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}.
\]

But as Cardano accurately pointed out, this is *nothing but 2*—test it in your calculator if you have any doubts. Hence, although we get a solution by the formula, our understanding of that solution is very poor.

**Bombelli**, who wrote another excellent algebraic treatise, belonged to the generation after Cardano. He gave an example that exhibited even further problems with the formula. The cubic he considered is the following:

\[ x^3 = 15x + 4. \]

Here \( p = -15 \), \( q = -4 \). Observe first that 4 is in fact a solution of this equation. However, if we apply the formula, we get

\[
x = \sqrt[3]{\frac{4 + \sqrt{16 - 13500}}{2}} - \sqrt[3]{\frac{-4 + \sqrt{16 - 13500}}{2}},
\]

which after simplifying becomes:

\[
x = \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{-2 + \sqrt{-121}}
\]

and although \( x \) is 4, the form in which it is written is remarkable.

Cardano was aggressive enough to consider square roots of negative expressions, however their (*Tartaglia-Cardano*) formula for the solution of the cubic was particularly mystifying when we become confronted with such square roots, for then we are
compelled to consider cube roots of complex numbers, as in the previous example. Shortly after this period, one encounters Viète, who will understand the situation better, and give an alternative to dealing with square roots of negative numbers.

Needless to say, there were similar problems with the quartic equation. Nevertheless, the solutions of the cubic and quartic equations are quite an accomplishment, and they were deservedly admired. Naturally, the next equation to be tackled then was the **quintic equation** (degree five), and that puzzle is going to become quite a trophy in the eighteenth century, until finally in the early part of the nineteenth it is shown to not be solvable in the same fashion—by using **radicals only**.

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**Viète**

From one point of view, François Viète (1540-1603) was an amateur mathematician since his profession was diplomacy at the service of the French king. However, his impact on mathematics, although subtle, has been long lasting.

It is through him that algebra grows in abstraction and power. It is possible that he may have contributed the idea of **parameter**, or constant, in an algebraic expression. Thus, when we write \( ax + by = c \) for the equation of a line we are using \( a \), \( b \) and \( c \) as parameters. Prior to this time in history, people were confined to writing specific equations for lines, or quadratics, or whatever, hoping that by writing enough of them whatever pattern was to be established was successfully communicated. The idea of a parameter shifts the level of abstraction one notch higher. We inherited much of our analytic geometry notation from Descartes; however he was definitely influenced by Viète, as was also Fermat, and so some of the notation of Viète has survived until the present.

Viète was also very interested in the cubic equation. He observed that there was a definite difference in the behavior of the Cardano-Tartaglia method depending on whether the cubic had one root or three roots—or as we would say in modern times, whether the cubic had one real and two complex roots, or whether it had three real roots.

In fact, consider the cubic \( x^3 + px + q = 0 \) where we have already eliminated the quadratic term. The main difficulty from the point of view of the sixteenth century was the extraction of the square root when the formula was to be applied. Hence the difficulty was on whether \( \sqrt{q^2 + \frac{4p^3}{27}} \) was real or not, which is equivalent to whether \( q^2 + \frac{4p^3}{27} \) was positive or not.

Using more modern points of view, a cubic is going to have three real roots when it turns two times, in other words, when the derivative has two real roots, and the cubic polynomial is positive on the one root and negative on the other root of the derivative.
The derivative of our cubic is
\[ 3x^2 + p = 0, \]
so if we let \( \pm \alpha \) be its roots, then we know
\[ \alpha^2 = (-\alpha)^2 = \frac{-p}{3}. \]

Our original polynomial evaluated at either of these two roots of the derivative, gives
\[ \frac{-p\alpha}{3} + p\alpha + q = \frac{2p\alpha}{3} + q \]
and
\[ \frac{p\alpha}{3} - p\alpha + q = -\frac{2p\alpha}{3} + q. \]

To see whether these have different signs, all we need is for their product to be negative. But their product is
\[ q + 2\frac{p\alpha}{3} \left( q - 2\frac{p\alpha}{3} \right) = q^2 - \frac{4p^2\alpha^2}{9} = q^2 + \frac{4p^3}{27}. \]

And thus we have three real roots exactly when the expression
\[ q^2 + \frac{4p^3}{27} \]
is negative.

This quantity \( q^2 + \frac{4p^3}{27} \) is called the \textbf{discriminant} of the cubic, and its significance had already been observed by Khayyam.

Viète decided to take a different approach to this difficult case of the cubic by using \textbf{trigonometric identities}! He knew, as Ptolemy knew, the formula for the cosine of the sum of two angles, in particular he knew that
\[ \cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta). \]

He used this relation to develop a method for tackling the cubics that had three real roots.

Consider the cubic
\[ x^3 + px + q = 0, \]
and suppose it will have three real roots. Thus, in particular, \( p \) \textbf{will be negative}. Let
\[ m = \sqrt{-\frac{4p}{3}} \]
and let
\[ x = m\cos(\theta). \]

When we substitute into the cubic we get
\[ m^3 \cos^3(\theta) + pm\cos(\theta) + q = 0, \]
which simplifies to:
\[
\cos^3(\theta) + \frac{p}{m^2} \cos(\theta) = -\frac{q}{m^3},
\]
or equivalently
\[
\cos^3(\theta) - \frac{3}{4} \cos(\theta) = \frac{3q}{4pm},
\]
which becomes
\[
4 \cos^3(\theta) - 3 \cos(\theta) = \frac{3q}{pm},
\]
which by the trigonometric identity above, simplifies to
\[
\cos(3\theta) = \frac{3q}{pm},
\]
which is easily solved via a set of trigonometric tables. Note that since if \( \phi = 3\theta \) satisfies this equation, then so do \( 360^\circ - \phi \) and \( 360^\circ + \phi \), and so we get three angles for \( \theta \), and thus we also get all three roots of the cubic at once. We illustrate it with two examples.

Consider first the cubic we saw in the Cardano section: \( x^3 = 15x + 4 \). We have \( p = -15, \ q = -4 \). And hence \( m = \sqrt{20} \). Thus,
\[
\cos(3\theta) = \frac{12}{15\sqrt{20}} = .1788854382,
\]
and so
\[
3\theta = 79.695153531234^\circ, \ 360^\circ - 79.695153531234^\circ, \ 360^\circ + 79.695153531234^\circ
\]
thus, (observe that we are working with degrees),
\[
3\theta = 79.695153531234^\circ, \ 280.304846468766^\circ, \ 439.695153531234^\circ
\]
and so
\[
\theta \approx 26.565051177078, \ 93.434948822922, \ 146.565051177078
\]
which leads to three values of \( x = m \cos(\theta) \):
\[
x \approx 4.00000000, \ -0.26794919243112, \ -3.73205080756888.
\]
Similarly, if we take the cubic \( x^3 - 3x + 1 = 0 \), so \( p = -3, q = 1 \). Since \( q^2 + \frac{4p^3}{27} = -3 \), we know that the Cardano-Tartaglia formula will involve complex numbers. Let us apply Viète's method: \( m = 2 \), and \( x = 2 \cos(\theta) \), and then we have \( \cos(3\theta) = -\frac{1}{2} \), and so
\[
3\theta = 120^\circ, \ 240^\circ, \ 480^\circ
\]
\[
\theta = 40^\circ, \ 80^\circ, \ 160^\circ
\]
\[
x \approx 1.53208888623796, \ 0.34729635533386, \ -1.87938524157182
\]
Being involved intensely with polynomials, it is not surprising then that Viète considered what are referred to as Viète's Relations, and which we now examine closely. What is the relation between the coefficients of a quadratic polynomial and its roots?
Consider \( x^2 - ax + b = 0 \) (it will be clear in a minute why we are writing the polynomial with the alternating sign pattern), and suppose \( \lambda_1 \) and \( \lambda_2 \) are its roots. Although not fully clear at this time, the idea of the factorization of a polynomial by its roots is already in the air and just around the corner, and we will adopt a modern perspective. Then we know that
\[
x^2 - ax + b = (x - \lambda_1)(x - \lambda_2),
\]
so if we expand the right-hand side and equate coefficients we get that
\[
a = \lambda_1 + \lambda_2 \quad \text{and} \quad b = \lambda_1\lambda_2.
\]

Hence what is given in a quadratic equation is the **sum of the two roots and the product of the two roots**, and thus, to solve a quadratic equation is tantamount to answering the question: **If I give you the sum of two numbers and the product of two numbers can you find the two numbers?** As a matter of fact, many problems from antiquity were phrased in exactly that manner, and Diophantus was certainly aware of this fact as we saw before.

What happens in a cubic equation: \( x^3 - ax^2 + bx - c = 0 \)? Again if we let \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) be the roots, then by the factorization theorem, when we expand \( (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) \), we get that
\[
a = \lambda_1 + \lambda_2 + \lambda_3, \\
b = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \\
\text{and} \\
c = \lambda_1\lambda_2\lambda_3.
\]

In order to see this readily, let us see the expansion the correct way. What is the product of the three binomials? It consists of all possible terms we obtain when we multiply one term from each of the factors. For example, the only way to obtain \( x^3 \) is then to take \( x \) from each of the multiplicands.

How can we obtain a term with an \( x^2 \)? It must have been that two of the factors contributed an \( x \) while the remaining factor contributed the \(-\lambda\) term, and hence when we collect terms, we get that the coefficient of \( x^2 \) is \(-\lambda_1 + \lambda_2 + \lambda_3\). How do we get an \( x \) term? One of the factors gave the \( x \) while the other two gave \(-\lambda\)'s, and thus we get, when we gather coefficients, the coefficient of \( x \) is the sum of all the products of the \(-\lambda\)'s taken two-at-a-time: \( \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \). Finally, the constant term is the product of the \(-\lambda\)'s, so we get \(-\lambda_1\lambda_2\lambda_3\).

Hopefully, the pattern for the signs is understood.

Let us consider again the example above, \( x^3 - 3x + 1 = 0 \)—note \( a = 0 \), \( b = -3 \) and \( c = -1 \). If we let \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) be 1.53208888624, .347296355334 and \(-1.87938524157\), respectively (the three roots), we then have
\[ 0 = a = \lambda_1 + \lambda_2 + \lambda_3, \]
\[ -3 = b = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 0.532088886239 - 2.87938524157 - 0.652703644666 \]
and
\[ -1 = c = \lambda_1\lambda_2\lambda_3. \]

A little reflection will help explain the quartic:
\[ x^4 - ax^3 + bx^2 - cx + d = 0. \]

If \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) are the roots, then
\( a \) is the sum of the roots:
\[ a = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \]
\( b \) is the sum of the products of the roots taken two-at-a-time:
\[ b = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4, \]
\( c \) is the sum of the products of the roots taken three-at-a-time:
\[ c = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4, \]
and \( d \) is the sum of the products of the roots taken four-at-a-time, of which there is only one term:
\[ d = \lambda_1\lambda_2\lambda_3\lambda_4. \]

This pattern is important, and it is easily seen to extend to the general case of an \( n \)th degree polynomial. Note that in the case of the quartic we needed 6 terms for the computation of \( b \). In general, the number of terms to be computed for a given coefficient is a natural and interesting question, and one that will receive much attention from several points of view. We will return to it in the next chapter.

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**Napier**

During the early part of the seventeenth century, a revolution in computation—logarithms—came about. Their power came from their ability to replace doing a multiplication by doing an addition. The idea was not exactly new. Prosthaphaeresis, which goes back to at least the Arab mathematicians of the eleventh century, was a precursor, using trigonometric identities (remember Ptolemy) such as
\[ \cos(x)\cos(y) = \frac{\cos(x + y) + \cos(x - y)}{2} \]
to effectively substitute doing a multiplication by doing an addition with the aid of cosine tables.

Let us examine how this works. Suppose we have a good table of cosines at our disposal.

And suppose we are to multiply
\[ 0.1654235 \times 0.458369. \]
Then looking up in our cosine tables we find that
\[
\cos(80.4782^\circ) \approx 0.1654235 \quad \text{and} \quad \cos(62.7181^\circ) \approx 0.458369
\]
(the accuracy of course depended in how good our tables were). Using the trigonometric identity, and the tables, we get
\[
0.1654235 \times 0.458369 = \frac{\cos(143.1963^\circ) + \cos(17.7601^\circ)}{2} \approx 0.0758246789385
\]
while the exact answer (using a modern day sophisticated calculator) is .0758250042715, and we have not done anything that is very difficult. Naturally, the more accuracy on the measurement of the angles, the more accuracy in our answers.

**John Napier** (1550-1617), who is credited with the creation of modern logarithms thought of them in terms very different from our own. His description was as follows.

Consider a line segment \( AB \) (Napier used 10 million units for the length of \( AB \)) and an infinite ray \( DE \). At time \( T=0 \), a man and a woman stand at the beginning points \( A \) and \( D \), respectively.

Immediately they both begin moving with the same initial velocity: \( AB \). The woman in ray \( DE \) remains moving with this uniform velocity, while the man in segment \( AB \) always moves with velocity equal to his distance to \( B \). At any time \( t \), the woman is somewhere along the ray \( DE \) while the man is somewhere in the segment \( AB \).

Then Napier lets \( DF \) be the logarithm of \( CB \). As it turns out, this definition is more closely related to the natural logarithm, \( \ln \), than to logarithms base 10—\( \log_{10} \)—to which he switched shortly afterwards with the help of Briggs.

What Napier had in mind was the contrast of something moving at an arithmetic rate (the woman) versus something moving at a geometric rate (the man). And this is the basic fact behind being able to substitute a multiplication by an addition, namely, as we multiply numbers, we add exponents. Archimedes, in the Sand Reckoner, almost two thousand years before, had had the same idea except he had not had the notation to implement it so successfully.

After his definition, Napier proceeded to build a table of logarithms as well as to argue their basic properties. This was all included in his famous book, *Mirifici Logarithmorum Canonis Descritio*, which he published in 1614—three years before his death.
Shortly after the sensational publication of the Mirifici, Briggs, a professor at Oxford traveled north to Scotland, and together they reworked the concept to base 10 logarithms, and closer to the modern high-school definition of logarithms—\(\log_{10} a = x\) means \(10^x = a\).

In modern times, we also define logarithms by areas (or integrals), namely
\[
\ln(x) = \int_1^x \frac{1}{t} \, dt,
\]
which is the area under the hyperbola. The connection between hyperbolic areas and Napier's version was made a generation after Napier (~1647). The crucial observation was the fact that indeed hyperbolic areas satisfy the fundamental property of logarithms:
\[
\ln(xy) = \ln(x) + \ln(y).
\]

In pictures,

Before we give an analytic proof of this theorem, we need the lemma that does most of the work. First a picture of the Lemma:

Next we give a heuristic, geometric argument of why this is so. By this time, the technique of approximating areas by consecutively dividing into rectangles is well understood. And that is exactly the way to compute the area. What happens when we break the interval from 1 to \(y\) into four pieces? We get the four intervals: \(\left[1, \frac{y+3}{4}\right]\),...
\[
\left[ \frac{y+3}{4}, \frac{1}{4} \right], \left[ \frac{2y+2}{4}, \frac{3y+1}{4} \right] \text{ and } \left[ \frac{3y+1}{4}, y \right], \text{ each of width } \frac{y-1}{4}. \]
With each of these intervals we can associate an underestimate and an overestimate. The underestimate of the area for the region then becomes
\[
\frac{1}{y+3} + \frac{1}{2y+2} + \frac{1}{3y+1} + \frac{1}{y} \left( \frac{y-1}{4} \right). \]
Similarly, what happens when we divide the interval \( x \) to \( xy \) into four pieces? The width of each interval then becomes \( \frac{xy-x}{4} \), while the heights of the underestimates are respectively
\[
\frac{1}{xy+3x} = \frac{1}{x} \left( \frac{y+3}{4} \right), \quad \frac{1}{2xy+2x} = \frac{1}{x} \left( \frac{2y+2}{4} \right), \quad \frac{1}{3xy+x} = \frac{1}{x} \left( \frac{3y+1}{4} \right) \quad \text{and} \quad \frac{1}{xy}. \]
Thus when each multiplied by the width, the \( x \)'s will cancel, and we will have identical underestimates for the areas. Similarly, the overestimates are the same.

**Geometric Observation on Why** \( \int_x^{xy} \frac{1}{t} \, dt = \int_1^y \frac{1}{t} \, dt \)

As we proceed to do the partition into rectangles of the two areas, the respective rectangles have equal areas. A rectangle in the \( x \)-to-\( xy \) range has base \( x \times \) the base of the respective rectangle in the \( 1 \)-to-\( y \) range, but its height is only \( \frac{1}{x} \times \) the height of the respective rectangle in the \( 1 \)-to-\( y \) range. But what is true when the interval is divided into 4 parts is always true, and so we give a rigorous proof of the lemma:

**Lemma.** \( \int_x^{xy} \frac{1}{t} \, dt = \int_1^y \frac{1}{t} \, dt \) for \( 1 < x < y \).

**Proof.**
\[
\int_x^{xy} \frac{1}{t} \, dt = \lim_{n \to \infty} \sum_{k=0}^{n-1} \frac{1}{x+k \left( \frac{xy-x}{n} \right)} \left( \frac{xy-x}{n} \right) \quad \text{Riemann Sums for } \frac{1}{t} \text{ from } x \text{ to } xy.
\]
\[
= \lim_{n \to \infty} \sum_{k=0}^{n-1} \frac{1}{x \left( 1+k \left( \frac{y-1}{n} \right) \right)} \left( \frac{xy-x}{n} \right) \quad \text{Factor the } x \text{'s.}
\]
\[
\lim_{n \to \infty} \sum_{k=0}^{n-1} \left( \frac{1}{1 + k \left( \frac{y-1}{n} \right)} \right) (y-1) = \int_1^y \frac{1}{t} \, dt.
\]

Cancel the \(x\)'s.

We now prove

**Theorem.** \( \ln(xy) = \ln(x) + \ln(y) \)

**Proof.**

\[
\ln(xy) = \int_1^y \frac{1}{t} \, dt \quad \text{by definition;}
\]

\[
= \int_1^x \frac{1}{t} \, dt + \int_x^y \frac{1}{t} \, dt \quad \text{by basic fact about integrals & areas;}
\]

\[
= \int_1^x \frac{1}{t} \, dt + \int_1^y \frac{1}{t} \, dt \quad \text{by the lemma;}
\]

\[
= \ln(x) + \ln(y) \quad \text{by definition.}
\]

It is also the generation after Napier that begins to make the connection between the inverse function to the logarithm and exponents since exponents always satisfy the law of exponents: \( a^{x+y} = a^x a^y \). So if there is a number \( a \) such that \( a^{\ln(x)} = x \) for all \( x \)'s, then clearly \( a \) satisfies \( \ln(a) = 1 \). By the continuity of area, there must be such a number, and actually if we overestimate, we see clearly that \( \ln(2) < 1 \), while if we divide the interval 1 to 3 into eight pieces and we underestimate the area, we get

\[
\ln(3) > \frac{1}{4} \times \left( \frac{1}{1.25} + \frac{1}{1.5} + \frac{1}{1.75} + \frac{1}{2} + \frac{1}{2.25} + \frac{1}{2.5} + \frac{1}{2.75} + \frac{1}{3} \right) = 1.019877 > 1
\]

so we know the number lies between 2 and 3, \( e = 2.7182818 \ldots \). This number will not be known as \( e \) until the 18th century when Euler names it so, and that appellation becomes be widely accepted (as most of the notation that Euler introduced).

Soon afterwards that \( e = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n \) will also be noted (although the convergence is slow as the table suggests), and thus multiple connections around that number are established by the 18th century, and hence it will become a constant second only to \( \pi \) in importance, superseding \( \phi \), the golden ratio, \( \phi = \frac{1 + \sqrt{5}}{2} \).

Of course, from the geometric series,

\[
\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots,
\]

the next generation of mathematicians would soon integrate to obtain
\[ \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots. \]

Additionally, as soon as differentials were developed and the fundamental theorem of calculus noted, if we let \( y = \ln(x) \), then \( dy = \frac{dx}{x} \), and so \( \frac{dx}{dy} = x \), and so the inverse function, the exponential is its own derivative.

We mentioned above that Euler was the first to name the fundamental constant as \( e \). He also proved that it was an irrational number.

**Theorem.** Euler. \( e \) is irrational.

**Proof.** We know by the Taylor series expansion of the exponential function that 
\[
e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots. \]

Let us suppose, by way of contradiction, that \( e \) is not irrational, and hence there exist positive integers \( h \) and \( k \) such that \( e = \frac{h}{k} \). But then, consider the series above as divided into two pieces, a finite sum consisting of those terms whose denominators are less than or equal to \( k \), and an infinite sum of all those terms whose denominators exceed \( k \), that is
\[
e = \sum_{i=1}^{k} \frac{1}{i!} + \sum_{i=k+1}^{\infty} \frac{1}{i!} + \cdots,
\]
and let us refer to these as \( F \) and \( R \) respectively. Thus \( e = F + R \), or equivalently, \( R = e - F \). Consider multiplying this equation by \( k! \) Clearly, \( k!e = (k-1)!h \) is an integer. Also, \( k!F = k! + \frac{k!}{2!} + \frac{k!}{3!} + \cdots + \frac{k!}{k!} \), being the sum of positive integers, is also a positive integer. Thus we have to conclude that \( k!R = \sum_{i=1}^{\infty} \frac{k!}{(i+1)!} + \frac{k!}{(i+2)!} + \frac{k!}{(i+3)!} + \cdots \) is an integer too. Since it is positive, if we can argue that it is less than 1, we will have arrived at a contradiction. But
\[
k!R = \sum_{i=1}^{\infty} \frac{k!}{(i+1)!} + \frac{k!}{(i+2)!} + \frac{k!}{(i+3)!} + \cdots = \frac{1}{k+1} + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} + \cdots.
\]

However, since \( k + 2 > k + 1 \), 
\[
\frac{1}{(k+1)(k+2)} < \frac{1}{(k+1)^2}, \text{ and, similarly,}
\]
\[
\frac{1}{(k+1)(k+2)(k+3)} < \frac{1}{(k+1)^3}
\]
because \( k + 1 \) is the smallest of the three factors.

In a continuing fashion, we obtain
\[
k!R < \frac{1}{k+1} + \frac{1}{(k+1)^2} + \frac{1}{(k+1)^3} + \frac{1}{(k+1)^4} + \cdots \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1
\]
since \( k \) is a positive integer, and so we have arrived at a contradiction. \( \blacksquare \)
Another interesting issue associated with $e$ was whether $e$ could satisfy some polynomial equation with integer coefficients. This was eventually answered (in the negative) during the second half of the nineteenth century. And a similar question for $\pi$ needed to be answered in order to put the Quadrature of the Circle to rest forever.

We end the chapter with two of the giants on whose shoulders Newton stood. Both Kepler and Galileo have permanently engraved places in both physics and astronomy since they made major contributions to both subjects. But they also have a place in the history of mathematics. Both were early supporters of the Copernican theory, which stated that the Earth revolved about the Sun—echoes of Aristarchus, and they both helped tremendously toward the general acceptance of the theory with their contributions.

Johann Kepler (1571-1630) contributed with his Laws of Planetary Motion. Later, the derivation by Newton of these laws from his simple gravitational law by the use of calculus constitutes a decisive step in the history of astronomy, physics and mathematics.

1. The orbit on which a planet travels is elliptical with the sun as one of the foci of the path.

Although the orbits are nearly circular, Kepler made a major breakthrough by parting from the constricting circles. In the picture, in addition to the ellipse, we have also drawn a circle to illustrate the contrast.

2. As a planet travels around the sun, the ray connecting the planet to the sun will sweep equal areas in equal times.

It is actually this law that Newton first proves by using his Universal Law of Gravitation.

and

3. The squares of the periods of revolution of the planets are proportional to the cubes of their mean distances to the sun.
Kepler does not state the third law until many years after the first two, and undoubtedly, it is his faith in simple patterns, and his intense belief in geometry, in particular the conics and the regular solids, which helped him through the massive amounts of data gathered by him and by his illustrious predecessor Tycho Brahe.

On the mathematical side, Kepler was interested in lengths, areas and volumes. He contributes to the then growing literature on infinitesimals of which we will see more in the next chapter.

For example, he used infinitesimals to compute the area of an ellipse (Archimedes's computation of it was unavailable to him). Kepler developed a keen curiosity in the volumes of barrels, and again, contributed to the growing interest and literature on the subject.

Galileo

Galileo Galilei (1564-1642) set out to reinforce Copernican theory by quantifying how objects fell. He was very creative in doing so. Since he wanted to slow the process of free fall so he could measure it, he used an inclined plane to study falling. This plane had removable pegs that he used to either lengthen or shorten the fall. Since a reliable clock was not available to him, he used his pulse to measure time—fortunately his father had been a musician so Galileo had an impeccable sense of timing. What were some of his observations? One of them was the fact that it took approximately the same time for the ball to fall from the beginning to the first peg as from the first peg to the second peg, or from the second peg to the third, etceteras.

Another type of observation he could make would be to take some unit of time measurement, for example 10 heartbeats. Then he could estimate how much the ball would fall for each period of 10 heartbeats. Thus, for example, he could have the sample (which is manufactured at our convenience):

<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>distance</td>
<td>0</td>
<td>16</td>
<td>64</td>
<td>144</td>
<td>256</td>
<td>400</td>
<td>576</td>
<td>784</td>
<td>1024</td>
</tr>
<tr>
<td>velocity</td>
<td>16</td>
<td>48</td>
<td>80</td>
<td>112</td>
<td>144</td>
<td>176</td>
<td>208</td>
<td>240</td>
<td></td>
</tr>
<tr>
<td>acceleration</td>
<td>32</td>
<td>32</td>
<td>32</td>
<td>32</td>
<td>32</td>
<td>32</td>
<td>32</td>
<td>32</td>
<td></td>
</tr>
</tbody>
</table>

where distance is being measured in some unit of measurement. By looking at the changes in distances covered, we obtain the next row—velocity, and from it, it would just be a similar step to an important observation on the change of velocities, or the acceleration:

**Acceleration is constant!**

Most authors also attribute to Galileo the first clear statement of the law of inertia, of which Newton will make powerful use. In 1632, he published *Dialogue Concerning the*
Two Chief Systems of the World. This book is fundamental in the history of astronomy and science. Its publication caused the trial and incarceration of Galileo.

For now we see an unusual side of Galileo: his contribution to probability. We saw in a previous section how Cardano had begun the taming of uncertainty that occurred in the 16\textsuperscript{th} and early part of the 17\textsuperscript{th} centuries by a very simple probabilistic statement. Why it took so long to even develop to the extent it did in those early centuries is indeed interesting, but not for us to speculate (gambling is very old, definitely thousands of years old.) What is relevant to us is that by the year 1600 it was reasonably clear in many people's minds what some aspects of probability were about. Galileo was posed the following puzzle by a gambling gentleman that was disturbed by the fact that common sense did not seem to quite fit reality. As everything else he did, Galileo analyzed the problem correctly.

Suppose two people are to play the following game (those early years were mostly concerned with gambling questions):

Person A rolls three dice, if a 9 shows up, Person A pays Person B $1, while if a 10 shows up, Person B pays Person A $1. If anything else shows up, Person A restarts the game.

At first thought it seems like a fair and reasonable game. Both numbers can be rolled in six different ways, as the two lists of possibilities indicate. Thus it seemed like an equitable situation to the gentleman who thus so stated. However, he had noticed that 10's were more common than 9's and thus the puzzle emerged. But what is the common sense behind this conclusion?

It has something to do with the number of ways of doing something and if one thing has more ways of occurring than another, then it is more likely to occur.

After all nobody would play the previous game if the competition were between rolling a 3 and rolling a 10 since intuitively one feels that a 3 is much rarer than a 10. Although there is logic behind this, it is not quite correct. It needs to be improved upon. And Galileo did just that. The basic principle that Galileo is going to use for his calculations is nothing but the same principle stated decades before by Cardano:

suppose an activity or experiment is to be performed, and we have equally feasible outcomes, then the probability for a given event to occur is the number of outcomes that give the desired event divided by the total number of outcomes.
But extra emphasis needs to be made on the premise of the principle: one must first reduce the outcomes of the activity to \textit{equally feasible outcomes}. What are equally feasible outcomes can be in itself a polemic. How do you know a coin is fair? But we will be naive about the subtleties of statistical analysis, and only insist that, from what we know, we can honestly claim that the outcomes we are taking are equally feasible.

Then you can start looking at the probability of the event that you are interested in. Going back to the game in question.

What is the activity in this example? Rolling 3 dice. What are the outcomes? It seemed acceptable to the gentleman to say that the outcomes were, in addition, to the 2 listed above:

\begin{itemize}
  \item 3
  \item 4
  \item 5
  \item 6
  \item 7
  \item 8
  \item 11
  \item 12
  \item 13
  \item 14
  \item 15
  \item 16
  \item 17
  \item 18
\end{itemize}

and thus there would be a total of \textbf{56 outcomes}, so the probability of a 9 would be \( \frac{6}{56} \), and a 10 would have the same probability, so the game is \textit{seemingly} fair.

However, if we apply the same reasoning, then the probability of a 3 is \( \frac{1}{56} \), and a 4 has the same probability. So if we keep rolling the three dice for a long time, the number of
3's occurring should roughly be the same as the number of 4's. It does not take much experimentation to perhaps start doubting our premise, and maybe we should question why did we label those outcomes as equally feasible? So let's rethink a bit. Is a \( \bullet \bullet \bullet \) as equally feasible as a \( \bullet \bullet \bullet \) ?

Suppose we had a yellow die, a white die and a blue die. Then, to roll a 3, we would have to have \( \bullet \bullet \bullet \), we need to show a \( \bullet \) in each die, but to get a 4, we can do it by: \( \bullet \bullet \bullet \), or \( \bullet \bullet \bullet \), or \( \bullet \bullet \bullet \), there are three ways since any of the three dice could show a \( \bullet \) and the other two need to show a \( \bullet \). It seems like we have some more choices in the latter situation. Three times as many, actually.

A way out of the quagmire is to take for our outcomes the 216 different ways there are to roll three dice if one of them is yellow, other one white and the third one blue. We get the 216 from: \( 6 \times 6 \times 6 \). Nobody can argue on the equal feasibility of these 216 outcomes. So we start now from there. How many ways can we roll a 3? As before, only one way, so the probability is \( \frac{1}{216} \), not \( \frac{1}{56} \). But how many ways can we roll a 4? Three ways as we saw above, so the probability of a 4 is \( \frac{3}{216} \). So 4's should occur about three times more often than 3's.

Let's go back to the 9 and the 10 of our game. Of the 216 ways, how many ways can we roll a \( \bullet \bullet \bullet \)? Easily, we have three decisions, which die shows the \( \bullet \), which the \( \bullet \) and which the \( \bullet \): \( 3 \times 2 \times 1 = 6 \) ways:

\[
\begin{array}{cccc}
\text{Yellow} & \text{White} & \text{Blue} \\
\bullet \bullet \bullet & \bullet \bullet \bullet & \bullet \bullet \bullet
\end{array}
\]

By identical reasoning, there are 6 ways to roll a \( \bullet \bullet \bullet \). But what about a \( \bullet \bullet \bullet \)? Here our tree of options has only two stages, since once we have decided which die shows the \( \bullet \), the other two dice must show a \( \bullet \) and we have nothing left to decide (or equivalently, we only have one option, once we have placed the \( \bullet \)), so there are only 3 ways to roll a \( \bullet \bullet \bullet \):

\[
\begin{array}{ccc}
\text{Yellow} & \text{White} & \text{Blue} \\
\bullet \bullet \bullet & \bullet \bullet \bullet & \bullet \bullet \bullet
\end{array}
\]

Identically, there are 3 ways to roll a \( \bullet \bullet \bullet \), and 6 ways to roll a \( \bullet \bullet \bullet \).
Finally, there is only 1 way: ☐️ ☐️ ☐️ to roll a ☐️ ☐️ ☐️ (all 3 dice have to show ☐️'s.) So how many ways can we roll a 9? Totaling our options we obtain: $6 + 6 + 3 + 3 + 6 + 1 = 25$ ways to roll a 9. Putting it in a table, together with the similar calculations for 10.

<table>
<thead>
<tr>
<th>Roll of 9</th>
<th># of Ways</th>
<th>Roll of 10</th>
<th># of Ways</th>
</tr>
</thead>
<tbody>
<tr>
<td>☐️ ☐️ ☐️</td>
<td>6</td>
<td>☐️ ☐️ ☐️</td>
<td>6</td>
</tr>
<tr>
<td>☐️ ☐️ ☐️</td>
<td>6</td>
<td>☐️ ☐️ ☐️</td>
<td>3</td>
</tr>
<tr>
<td>☐️ ☐️ ☐️</td>
<td>3</td>
<td>☐️ ☐️ ☐️</td>
<td>6</td>
</tr>
<tr>
<td>☐️ ☐️ ☐️</td>
<td>3</td>
<td>☐️ ☐️ ☐️</td>
<td>6</td>
</tr>
<tr>
<td>☐️ ☐️ ☐️</td>
<td>6</td>
<td>☐️ ☐️ ☐️</td>
<td>3</td>
</tr>
<tr>
<td>☐️ ☐️ ☐️</td>
<td>1</td>
<td>☐️ ☐️ ☐️</td>
<td>3</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>25</strong></td>
<td><strong>Total</strong></td>
<td><strong>27</strong></td>
</tr>
</tbody>
</table>

Hence the probability of a 9 is $\frac{25}{216}$ while the probability of a 10 is $\frac{27}{216}$. So on the average, after 216 rolls of the dice, Person A would have lost 25 times, but would have won 27 times, and the rest would have been draws. So Person A would have a net outcome of a $2 profit after 216 rolls. Hence the gambler's observation was totally justified by Galileo's explanation. What is remarkable is that the gentleman would have observed the discrepancy when the difference in probabilities was so small!

One of the common errors made in the past by mathematicians (including some of the best like Leibniz, D'Alembert and others) is the one of presuming equally feasible outcomes to an experiment without further analysis.
Chapter 13
Gateway to Our Times

Number, place, and combination...the three intersecting but distinct spheres of thought to which all mathematical ideas admit of being referred—Sylvester

As we enter the seventeenth century, we enter an age where present day mathematics was formed. We will see the creation of a formidable weapon: analytic geometry, and through it, we see algebra begin to dominate as a most powerful tool, and when extended, we see modern analysis around the corner. It is the mathematics of this period together with the notation that will be developed in the next two centuries that shape the mathematics of today throughout the world.

The symbolic form of mathematics that started with Cardano and Viète, takes such control of the subject that, by the beginning of the nineteenth century, books will be written without a single picture in them! Of course, as with all extremes, without taking any power away from algebra and symbolism, there is also much power to geometry and graphical and pictorial reasoning.

European History Highlights

<table>
<thead>
<tr>
<th>Date</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>1609</td>
<td>The Netherlands becomes an independent nation.</td>
</tr>
<tr>
<td>1618</td>
<td>Thirty Year War starts.</td>
</tr>
<tr>
<td>1620</td>
<td>The voyage of the Mayflower.</td>
</tr>
<tr>
<td>1628</td>
<td>William Harvey publishes on the circulation of blood.</td>
</tr>
<tr>
<td>1635</td>
<td>Academie Francaise founded by Richelieu.</td>
</tr>
<tr>
<td>1638</td>
<td>Peter the Great is born.</td>
</tr>
<tr>
<td>1642</td>
<td>Civil War starts in England, Scotland and Ireland.</td>
</tr>
<tr>
<td>1643</td>
<td>Louis XIV inherits the French throne. Torricelli invents the barometer.</td>
</tr>
<tr>
<td>1648</td>
<td>Peace of Westphalia ends Thirty Year’s War.</td>
</tr>
<tr>
<td>1658</td>
<td>Oliver Cromwell dies.</td>
</tr>
<tr>
<td>1661</td>
<td>Louis XIV, the Sun King, assumes power in France, he will reign until 1715.</td>
</tr>
</tbody>
</table>

We concentrate now on the first 50 years of the seventeenth century, a glorious time indeed in the history of mathematics. We will study Cavalieri from Italy, while from France we will see three prestigious names in the history of our subject Descartes, Fermat and Pascal, as well as Huygens who comes from the Netherlands and finally, Wallis, who is English in origin.
Bonaventura Cavalieri (1598-1647) is most famous for his often-referred-to principle concerning the computation of areas and volumes. And although we saw Zu Chongzhi proposed the same principle 1,000 years before, this principle is most commonly known in the West by Cavalieri’s name. One version of this principle is given in:

If two solids have equal altitudes, and if sections made by planes parallel to the bases and at equal distances from them are always in a given ratio, then the volumes of the solids are also in this ratio.

Naturally, one could state a similar statement concerning areas: If two (flat) surfaces have equal bases, and if sections made by lines parallel to the bases and at equal distances from them are always in a given ratio, then the areas of the surfaces are also in this ratio. The idea behind this principle is quite intuitive, and although later generations would judge Cavalieri for his lack of rigor, all we are doing is thinking of the volume as consisting of the slices of area, a stack of pancakes, or we think of an area as made up of the lines, a collection of sticks, etcetera. It was powerful enough to allow him not only to do many calculations from Greek times in a much easier fashion, but also break new ground, and do some new computations.

We can for example use Cavalieri’s principle to derive the volume of the cone given that we know the volume of a pyramid—recall that Eudoxus was the first one to have evaluated the volume of a cone. We compare a cone of height \( h \) and radius \( r \) with a pyramid of square base of side 1 and of the same height \( h \). We take a slice of either solid at distance \( x \) from the top. The slice of the cone, because of similarity of triangles, has radius \( a \) where \( \frac{a}{r} = \frac{x}{h} \), and thus its area is \( \pi a^2 = \frac{\pi x^2 r^2}{h^2} \). Also by similarity, the slice of the pyramid has side \( s \) where \( \frac{s}{1} = \frac{x}{h} \), so its area is \( s^2 = \frac{x^2}{h^2} \). Thus the ratio of the slices is \( \frac{\pi x^2 r^2}{h^2} \) which is constant, independent of where the slice was taken. Since we know the volume of the pyramid to be \( \frac{1}{3} \) (in other words, \( \frac{1}{3} \times \text{area of the base} \times \text{height} \)), the volume of the cone is then \( \frac{1}{3} \times \pi r^2 \times h \) which is also \( \frac{1}{3} \times \text{area of the base} \times \text{height} \).
Now we revisit, à la Cavalieri, the computation of the area of a parabolic segment (originally done by Archimedes). We will use modern notation, but the basic ideas are his.

Consider a right triangle with both legs equal to \(a\), and an arbitrary slice \(x\). Since we know the area of this triangle, then we know that the area generated by the slices \(x\) is \(\frac{a^2}{2}\), and we can use \(\sum x = \frac{a^2}{2}\) to express this fact.

Suppose we wanted now \(\sum x^2\), which is basically equivalent to our modern computation \(\int_0^a x^2 \, dx\). We know the answer to be \(\frac{1}{3}a^3\).

Consider a square with side \(a\), the slices \(x\) and \(y\) as in the picture. Then we know, by symmetry, \(\sum x = \sum y = \frac{a^2}{2}\). But then by simple algebra \(a^3 = \sum a^2 = \sum (x + y)^2 = \sum x^2 + 2\sum xy + \sum y^2\), but by symmetry, \(\sum x^2 = \sum y^2\), so \(\frac{a^3}{2} = \sum x^2 + \sum xy\). So if we can compute \(\sum xy\), then we will have \(\sum x^2\). But what is the shape of \(\sum xy\)? \(xy\) represents a rectangle with base \(x\) and height \(y\), so \(\sum xy\) represents the volume of the shape consisting of all these rectangles stacked together which is a pyramid, as the picture illustrates, with base \(\frac{1}{2}a^2\) and height \(a\).

Thus \(\sum xy = \frac{a^3}{6}\). And hence \(\sum x^2 = \frac{a^3}{2} - \frac{a^3}{6} = \frac{a^3}{3}\) as expected. Cavalieri pushed this reasoning further to compute \(\sum x^3\) as follows:

\[a^4 = \sum a^3 = \sum (x + y)^3 = \sum x^3 + 3\sum x^2 y + 3\sum xy^2 + \sum y^3 = 2\sum x^3 + 6\sum x^2 y,\]

so all we need is \(\sum x^2 y\). To evaluate the latter, we use the computation above,

\[a^4 = \sum a^3 = a\left(\sum a^2\right) = a\left(2\sum x^2 + 2\sum xy\right) = a\left(\frac{2}{3}a^3 + 2\sum xy\right).\]

Thus,

\[\frac{1}{3}a^4 = a\left(2\sum xy\right) = (x + y)\left(2\sum (xy)\right) = 2\sum x^2 y + 2\sum xy^2 = 4\sum x^2 y.\]

Hence \(\sum x^2 y = \frac{1}{12}a^4\), from which we easily get that \(\sum x^3 = \frac{1}{4}a^4\).

Cavalieri continued this way and computed \(\sum x^n\) for \(n \leq 9\), and from his computations,
he correctly conjectured that it would equal \( \frac{a^{n+1}}{n+1} \), the equivalent of our modern expression:

\[
\int_0^a x^n \, dx = \frac{a^{n+1}}{n+1}
\]

As this formula represents, there is a great deal of interest in this early part of the seventeenth century in areas, volumes, lengths and also tangents. And it is perhaps these speculations that lead two Frenchmen, Descartes and Fermat, to a major contribution—analytic geometry, the most important mathematical development of this pre-calculus era.

In particular, they were interested in seeing in a different way why the formula above was correct. They approached the computation in a modern fashion, namely they set out to approximate the area by dividing the interval into more and more pieces, and using rectangles to approximate the areas. Thus, the geometric intuition required for the proof would be replaced by arithmetic.

To compute the area under \( y = x^n \), first we divide the interval into \( k \) equal pieces, thus the length of each interval is \( \frac{a}{k} \), so the \( x \)-coordinates are given by \( 0, \frac{a}{k}, \frac{2a}{k}, \ldots, \frac{(k-1)a}{k}, \frac{ka}{k} \). The height of the graph at any one point is given by \( x^n \). If we use the lower approximation (the left-end point of each interval), since we are underestimating by the shaded area, our area is at least

\[
\frac{a^n}{k} \left( 0^n + \left( \frac{a}{k} \right)^n + \left( \frac{2a}{k} \right)^n + \cdots + \left( \frac{(k-1)a}{k} \right)^n \right) = a^{n+1} \left( \frac{0^n + 1^n + 2^n + 3^n + \cdots + (k-1)^n}{k^{n+1}} \right).
\]

While if we use the upper approximation, the right-end point of each interval, we get that our area is at most, shaded area represents now an overestimate:

\[
\frac{a^n}{k} \left( \left( \frac{a}{k} \right)^n + \left( \frac{2a}{k} \right)^n + \cdots + \left( \frac{ka}{k} \right)^n \right) = a^{n+1} \left( \frac{1^n + 2^n + 3^n + \cdots + k^n}{k^{n+1}} \right).
\]

And thus our area satisfies,

\[
a^{n+1} \left( \frac{0^n + 1^n + 2^n + 3^n + \cdots + (k-1)^n}{k^{n+1}} \right) \leq \text{Area} \leq a^{n+1} \left( \frac{1^n + 2^n + 3^n + \cdots + k^n}{k^{n+1}} \right).
\]

Since the difference between the right-side of the inequality and the left-hand side is \( \frac{a^{n+1}}{k} \), which becomes arbitrarily small for \( k \) large enough, both sides converge to the area we are seeking. Thus, the crucial computation was
which they successfully argued in various fashions, it being a typical computation of the times. However, for us at present, it would lead us too far afield.

As we see from the discussion above, polynomials are coming to be accepted: the notation is in place, the limitations of geometric dimensions are long gone, although not forgotten, and with analytic geometry comes the fascination of searching for the shape of a given equation. As a matter of fact, negative coefficients in polynomials occur even before isolated negative numbers since the latter could be justified as being from the other side of the equation. Actually, of the three great Frenchmen of the period, at least two of them felt strongly that negative numbers were nonsense.

**Descartes**

Soldier, philosopher and mathematician, René Descartes (1596-1650) will always be associated with the discovery of analytic geometry—or Cartesian geometry as is often called. However, Fermat’s also made meaningful contributions to the subject. In the history of philosophy, Descartes is well known for his *Cogito Ergo Sum* dictum, and had great impact in accentuating mathematics as the language of science and knowledge.

It was in an appendix, *La Géométrie*, to Descartes’ main philosophical work, *Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences* (1637) (often referred to as the Method) where he presented his ideas for the arithmetization of geometry. Since we have all received training in this subject, we will not dwell on it except to mention that one of the motivations of Descartes was a problem of Pappus. However, since Descartes was opposed to negative numbers, all he would accept would be the first quadrant of what we refer to as the Cartesian plane.

Both he and Fermat were very interested in finding tangent lines to arbitrary curves, and Descartes referred to the construction of tangents as

*the most useful and general problem that I know but even that I have ever desired to know in geometry.*

After we review polynomials, a favorite topic of his, which he discussed in the third section of *La Géométrie*, we will illustrate his circle method for drawing tangents.

As mentioned above, we use this section to introduce many basic facts about polynomials that stem from the period. By now the connection between being a root and factorization was becoming clear, a fundamental fact:

1. \( a \) is a root of \( p(x) \) if and only if \( x - a \) is factor of \( p(x) \).

Recall that \( a \) being a root of \( p(x) \) means \( p(a) = 0 \). Thus, 1 claims \( x^3 + 6x - 20 \) has 2 for a root is equivalent to \( x - 2 \) being a factor: \( x^3 + 6x - 20 = (x - 2)(x^2 + 2x + 10) \).
Actually, \( n \) is a special case of a more general situation, namely what \( p(a) \) is in terms of division and factorization. We review this fact quickly as well as what still is the most effective way to evaluate a polynomial at a point. Namely, if \( p(x) \) is a polynomial and \( a \) is an arbitrary number, then the most efficient way to compute \( p(a) \) is to use synthetic division. We exemplify with the polynomial above \( x^3 + 6x - 20 \) and with \( a = 5 \). The first step is to write all coefficients of the polynomial including the zero coefficients, and also the number at which we are evaluating the polynomial, and do the table, which means that \( p(5) = 135 \) and, furthermore, the quotient when \( p(x) \) is divided by \( x - 5 \) is \( x^2 + 5x + 31 \), as we read from the top of the division box.

In addition, the notion of multiplicity of a root is becoming well understood in terms of algebra by this time. Consider, for example, the family of polynomials \( p_c(x) = x^3 + 6x^2 + c \). Now, regardless of what \( c \) is, the shape of the function is where the turning points occur at the zeros of the derivative, 0 and \(-4\). If \( p_c(0) > 0 \), we only have one root, and it is less than \(-4\), while if \( p_c(-4) < 0 \), then we again have only one root, and it is bigger than 0.

So the interesting \( c \) occur when \( p_c(0) = c \leq 0 \) and \( p_c(-4) = 32 + c \geq 0 \), in other words when, \(-32 \leq c \leq 0 \). In this case we have three roots, \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \) that are interlaced by 0 and \(-4\):

\[
\lambda_1 \leq -4 \leq \lambda_2 \leq 0 \leq \lambda_3.
\]

There are two special cases: \( c = 0 \) and \( c = -32 \). If \( c = 0 \), then \( p_0(x) = x^3 + 6x^2 \), and \( \lambda_1 = -6 \); and \( \lambda_2 = \lambda_3 = 0 \) and we would say that 0 is a root of multiplicity two (or is a double root). Note the factorization,

\[
x^3 + 6x^2 = x^2(x - 6) = (x - 0)^2(x - 6).
\]

While if \( c = -32 \), then \( p_{-32}(x) = x^3 + 6x^2 - 32 \), and then \( \lambda_1 = \lambda_2 = -4 \), and \( \lambda_3 = 2 \). We also have \( p_{-32}(x) = (x + 4)^2(x - 2) \), and \(-4 \) has multiplicity two.
By the end of the eighteenth century, the basic theorem concerning the number of roots will eventually be proven by Gauss—although many attempt the proof. The theorem is called the **Fundamental Theorem of Algebra**, and it states that every polynomial has complex roots, and the correct number of them:

2. A polynomial of degree $n$ has exactly $n$ complex roots, including multiplicities.

Although polynomials of arbitrary degree do not become widely accepted until this period, several techniques for finding roots of polynomials were developed from early times. In particular, two of them came to us from as early as the time of the Arabs, if not earlier: the **bisection method** and the **method of false position**—from which another variant came about later called the **secant** method.

**They both start with an $a$ and a $b$ given where $p(a)$ and $p(b)$ disagree in sign, one is positive, one is negative.** Hence one expects a root between $a$ and $b$. **Both procedures shrink the interval where the root is located.**

The bisection method simply goes to the midpoint between $a$ and $b$, $\frac{a+b}{2}$, and decides by checking the sign at the midpoint in which of the two halves the root is located in. We will illustrate both methods with an example:

$$p(x) = x^4 - 4x^3 + 12.$$  
with $a = 1$, $p(1) = 9$ and $b = 2$, $p(2) = -4$.

Note it was our job to produce the $a$ and the $b$.

The bisection method table is on the right.

The method of false position is more complicated in that it uses the secant to the graph given by the two point $a$ and $b$. In more detail, it takes as its new guess the intersection of the $x$-axis with the line going through the points $(a, p(a))$ and $(b, p(b))$. Thus, for the method of false position method, the new point is given by the expression:

$$a - \frac{(a-b)p(a)}{p(a) - p(b)} = \frac{bp(a) - ap(b)}{p(a) - p(b)}.$$
Obviously, the computation of the new guess is harder, but as we can see, at least in this case, the method of false position works considerably faster than the bisection method. When we discuss Newton we will study a considerable improvement on both of these.

<table>
<thead>
<tr>
<th>Method of False Position</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive</td>
</tr>
<tr>
<td>1.000000</td>
</tr>
<tr>
<td>1.692308</td>
</tr>
<tr>
<td>1.744416</td>
</tr>
<tr>
<td>1.746119</td>
</tr>
<tr>
<td>1.746169</td>
</tr>
</tbody>
</table>

The following is a qualitative improvement on the number of roots.

**3** Suppose \( p(a) \neq 0 \) and \( p(b) \neq 0 \). Then the number of real roots of \( p(x) \) between \( a \) and \( b \) (including multiplicities) is odd if and only if \( p(a)p(b) < 0 \).

The next of the statements is very much of this era, since derivatives have just been introduced, mainly to draw tangent lines. The result is very intuitive as the picture indicates, but the main analytical tool was what we referred to today as **Rolle's Theorem**:

**Between two consecutive roots, the derivative must have a root.**

**4** Let \( p(x) \) have \( m \) real roots (including multiplicities). Then the derivative of \( p(x) \), \( p'(x) \) has at least \( m-1 \) real roots (including multiplicities), which interlace the roots of \( p(x) \).

For example, let \( p(x) = (x^2 - 1)^{12} \). Then \( p(x) \) is of degree 24 and has 24 real roots, 1 with multiplicity 12 and \(-1\) with multiplicity 12. Then \( p'(x) = 12(x^2 - 1)^{11}(2x) \) has 23 real roots, 1 with multiplicity 11, \(-1\) with multiplicity 11 and 0 (which is between \(-1\) and 1). Continuing, 1 is a root of multiplicity 10 in \( p''(x) \) and so is \(-1\), the remaining two roots \( \lambda_1 < \lambda_2 \) satisfy \(-1 < \lambda_1 < 0 \) and \( 0 < \lambda_2 < 1 \) because of the interlacing property. And actually, because of Viète's relations, \( \lambda_1 = -\lambda_2 \). The third derivative, \( p^{(3)}(x) \) is of degree 21, its roots are \(-1\) of multiplicity 9, 1 of multiplicity 9, and \( \delta_1 < \delta_2 < \delta_3 \) which satisfy \(-1 < \delta_1 < \lambda_1 < \delta_2 < \lambda_2 < \delta_3 < 1 \).

We see these facts nicely reflected in the picture.

If we were to continue in this fashion, we could conclude that \( p^{(12)}(x) \), the twelfth
derivative, has 12 different real roots, all between $-1$ and $1$. In fact, $p^{(12)}(x)$ obtains some high values, for example, $p^{(12)}(1) = 1,961,990,553,600$, but as we see in the graph, there are twelve roots between $-1$ and $1$.

Later on, a finer theorem concerning the locations of the roots of the derivative would be called the **Gauss-Lucas Theorem** and its statement is very elegant. It states that if we consider the complex roots of a polynomial $p(x)$ as points in the plane (see the first chapter), then the roots of the derivative, $p'(x)$, are all contained in the convex polygon made by the roots.

For example, consider one of the polynomials from the previous chapter, $p(x) = x^3 + 6x - 20$, its roots are $2$, $-1 + 3i$ and $-1 - 3i$, so if we plot those three points in the Cartesian plane, we get the three black dots in the picture:

But the derivative of $p(x)$, $p'(x) = 3x^2 + 6$, which has $\pm \sqrt{2}i$ for its roots, the two (blue) points on the $y$-axis. And we see, as the theorem asserts, that the roots of the derivative are contained in the triangle whose vertices are the roots of the original polynomial.

The name of Fermat is sometimes associated with the next result

\* Rational Root Theorem. Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ have integer coefficients. Suppose $\frac{k}{m}$ is a (reduced) rational root of $p(x)$. Then $k$ is a factor of $a_0$ and $m$ is factor of $a_n$.

For example, if $p(x) = 6x^4 + x^3 + 4x^2 + x - 2$, then the only possible rational roots have to be of the form $\frac{k}{m}$ where $k$ is a factor of 2, the constant term, and $m$ is a factor of 6, the leading term. Hence $k = \pm 1, \pm 2$, and $m = \pm 1, \pm 2, \pm 3, \pm 6$. Indeed, $p(x)$ factors as

$$6\left(x - \frac{1}{2}\right)\left(x + \frac{2}{3}\right)(x^2 + 1) = (2x - 1)(3x + 2)(x^2 + 1),$$

in other words its rational roots are $\frac{1}{2}, \ -\frac{2}{3}$.

The last result on polynomials we look at has the name of Descartes associated with it.
Descartes’ Rule of Signs. Let \( n \) be the number of positive roots of \( p(x) \) (including multiplicities). Let \( v \) be the number of variations in sign among the coefficients of \( p(x) \) (from positive to negative or vice versa, zeros do not count). Then \( n \leq v \) and \( v - n \) is always even.

For example, consider \( p(x) = 6x^4 + x^3 + 4x^2 + x - 2 \). Here \( v = 1 \), so it can have at most one positive real root, and since \( v \) is odd, \( n \) is odd, and so \( n = 1 \). As we saw in \( \Theta \), \( p(x) \) has one positive root, \( \frac{1}{2} \). Since \( \lambda \) is a root of \( p(x) \) if and only if \( -\lambda \) is a root of \( p(-x) \), we can use Descartes’ Rule of Signs to count the negative roots as well (something he would have not necessarily approve of). Because all we need then is to substitute \(-x\) for \( x \), and count the number of variations in sign. For example, in the example from \( \Theta \), \( p(-x) = 6x^4 - x^3 + 4x^2 - x - 2 \), and thus we have 3 variations in sign, and the Rule of Signs then says we have either 3 or 1 negative roots. In fact, we have one as done in \( \Theta \).

As another example, consider
\[
p(x) = x^8 + 3x^7 - 29x^6 - 61x^5 + 208x^4 + 170x^3 + 60x^2 + 608x - 960.
\]
Here \( v = 3 \), and indeed we have three positive roots. Searching for negatives,
\[
p(-x) = x^8 - 3x^7 - 29x^6 + 61x^5 + 208x^4 - 170x^3 + 60x^2 - 608x - 960.
\]
Here \( v = 5 \), so the number of negative roots is either 5, 3 or 1. We see in the picture that 3 is the correct number of negative roots.

In a different vein, we mention an original geometrical result due to Descartes, which could have been stated in classical times, yet it was not. The following is a direct quotation (translated, of course):

\textit{As in a plane figure all the exterior angles, taken together, equal four right angles, so in a solid body all the exterior solid angles, taken together, equal eight solid right angles.}

Thus, if we consider a cube, each corner has \( 270^\circ \) (3 squares of \( 90^\circ \) each), so the exterior angle is \( 90^\circ \) (one right angle), and of course there are exactly 8 corners (as his theorem predicts). A tetrahedron has at each corner 3 triangles of \( 60^\circ \) each, for a total of \( 180^\circ \), so the exterior angle is also \( 180^\circ \), and since there are four corners, we have a total of \( 720^\circ \) which is eight right angles.

The last topic on Descartes that we will discuss is his \textit{circle method of drawing tangents}, and as usual, we give a simple example. Suppose we want to draw the tangent to the graph of \( y = x^4 \) at the point \((a, a^4)\).

Certainly, it is sufficient to find the \textit{slope of the tangent line}, or equivalently, to find the \textit{slope of the normal line}. 

}\[ \begin{array}{c}
\text{Normal Line}
\end{array} \]
First, we take an arbitrary point on the $x$-axis, $(b,0)$, and we draw the circle with center $(b,0)$ and going through $(a,a^4)$: the equation of which is

$$(x-b)^2 + y^2 = (a-b)^2 + a^8.$$  

What we are looking for is the point $(b,0)$ so that this circle is tangent to the curve, that is, it intersects the curve at only $(a,a^4)$.

The example in the picture has the wrong center since the circle intersects the curve in two points, so it is not tangent to it.

Let us compute the intersection of the circle with the curve:

$$(x-b)^2 + (x^4)^2 = (a-b)^2 + a^8,$$

which simplifies to

$$x^2 - 2bx + b^2 + x^8 = a^2 - 2ab + b^2 + a^8,$$

$$(x^8 - a^8) + (x^2 - a^2) - 2b(x - a) = 0.$$

We know $a$ is a root, or equivalently, $x - a$ is a factor—as we can clearly see in the factorization. But in order for no other point of intersection, we want $a$ to be a double root, to be of multiplicity two, and thus we want another factor of $x - a$. Factoring the expression we get

$$(x-a)(x^7 + ax^6 + a^2x^5 + a^3x^4 + a^4x^3 + a^5x^2 + a^6x + a^7) + (x-a)(x+a) - 2b(x - a) = 0$$

$$(x-a)(x^7 + ax^6 + a^2x^5 + a^3x^4 + a^4x^3 + a^5x^2 + a^6x + a^7 + x + a - 2b) = 0$$

And what we want is for $a$ to be root of

$$p(x) = x^7 + ax^6 + a^2x^5 + a^3x^4 + a^4x^3 + a^5x^2 + a^6x + a^7 + x + a - 2b.$$  

But $p(a) = 8a^7 + 2a - 2b$. Thus, $a$ will be a double root if $b = a + 4a^7$. But then the slope of the line from $(b,0)$ to $(a,a^4)$—the normal line—is $\frac{a^4}{a - a - 4a^7}$, which equals $\frac{-1}{4a^3}$, and so the slope of the tangent line is $4a^3$, as we all know.

Since we are a long way from the notion of function, it was very common in this period to see equations involving $x$’s and $y$’s. These letters we owe very much to Descartes who so labeled them from the beginning (some of his notation came from Viète).

One example was his curve: the folium, with equation

$$x^3 + y^3 = 6xy,$$

which was originally a challenge to Fermat for finding its tangents. It is Fermat in fact that many consider the founder of differentiation, and we will add to this remark when we discuss him below.

In closing our discussion on Descartes, we should remark that mathematics played a very important role in his influential philosophy. He was one of the original advocates of a mechanistic world, and ironically, he is both crowned and dethroned by Newton.
Next we look at a Parisian contemporary of Descartes. He has been referred to as the greatest mathematical **might-have-been**, since he voluntarily cut his mathematical career short in addition to his early death at 39. But in fact he accomplished much in his short life.

**Pascal**

The father of **Blaise Pascal** (1623-1662) was a mathematician, and the young Pascal grew up in an intellectually stimulating environment. But this does not explain fully the tremendous talent he showed as a young person. At 16, he had written *Essay pour les coniques* in which he proved an original theorem, still referred to as Pascal's Theorem.

It was known that **any 5 points lie on a unique conic**. But when do 6 points lie on a conic? Not every collection of six points lies on such a curve. The answer is

**Pascal's Theorem.** Six points, arbitrarily labeled into three pairs: A, A, B, B, C and C, lie on a conic if and only if the three points obtained when we intersect the lines AB and AB, the lines AC and AC, and the lines BC and BC are themselves collinear.

The theorem is closely associated with Pappus' Theorem, and **Desargues'** Theorem, who was a contemporary and friend of Pascal's father.

At 18, Pascal invented a calculating machine of which, in partnership with his father, he sold at least 50.

But the area of Pascal's interest that we will concentrate on carries his name. He called it the **arithmetic triangle**, but we often referred to it as **Pascal's Triangle**. As usual, we will adopt modern notation. The triangle had been known for a long time. The Chinese author **Zhu Shijie** wrote on it in the thirteenth century. However, Pascal gave one of the first formal treatments of it in Europe and he did prove facts about the triangle that were unknown or unproved before. As he did this, he began the formalization of what we call **mathematical induction**, and which Fermat (in the next section) had touched upon with his **method of infinite descent**. There is no doubt, as we will see below, that Pascal possessed great recursive ability, the power to simplify a problem to a simple single step. And the triangle represents one of the most important examples of a two-dimensional recursive array.

When we discussed Fibonacci, we discussed the recursive nature of his sequence. The triangle is also a recursion, but it is two-dimensional.
Let \( n \) and \( k \) be nonnegative integers. Then the number of subcollections of size \( k \) (or \( k \) - subsets) from a collection of \( n \) objects (an \( n \) - set) is called \( n \) choose \( k \) and is denoted by \( \binom{n}{k} \), although another common notation is \( _n C_k \). The notation we use is from the \( 19^{\text{th}} \) century. Why this name? What you are doing is choosing out of \( n \) friends that you have, \( k \) of them to come to a party, and you are counting the number of ways of doing that. Or from \( n \) different balls, you are choosing \( k \) to put in a bucket. The expression \( \binom{n}{k} \) is also the number of ways of making a committee of \( k \) people out of \( n \) eligible candidates. The numbers \( \binom{n}{k} \) are also called \textbf{binomial coefficients} (the reason for this name will be clarified later), and many of us find them among the most charming of numbers.

We do the example with \( n = 5 \) and \( k = 2 \) in the discussion below. We are starting with five balls of different colors, and we are going to choose two to go into a bucket (for whatever reason).

If we label the balls 1, 2, 3, 4 and 5, then we easily come up with exactly 10 pairs: \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\} and \{4,5\}, and so we would state that \( \binom{5}{2} = 10 \).

A similar computation yields \( \binom{5}{3} = 10 \) Is this a coincidence? No.

Think about it this way. \textbf{Anytime you choose 2 balls to put in the bucket, you are automatically choosing 3 not to be placed so.}

Hence, \textbf{the number of ways of choosing 3 out of 5 is the same as the number of ways of choosing 2 out of 5} (compare pictures).

By similar reasoning, \( \binom{5}{1} = \binom{5}{4} = 5 \), and finally \( \binom{5}{0} = \binom{5}{5} = 1 \).

Of course, if \( k > 5 \), \( \binom{5}{k} = 0 \) since there is no way to select more than 5 balls from the collection. Finally, since every collection of balls is accounted for, we must have:
since that is the total number of subcollections. In fact, \( 1+5+10+10+5+1 = 32 \). The same observations from the example can be generalized to arbitrary numbers:

1. \( \binom{n}{k} = 0 \) whenever \( k > n \) since there is no way to choose more objects than what is available.

2. \( \binom{n}{k} = \binom{n}{n-k} \) whenever \( 0 \leq k \leq n \) since to choose \( k \) friends to come to the party is tantamount to choosing \( n-k \) not to come.

3. \( \binom{n}{0} = \binom{n}{n} = 1 \) since there is only one way to give a party where either nobody comes, or everybody comes.

4. \( \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n \) since we are partitioning the subsets of an \( n \) set by their size.

More importantly, there is a very nice recursion that they satisfy—which is due to Pascal.

**Theorem. Pascal's Recursion.** Let \( 1 \leq k \leq n \), then \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

**Proof.** It is very simple indeed. Among your \( n+1 \) friends, from which you have to choose \( k \) for the party, there is a special one: Otto, the Brute. Partition your parties into two types, the ones with Otto and the ones without Otto. How many parties with \( k \) people do you have if Otto is coming? Since Otto is one of the guests you have yet to choose \( k-1 \) friends out of your remaining \( n \) friends, hence the answer is \( \binom{n}{k-1} \). On the other hand, if Otto is not coming, you have to choose all \( k \) guests out of the remaining \( n \) friends, hence the answer is \( \binom{n}{k} \). By simple counting then, since we partitioned the set of parties into two pieces and we have counted each of the pieces, we are done. \( \blacksquare \)

With this recursion, together with the conditions before the theorem, we can now build the table of binomial coefficients. Although this table has been known to many people and many cultures from way before the time of Pascal, it is known, in the Western world at least, as Pascal's triangle. We are going to let \( n \) index the rows of our array while \( k \) will index the columns. We will start with \( n = k = 0 \), and grow from there. By the conditions before the theorem we know our array looks like:
with zeros above the main diagonal.

But now with the recursion we can fill in the rest of the array. Namely to fill a new row, one adds the position just above it to the one above and to the left:

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
0 & 1 & & & & & & & & & \\
1 & 1 & 1 & & & & & & & & \\
2 & 1 & 2 & 1 & & & & & & & \\
3 & 1 & 3 & 3 & 1 & & & & & & \\
4 & 1 & 4 & 6 & 4 & 1 & & & & & \\
5 & 1 & 5 & 10 & 10 & 5 & 1 & & & & \\
6 & 1 & 6 & 15 & 20 & 15 & 6 & 1 & & & \\
7 & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & & \\
8 & 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & \\
9 & 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 \\
10 & 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 1 \\
\end{array}
\]

Pascal's Triangle

Observe the example of the recursion in the table. Besides the observations we made before the theorem, there are many nice features in Pascal's triangle. One of the important ones is that the coefficients increase in each row up to the middle, and then, because of the symmetry, they decrease. For example, the last row in our table went

1 → 10 → 45 → 120 → 210 → 252

(which is the exact middle).

But as with every recursion, sometimes a closed expression is preferable. This formula when used wisely is computationally superior to the recursion, but the key word is wisely. The well-known expression is due to none other than Newton.

**Theorem. Newton's expression.** Let \(0 \leq k \leq n\), then

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}.
\]

In order for this expression to be correct, it is necessary to define 0! to be 1, and we shall agree to this (there are many reasons for this to be the case). Thus, for example,

\[
\binom{100}{5} = \frac{100!}{5!95!} = \frac{100(99)(98)(97)(96)}{120} = 75,287,520.
\]

This example points out the folly of pretending to use the formula with total abandon. First, there is no doubt the binomial coefficients get large: for example, \(\binom{100}{50}\) has 29 digits; second, that in order to use the formula wisely, one has to be careful with the
computations.

One can use Pascal’s recursion to derive Newton’s expression (see exercises).

Now we return to the very important issue of why they are called the binomial coefficients. Suppose I ask you to expand \((x + y)^6\)? Everybody knows what this stands for. Namely, \((x + y)(x + y)(x + y)(x + y)(x + y)(x + y)\), a total of 6 times.

If we proceed without any thought, we can just multiply this out. Not only boring, but very inefficient. Instead let’s think about the process of the multiplication and the powerful distributive law. **What we are doing then is taking one of the two summands from each one of the factors in order to get one of our terms.** (Let us take the opportunity to recall that when Viète relations were discussed, we used very similar reasoning.)

So each of our terms is of the form \(x^i y^j\) where \(i\) and \(j\) are nonnegative integers and \(i + j = 6\). How many terms do we have? In order to build a term, we have six stages or decisions (one for each of the factors) and two options for each of the decisions, so we will have \(2^6 = 64\) terms. One term is, clearly, \(x^6\) which comes up by taking an \(x\) from each of the factors, and that is the only way to obtain \(x^6\). But another term is \(x^4 y^2\) (we are indicating by the position what each factor contributed) which equals \(x^4 y^2\). But when we collect terms to simplify, \(x^4 y^2\) occurred several times; we have just seen one of those occurrences. Another is \(x^3 y^3\). In total, how many times does \(x^4 y^2\) occur? We have to decide which of the six factors will contribute \(y\)'s, the others will contribute \(x\)'s. So we have to choose 2 out of 6, so there are \(\binom{6}{2} = 15\) terms that equal \(x^4 y^2\), so its coefficient is 15. This is why these numbers are called **binomial coefficients.** If we just extend our reasoning to all the terms, we get

\[(x + y)^6 = x^6 + 6x^5 y + 15x^4 y^2 + 20x^3 y^3 + 15x^2 y^4 + 6xy^5 + y^6.\]

Of course, since this is an algebraic identity, it is valid for arbitrary \(x\) and \(y\). Thus, if what we wanted was \((2a - b)^6\), then all we would have to do is substitute \(x\) by \(2a\) and \(y\) by \(-b\) (watch that minus sign), in order to obtain

\[(2a - b)^6 = 64a^6 - 192a^5 b + 240a^4 b^2 - 160a^3 b^3 + 60a^2 b^4 - 12ab^5 + b^6.\]

Or we could have let \(x = y = 1\) to get 64=1+6+15+20+15+6+1, an identity we had seen before since we are adding a row of Pascal’s triangle; the number 64 is the same as the number of terms, of course. The reasoning above clearly generalizes to any exponent (as long as it is a **positive integer**), so we can state


The Binomial Theorem. \((x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i\).

The sigma notation for sums can be a bit intimidating, but by just writing the terms one-by-one all fears are conquered:

\[(x + y)^n = x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \binom{n}{3} x^{n-3} y^3 + \cdots\.

We can see why this theorem is true by using (again) Pascal's recursion. The proof is by induction (the theorem clearly holds for \(n = 1\)), so assuming it holds for \(n\), one needs to show it holds for \(n + 1\). Instead of obscuring the idea, which is fairly simple, by the general case notation, we will simply go from \(n = 6\) to \(n = 7\). So we assume the theorem holds for \(n = 6\), and we need to show it holds for \(n = 7\).

We have then:

\[(x + y)^6 = x^6 + \binom{6}{1} x^5 y + \binom{6}{2} x^4 y^2 + \binom{6}{3} x^3 y^3 + \binom{6}{4} x^2 y^4 + \binom{6}{5} x y^5 + y^6.

We need

\[(x + y)^7 = x^7 + \binom{7}{1} x^6 y + \binom{7}{2} x^5 y^2 + \binom{7}{3} x^4 y^3 + \binom{7}{4} x^3 y^4 + \binom{7}{5} x^2 y^5 + \binom{7}{6} x y^6 + y^7.

But

\[(x + y)^7 = (x + y)^6 (x + y) = \left(\binom{6}{1} x^5 y + \binom{6}{2} x^4 y^2 + \binom{6}{3} x^3 y^3 + \binom{6}{4} x^2 y^4 + \binom{6}{5} x y^5 + y^6\right) (x + y).

So

\[(x + y)^7 = \left(\binom{6}{1} x^5 y + \binom{6}{2} x^4 y^2 + \binom{6}{3} x^3 y^3 + \binom{6}{4} x^2 y^4 + \binom{6}{5} x y^5 + y^6\right) x + \left(\binom{6}{1} x^5 y + \binom{6}{2} x^4 y^2 + \binom{6}{3} x^3 y^3 + \binom{6}{4} x^2 y^4 + \binom{6}{5} x y^5 + y^6\right) y = x^7 + \binom{6}{1} x^6 y + \binom{6}{2} x^5 y^2 + \binom{6}{3} x^4 y^3 + \binom{6}{4} x^3 y^4 + \binom{6}{5} x^2 y^5 + \binom{6}{5} x y^6 + y^7.

After collecting terms:

\[(x + y)^7 = x^7 + \binom{6}{1} x^6 y + \binom{6}{2} x^5 y^2 + \binom{6}{3} x^4 y^3 + \binom{6}{4} x^3 y^4 + \binom{6}{5} x^2 y^5 + \binom{6}{5} x y^6 + y^7.

But what is \(\binom{6}{1} + \binom{6}{0}\)? By Pascal's recursion, it equals \(\binom{7}{1}\).
Similarly \( \binom{6}{2} + \binom{6}{1} = \binom{7}{2} \), \( \binom{6}{3} + \binom{6}{2} = \binom{7}{3} \) and \( \binom{6}{4} + \binom{6}{3} = \binom{7}{4} \), etcetera. And so we obtain the theorem for \( n = 7 \).

The following special case of the theorem deserves recognition:

**Corollary.** \((1 + x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i\)

This equation is, again, an algebraic identity so it lends itself to all kinds of manipulations and substitutions. For example, we can differentiate (with respect to \( x \)) to obtain a new identity:

\[ n(1 + x)^{n-1} = \sum_{i=0}^{n} \binom{n}{i} x^{i-1}, \]

thus if we let \( n = 6 \), we have

\[
\binom{6}{1} + 2 \binom{6}{2} x + 3 \binom{6}{3} x^2 + 4 \binom{6}{4} x^3 + 5 \binom{6}{5} x^4 + 6 \binom{6}{6} x^5 =
\]

\[
6 + 30x + 60x^2 + 60x^3 + 30x^4 + 6x^5 = 6(1 + x)^5.
\]

When we study Newton we will see a powerful generalization of the binomial theorem.

The next person we will study is Fermat. Although Fermat and Pascal never met, they did correspond, and although Fermat could never get Pascal interested in number theory, Fermat’s favorite subject, Pascal did get Fermat interested in a problem that had been proposed to him, and we end our discussion of Pascal with this problem and their two very different approaches. Pascal was proposed the following problem:

**Two parties, of equal ability, will play a game until one of them has won six hands. Each of them has placed 32 coins in the pot to be collected by the winner. For some unexpected reason they have to stop when one of them has won 5 games and the other 3. How should the 64 coins be divided?**

You may think of this problem as that of flipping a coin until you get a total of 6 heads or 6 tails. The problem had been around for a long time, and several proposed answers had been given, including 2:1 and 5:3. Pascal corresponded with Fermat on it, and they both solved it correctly, but in very different ways.

First we look at Fermat’s solution: let’s call the two players **A** and **B**. Then if they were to play 3 more hands (or flip the coin three more times), they would have decided for sure on the winner, for then either **A** would have at least won 1 or **B** would have won the needed 3. Of the possible, **equally feasible**, 8 outcomes to these 3 future games, 7 of them make **A** the winner, while one of them makes **B** the winner, so the probability of **A** winning is \( \frac{7}{8} \) so the stakes should be divided 7-to-1, or 56 coins for **A** and 8 for **B**. Note
that Fermat was very careful to make sure he had equally feasible outcomes to his claim.

As a matter of fact, a contemporary of them, Roberval, complained that the outcomes of the future would be: A, BA, BBA, or BBB. So the probability of A winning is $\frac{1}{4}$, so the stakes should be divided 3-to-1.

The problem, of course, is that Roberval had no reason to presume his outcomes to be equally feasible. And indeed they are not: A has probability $\frac{1}{2}$, BA has probability $\frac{1}{4}$, and both BBA and BBB have probability $\frac{1}{8}$, and the answer is correctly $\frac{7}{8}$.

<table>
<thead>
<tr>
<th>Game 1</th>
<th>Game 2</th>
<th>Game 3</th>
<th>Winner</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>A</td>
<td>A</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>A</td>
<td>B</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>A</td>
<td>B</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>B</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>B</td>
<td>A</td>
<td>B</td>
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</tr>
<tr>
<td>B</td>
<td>B</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>B</td>
<td>B</td>
</tr>
</tbody>
</table>

On to Pascal's solution. He solved the problem recursively. He argued as follows. If the players had each won 5 games, nobody would disagree with splitting the stakes 32-32. What would happen if A had won 5 and B had won only 4. Well, if they played one more hand they would either be 6-4 or 5-5. In one situation, A would receive all 64 coins while in the other, A would receive 32. Thus, we should average the two, and give A, $\frac{64 + 32}{2} = 48$ coins, and B would receive 16. So we understand 5-4.

How about 5-3? With one more hand, they would be at either 6-3 or 5-4. In one situation, A gets 64, while in the other, 48—averaging we get 56. The same answer as Fermat!

We finish our extensive visit to France by visiting the greatest French mathematician of the seventeenth century:

**Fermat**

In contrast to the widely traveled and adventurous Descartes, Pierre de Fermat (1601-1665) led a quiet life in Southern France where he was a judge and did mathematics for his entertainment. No doubt, of all amateur mathematicians, he was the all-time best, since he is often thought of as the best French mathematician of his century.

We have already mentioned his contributions to analytic geometry, and many consider the idea that a function attains an extremal value when its derivative is zero as being clear to Fermat. One of the reasons for this thought is the optical principle referred to as Fermat's Principle that basically states that light will always travel by an extremal path.

But of all areas of mathematics in which he was interested, the one he transformed the
most is number theory. Inspired by a recent translation (1621) of Diophantus' Arithmetica, Fermat discovered and conjectured patterns that have inspired countless mathematicians until the present day and into the future. Alas, he proved very little, and although he was correct a vast majority of the time, he made a mistake occasionally. And we will start our discussion of him with one such mistake.

The Fermat Primes.

Fermat was interested in primes of the form \(2^k + 1\), and after some thought (and some experimentation) he realized that for such a number to be prime, \(k\) itself had to be a power of 2. Hence he became interested in the numbers we call Fermat numbers, numbers of the form \(F_n = 2^{2^n} + 1\) for \(n = 0, 1, 2, 3, \ldots\). Thus, \(F_0 = 3\), \(F_1 = 5\), \(F_2 = 17\), \(F_3 = 257\), \(F_4 = 65,537\) and \(F_5 = 2^{32} + 1 = 4,294,967,297\). Clearly, the first 3 are primes. But with a little sieve work, we can find that \(F_3\) and \(F_4\) are also primes.

Unfortunately, Fermat believed all the \(F_n\)'s were primes. However, Euler, possibly using a fact due to Fermat, known as Fermat's Little Theorem (that Euler first proved and that we will see below) factored

\[ F_5 = 4,294,967,297 = 641 \times 6700417. \]

Until the present day, there are no other Fermat numbers that are also primes after \(F_4\).

More than 100 years after Fermat, and shortly after Euler's death, Gauss made a remarkable connection between Fermat primes and constructibility of a regular polygon by straightedge and compass alone. Namely, Gauss proved that we can construct the 17-gon, and the 257-gon and the 65537-gon because they are Fermat primes, and we can build the pentadecagon because it is the product of distinct Fermat primes, but we cannot build the heptagon because 7 is not a Fermat prime. Isn't this sort of amazing?

Fermat's Little Theorem. Let \(p\) be a prime. Then for any positive integer \(a\),

\[ a^p - a \text{ is a multiple of } p. \]

<table>
<thead>
<tr>
<th>(a)</th>
<th>(a^7 - a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>126</td>
</tr>
<tr>
<td>3</td>
<td>2184</td>
</tr>
<tr>
<td>4</td>
<td>18380</td>
</tr>
<tr>
<td>5</td>
<td>78120</td>
</tr>
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<td>279930</td>
</tr>
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<td>7</td>
<td>823536</td>
</tr>
<tr>
<td>8</td>
<td>2097144</td>
</tr>
<tr>
<td>9</td>
<td>4782960</td>
</tr>
<tr>
<td>10</td>
<td>9999990</td>
</tr>
</tbody>
</table>

Thus, for example \(a^7 - a\) is a multiple of 7 for any \(a\), as the following table indicates:

As we mentioned above, Euler was the first one to prove this important theorem. He also spent considerable energy in the generalization of it. Both Fermat's original and Euler's generalization are used every day today by both industry and government. We will see Euler's argument for this theorem.

Note the theorem as stated does not necessarily hold if \(p\) is not a prime: \(3^4 - 3 = 78\), which is not a multiple of 4.
Euler’s proof of Fermat’s Little Theorem is by induction, and it is very nice. First we discuss the crux of the argument. Consider an arbitrary binomial coefficient: \( \binom{m}{k} \) where \( m \) and \( k \) are arbitrary nonnegative integers. The question Euler needed to answer is when is such a coefficient \( \binom{m}{k} \) a multiple of \( m \). We know \( \binom{m}{0} = \binom{m}{m} = 1 \). But how about all the other coefficients?

For example, if \( m = 5 \), \( \binom{5}{1} = \binom{5}{4} = 5 \) and \( \binom{5}{2} = \binom{5}{3} = 10 \), all multiples of 5. However, it is not always so, \( \binom{4}{1} = 6 \), which is not a multiple of 4. Let us gather some more evidence.

How about \( \binom{17}{12} = \frac{17!}{12!5!} \)? This is a multiple of 17 since the numerator is a multiple of 17, and never will that factor be canceled by the denominator, since 17 is prime, and neither 12! nor 5! will have a factor of 17, and so it does not occur in the denominator. We will appeal to the intuition and state that \( \binom{m}{k} \) is a multiple of \( m \), for any \( 1 < k < m \) if \( m \) is a prime number. With that in hand we set out to prove:

**Theorem. Fermat’s Little Theorem.** If \( p \) is a prime, then \( n^p - n \) is a multiple of \( p \).

*Proof.* We proceed by induction on \( n \). If \( n = 1 \), it is trivial to aver that \( 1^p - 1 \) is a multiple of \( p \). So assume it holds for \( n \), so that \( n^p - n \) is a multiple of \( p \). We need to prove that \( (n+1)^p - (n+1) \) is also a multiple of \( p \). But by the binomial theorem we know that

\[
(n + 1)^p - (n+1) = n^p + \binom{p}{1}n^{p-1} + \binom{p}{2}n^{p-2} + \binom{p}{3}n^{p-3} + \cdots + \binom{p}{p-2}n^2 + \binom{p}{p-1}n + 1 - (n+1) =
\]

\[
= n^p + \binom{p}{1}n^{p-1} + \binom{p}{2}n^{p-2} + \binom{p}{3}n^{p-3} + \cdots + \binom{p}{p-2}n^2 + \binom{p}{p-1}n + 1 - n - 1 =
\]

\[
= n^p + \binom{p}{1}n^{p-1} + \binom{p}{2}n^{p-2} + \binom{p}{3}n^{p-3} + \cdots + \binom{p}{p-2}n^2 + \binom{p}{p-1}n - n - (n^p - n) =
\]

\[
= \binom{p}{1}n^{p-1} + \binom{p}{2}n^{p-2} + \binom{p}{3}n^{p-3} + \cdots + \binom{p}{p-2}n^2 + \binom{p}{p-1}n .
\]

Now, \( n^p - n \) is a multiple of \( p \), by the induction hypotheses, and

\[
\binom{p}{1}n^{p-1} + \binom{p}{2}n^{p-2} + \binom{p}{3}n^{p-3} + \cdots + \binom{p}{p-2}n^2 + \binom{p}{p-1}n
\]
is also a multiple of $p$ since by our remark above, each of the coefficients, $\binom{p}{1}, \binom{p}{2}, \binom{p}{3}, \ldots, \binom{p}{p-2}, \binom{p}{p-1}$ is a multiple of $p$, and our argument is done.

Euler eventually generalized this theorem to the non-prime case. In the long process, Euler developed the function that is referred to as the Euler $\varphi$-function. But to dwell much more on this lovely subject would take us too far afield. However, it should be pointed out Fermat's Little theorem can be restated in the form:

*if $n$ is relatively prime to the prime $p$, then $n^{p-1} - 1$ is a multiple of $p$.*

Euler's theorem is then a generalization of this statement:

*if $n$ is relatively prime to $k$, then $n^{\varphi(k)} - 1$ is a multiple of $k$.*

It is only natural that while reading the Arithmetica, Fermat would become interested in **Pythagorean triples**. As mentioned when we discussed both the Babylonians and Diophantus, if we take any $\alpha$ and $\beta$, $\alpha > \beta$, relatively prime, with exactly one of them even and one of them odd, then $a$, $b$ and $c$ given by: $a = 2\alpha\beta$, $b = \alpha^2 - \beta^2$ and $c = \alpha^2 + \beta^2$ form a Pythagorean triple. For example,

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
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<td>4</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
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<td>1</td>
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<td>15</td>
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<td>16</td>
<td>63</td>
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<td>41</td>
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<tr>
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<td>4</td>
<td>56</td>
<td>33</td>
<td>65</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>60</td>
<td>11</td>
<td>61</td>
</tr>
</tbody>
</table>

But then:

**Which primes then can be the legs of a right triangle?**

**Which primes can be the hypotenuse of such a triangle?**

Fermat answered both questions. We can see that if a prime is going to be the hypotenuse of a right triangle, then it has to be able to be written as the sum of two squares, and vice versa.

**\textbf{3} \textbf{ Pythagorean triples.}\)**

1. Every odd prime is the leg of a unique right triangle.
2. An odd prime is the hypotenuse of a right triangle if and only if it is of the form $4n + 1$. If this is the case, then there is exactly one such right triangle.
3. If $p = 4n + 1$ is a prime, then $p^m$ is the hypotenuse of exactly $m$ right triangles.

The table above provides examples of all three situations. First for 1: 3 is a leg in the (4,3,5) triangle, 5 in the (12,5,13), 7 in the (24,7,25), 11 in the (60,11,61). For 2, 3 never shows as a hypotenuse, while 5 occurs in (4,3,5), 11 does not show, but 13 does, 19 does not, but 17, 29, 41 and 53 do. For 3, we only have one example, $25 = 5^2$, and indeed 25 occurs as the hypotenuse of two triangles: (20,15,25) and (24,7,25)—the first one triple does not consist of relatively prime numbers.
We will manufacture another example: 125 = 5\(^3\) does appear in three triangles: (100,75,125), (120,25,125) and (44,117,125), the first two not being primitive.

We do not have the machinery to prove ②, nor ③, but we can prove ① (which was proven by Fermat himself).

**Proof of ①.** Let \(p\) be an odd prime. In order for \(p\) to be the leg of a triple, we must have \(p = \alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)\), but since \(p\) is prime, the only way to factor it is: \(\alpha + \beta = p\) and \(\alpha - \beta = 1\), and so \(\alpha = \frac{p+1}{2}\) and \(\beta = \frac{p-1}{2}\) constitute the unique possibility.

Another topic that intrigued Fermat was indeed Pythagorean: triangular, square pentagonal, … numbers. Recall that each of these types of numbers was so named because of their shape. Below we give the first fifteen values of each of the sequences:

| Triangular | 1 3 6 10 15 21 28 36 45 55 66 78 91 105 120 |
| Square     | 1 4 9 16 25 36 49 64 81 100 121 144 169 196 225 |
| Pentagonal | 1 5 12 22 35 51 70 92 117 145 176 210 247 287 330 |
| Hexagonal  | 1 6 15 28 45 66 91 120 153 190 231 276 325 378 435 |
| Heptagonal | 1 7 18 34 55 81 112 148 189 235 286 342 403 469 540 |
| Octagonal  | 1 8 21 40 65 96 133 176 225 280 341 408 481 560 645 |
| Nonagonal  | 1 9 24 46 75 111 154 204 261 325 396 474 559 651 750 |
| Decagonal  | 1 10 27 52 85 126 175 232 297 370 451 540 637 742 855 |

Fermat made the following wonderful observation:

③ **Figurate Numbers.**

Any number can be written as the sum of at most three triangular numbers.

Any number can be written as the sum of at most four square numbers.

Any number can be written as the sum of at most five pentagonal numbers.

And in general,

Any number can be written as the sum of at most \(n\) numbers of shape \(n\).

**Examples:**

\(\Delta:\) \(5 = 3 + 1 + 1, \ 8 = 1 + 1 + 6, \ 19 = 15 + 3 + 1.\)

\(\Box:\) \(7 = 1 + 1 + 1 + 4, \ 95 = 81 + 9 + 4 + 1, \ 105 = 64 + 36 + 4 + 1 = 100 + 4 + 1.\)

\(\bigstar:\) \(9 = 1 + 1 + 1 + 1 + 5, \ 11 = 5 + 5 + 1, \ 20 = 12 + 5 + 1 + 1 + 1.\)

Fermat did not prove any part of this claim, and ironically the hardest to prove \(\Delta\) are \(\Box\) and which were proven much later by Gauss, Euler and Lagrange, among others.

We end our discussion of Fermat with what had become one of the most celebrated open problems in mathematics. To motivate it, we again go back to Pythagorean triples: the
basic equation to be satisfied is: \( a^2 + b^2 = c^2 \), and we have found that there is an unbounded number of solutions. What happens if we increase the exponent? We find that solutions are not easy to find—indeed none have been found. Fermat himself on the margin of the Arithmetica claimed he had an admirable proof, but the margin is too narrow to contain it of the fact that there were no solutions. However, no one had been able to prove it, although many have tried, until 1993. Hence this became known as:

\[ \text{Fermat's Last Theorem (I).} \]

The equation \( x^n + y^n = z^n \) has no solutions in positive integers if \( n \geq 3 \). This had been verified for large \( n \), but no proof was known for over 300 years. Fermat himself settled the case \( n = 4 \). Euler independently settled \( n = 3 \) and 4. Dirichlet did 5 and Lamè, \( n = 7 \). Finally, in the summer of 1993, an announcement was made by Andrew Wiles of Princeton University that a proof was available. After a few minor repairs, the proof of more than 280 pages is now widely accepted as valid.

We now leave France to go to its Northern neighbor.

\[ \text{Huygens} \]

Although analytic geometry came out of France, the first best textbooks on the subject came out of Holland where great teachers such as Hudde and van Schooten mastered and explained the subject.

The latter's best student was Christian Huygens (1629-1695), who after his education moved to Paris where he met or corresponded with all the best scientists and mathematicians of the times. He has a place of prominence in science since he was an early advocate of the wave theory of light (in contrast to Newton's particle version), as well as the discoverer of one of Saturn's satellites, Titan. He followed Galileo, and he preceded Newton and Leibniz.

One of his major contributions to mathematical literature was, possibly the first book written on probability, De Ratiociniis in Ludo Alae (On Reasoning at Games of Chance), (1657), which quickly became the major textbook on the subject for the next fifty years. We will discuss this work first.

One crucial concept that shows up in Games clearly delineated is mathematical expectation (although Huygens referred to it by just value of the game).

We look at one of the problems in the book. To find how many times one may wager to throw a six with one die.
What is he asking? Suppose the wager is $a$ pieces of gold, and our main player will win it if at least one 6 is rolled when he rolls a dice $n$ times. What is the value of this game to our main player? Huygens works the problem recursively. Let $P_n$ denote the probability of winning when $n$ dice are rolled. So $P_1 = \frac{1}{6}$, and if we let $E_1$ denote his expectation in one throw, then we have $E_1 = \frac{a}{6}$. What is his expectation in two throws? He can expect $\frac{a}{6}$ from the first throw. With probability $\frac{5}{6}$, he gets to the second throw, which as we saw above is worth $\frac{a}{6}$ to him, so the second throw is worth $\left(\frac{5}{6}\right)\left(\frac{a}{6}\right)$ to him, so in total the value is $E_2 = \frac{a}{6} + \frac{5a}{36} = \frac{11a}{36}$, or equivalently, $E_2 = E_1 + \frac{5}{6}E_1 = E_1 + (1 - P_1)E_1$ and $P_2 = P_1 + (1 - P_1)P_1 = \frac{11}{36}$. What about three throws? The first two throws are worth $\frac{11a}{36}$ to him and he has $\frac{25}{36}$ of a chance to get to the third roll, which is worth $\frac{a}{6}$, for a total of $E_3 = E_2 + (1 - P_2)E_1 = \frac{11a}{36} + \left(\frac{25}{36}\right)\left(\frac{a}{6}\right) = \frac{91a}{216}$, and $P_3 = \frac{91}{216}$.

Similarly, $P_4 = P_3 + (1 - P_3)P_1 = \frac{671}{1296}$ and $E_4 = \frac{671a}{1296}$. Certainly, we can see similarities between this reasoning and Pascal's in the previous section. In modern days, we are more likely to argue in Fermat's style by counting directly the number of ways of not rolling a 6 in $n$ rolls, $\frac{5^n}{6^n}$, and so $P_n = 1 - \frac{5^n}{6^n}$, directly.

That the general public was rather unsophisticated when it came to probabilities and expectation can be appreciated since the following scam was somewhat popular.

Since one has a chance in six of rolling a 1 with a die, one has an even chance to roll at least one 1 when one rolls 3 dice. Hence I propose the following game to you. You will roll 3 dice. If three 1’s show up you win a wonderful $5$, if only two 1’s, you win $2$, while if only one 1 is rolled, you will still win $1$. If, unfortunately, on the other hand, no 1’s show up you pay me only $1$.

Naturally, you are suspicious of my proposition, but it is much better to pin point the reasons for your suspicions. What we need is to compute your expected value when you play this game—this is equivalent to what your average performance is going to be. Of course, if you are going to play this game just once, then it does not matter what you opt to do, but as a long range strategist, you need to compute.
The computation is just common sense. You are basically asking: **suppose I played the game so many times, what would happen?** In any one roll, you can win either 5, 2 or 1 dollars, or you can lose 1. Let’s say you played 216 times. Then on the average, three 1’s would show up once, while two 1’s would show up 15 times (\(15 = 3 \times 5\)), the 3 is the number of options of which two dice are going to show the two 1’s while the 5 is what the other dice is going to show). How many times does one 1 show up? Choose which die shows the 1 (3 options), and then choose what the other two dice show (5 × 5) for a total of 75 times. Finally, no 1’s will show 125 times (which is \(5 \times 5 \times 5 = 216 - 1 - 15 - 75\)).

So what are your winnings:

\[
5 \times 1 + 2 \times 15 + 1 \times 75 - 1 \times 125 = -15
\]

or equivalently, **your expectation** is

\[
5 \left( \frac{1}{216} \right) + 2 \left( \frac{15}{216} \right) + 1 \left( \frac{75}{216} \right) - 1 \left( \frac{125}{216} \right) = -\frac{15}{216}.
\]

So on the average you will lose $15 in 216 rolls. Naturally, you would rather not play a game when your expectation is negative, unless you will have so much fun you are willing to **pay the fee**.

In general, the computation for the expected value to a game consists of listing the different outcomes together with their probabilities, multiplying each outcome by its probability and then adding all these up.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
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<tbody>
<tr>
<td>🔄 ▪ ▪</td>
<td>(\frac{1}{216})</td>
<td>$5</td>
</tr>
<tr>
<td>🔄 ▪ ▪</td>
<td>(\frac{15}{216})</td>
<td>$2</td>
</tr>
<tr>
<td>▪ ▪ ▪</td>
<td>(\frac{75}{216})</td>
<td>$1</td>
</tr>
<tr>
<td>▪ ▪ ▪</td>
<td>(\frac{125}{216})</td>
<td>-$1</td>
</tr>
</tbody>
</table>

Huygens was also an inventor of the pendulum clock, on which he wrote, in 1673, **Horologium Oscilatorium (The Pendulum Clock)**. The second half of our investigation of Huygens involves his pursuit of the clock. So we take the opportunity to review a bit of its history up to this time. But for mathematics lovers, the most important is the **Pendulum Clock**. The image of a young **Galileo** using his pulse to time the swing back and forth of a hanging chandelier inside the cathedral in Pisa is well known.

Also well known are his conclusions:

1. **A pendulum is isochronous**: For a fixed length, the period of its oscillation does not depend on its amplitude.

2. **The square of the period is proportional to the length**.
Later on in life, with the tremendous need for an accurate portable clock for navigational purposes, Galileo would spend considerable time and energy in the development of a pendulum clock, but he would not succeed before his death. Instead it was Huygens who built it.

In his pursuit of more accurate clocks, Huygens became involved with a very favorite curve of his (and of many others), the cycloid (so popular, and quarrelsome, it has been called the Helen of Geometers). Ironically, the cycloid was baptized by Galileo—it had been called the roulette previously. Huygens’ attention was drawn to it by a contest in 1658 announced by none other than Pascal. Pascal participated under a pseudonym, and performed the best, but Huygens also did very well. He placed second in the contest.

As well known, the cycloid is defined as the path that a point on the circumference of a circle follows when the circle is rolled along a line. We are looking at a picture of an inverted cycloid, which is the more relevant to our discussion.

Some of the properties of the cycloid had been known for some time. Galileo himself had estimated the area of one arch of the cycloid to be three times the area of the generating circle by the use of wooden models.

Another property that was known was how to draw the tangent line to a cycloid—as done by the Frenchman Roberval. Namely, if a point A in the cycloid is given. Then we can draw the tangent to the cycloid at A by first finding the point B, the lowest point in the generating circle, and then joining A and B.

The proof that this is so is easy. The tangent to the cycloid at A is the sum of two vectors: one in the direction of the tangent to the circle and one in the horizontal direction both toward the direction of the roll, and the length of the two vectors is the same since the circle is rolling without slipping.

Thus, all we need is to argue that the parallelogram formed with a horizontal side, and the circle tangent as another side, and the segment AB as a diagonal is a rhombus. But since B is the lowest point in the circle, the line CB is the tangent to the circle, and so the length of CA is the same as the length of CB, and thus we have a rhombus.

One of the results from Pascal’s contest was the rectification of the cycloid:
the length of one arch being 4 times the diameter of the generating circle.

It was considered quite remarkable since rectification of curves was considered almost impossible. One other example, the semicubical parabola, \( y^2 = x^3 \), was also rectified by several mathematicians, including Fermat. Newton among others will become interested further in rectification. Huygens further development of involutes and evolutes (which will be discussed below) clarified why these two curves were so readily rectified.

Although Pascal’s contest had additionally asked some very difficult pre-calculus questions about the cycloid, Huygens soon became interested in a different property of the cycloid—or more appropriately, as we have been drawing it, the inverted cycloid, or upside-down cycloid as he referred to it.

He knew that Galileo’s claim concerning the independence of the period from the amplitude was only approximately so, in other words,

the length of the oscillation of a pendulum clock depended on the angle of the pendulum,

and that made clocks inherently inaccurate (although slightly so).

In pursuing a better path for a pendulum to follow, other than a circular path, he became interested in the tautochrone problem. The latter inquires what path can a bead follow (ignoring friction) so no matter where in the path the bead is dropped, it takes the same amount of time to reach the bottom, by the pure action of gravity. The answer is the cycloid. And that is what we prove next.

We will give some of Huygens original argumentation and we will see how his arguments are inherently geometrical in nature, using analytic geometric, but in a much more synthetic fashion.

We will start with the conservation of energy equation: namely, we will exploit the fact that kinetic energy (given by \( \frac{1}{2}mv^2 \)) added to potential energy (given by \( gmh \)) will remain constant throughout the fall:

\[
\frac{1}{2}mv^2 + gmh = \text{constant},
\]

where \( g \) is the gravitational constant, \( v \) is velocity and \( h \) denotes the height of the object. Clearly, \( v \) and \( h \) are variables depending on time \( t \).

We will consider an object of mass one (\( m = 1 \)) being dropped (so \( v(0) = 0 \)) at \( t = 0 \) at
the point on the cycloid with height $H$.

By substituting $t = 0$ in ①, we can evaluate our constant to be $g$. Hence, ① becomes

$$|v(t)| = \sqrt{2g(H - h(t))} \tag{2}$$

where the direction of the velocity at the point $A$ is the same as the direction $AB$ of the tangent line to the cycloid at that point.

We can evaluate the vertical component $v_y(t)$ of $v(t)$ as follows:

$$v_y(t) = |v(t)| \cos(\alpha), \tag{3}$$

where $\alpha = \angle CAB$.

But also because of the following picture:

we get $|AB| = 2r \cos(\alpha)$ and $|AB| = \frac{h(t)}{\cos(\alpha)}$, and thus we obtain

$$h(t) = 2r \cos^2(\alpha), \text{ which gives } \cos(\alpha) = \sqrt{\frac{h(t)}{2r}} \tag{4}$$

Substituting ② and ④ in ③, we have then

$$v_y(t) = \frac{g}{r} \sqrt{h(t)(H - h(t))} \tag{5}$$

We could then treat this as a simple differential equation:

$$h'(t) = \frac{g}{r} \sqrt{h(t)(H - h(t))}$$

but Huygens, a precalculus, geometric thinker, found another way to deal with ⑤.

Consider a circle with diameter $H$ and consider a particle starting at the top, traveling around that circle with uniform speed $|w|$, ($w$ unspecified for now) and arriving at point $\hat{A}$ at the same height as $A$--height $h(t)$--at time $t$.

Let us find the vertical component $w_y$ of $w$ at that point. From this picture:

$$w_y = |\hat{A}B|,$$

which equals

$$w_y = |w| \cos(\angle B\hat{A}C).$$
If we let \( O \) be the center of our circle with diameter \( H \), and we let \( P \) be the point on the diameter on the horizontal with \( \hat{A} \) (or \( A \)), then since \( OA \) is perpendicular to \( \hat{A}C \), and \( AP \) is perpendicular to \( \hat{A}B \), we have

\[ \angle OAP = \angle \hat{BA}C, \text{ and so} \]

\[ w_i = \frac{2|AP|}{H}. \]

By the Pythagorean Theorem,

\[ (|\hat{A}P|)^2 = \left(\frac{H}{2}\right)^2 - (|OP|)^2 = \left(\frac{H}{2}\right)^2 - \left(\frac{H}{2} - h(t)\right)^2 = h(t)(H - h(t)). \]

Thus we have

\[ w_i = \frac{2|w|}{H} \sqrt{h(t)(H - h(t))}. \]

If we choose \( 2|w| = H \sqrt{\frac{g}{r}} \), then \( w_i = h_i \). Thus the motions of the two particles are equivalent, and the particle moving on the cycloid will reach bottom at the same time as the particle traveling around the circle. But the latter will reach bottom when it has traveled the length of the semicircle, \( \frac{\pi H}{2} \), and since its speed is \( |w| = \frac{H}{2} \sqrt{\frac{g}{r}} \), this will occur at time:

\[ T = \pi \frac{r}{g}, \text{ independent of } H, \text{ and we have the cycloid to be a tautochrone.} \]

But, Huygens still had work to do. In order to get the pendulum to swing on the path of a cycloid, he had to build clocks with cheeks, and he had to find the shape of the cheeks that would force the path of the cycloid on the pendulum.

With this purpose in mind he developed the notion of involute and evolute. We will only glimpse at these concepts, giving no proofs of any of his propositions. What are involutes and evolutes that he used so effectively? In his own words:

\begin{center}
\textbf{When we consider that a thread or flexible line is laid along a curve concave to one side, and when we remove one end from it while the other end of the thread stays on the curve in such a way that the developed part remains taut, then this end of the thread will clearly describe another curve; this curve is called an involute.}
\end{center}

Involutes are sometimes better referred to as evolvents.
For example, if we consider the circle \((\cos(t), \sin(t))\) then its evolvent is given by
\[
(\cos(t) + t \sin(t), \sin(t) - t \cos(t))
\]
The first proposition he proves then is:

**Every tangent of the evolute intersects the involute at right angles.**

He later proved the converse of the first proposition, and also proves that the involute of a cycloid is also a cycloid.

In fact, he used all his studies of the cycloid to build clocks which were more accurate in theory, but which in practice, unfortunately, provided no discernible difference in accuracy. Later on, he would work on spring clocks, but he would not succeed in building one.

Finally, we cannot mention the tautochrone and time, and not discuss briefly another famous problem associated with free fall and time, the Brachistochrone and two Swiss brothers of the eighteenth century associated with the problem.

The most prestigious family in mathematical history is the Bernoulli’s. The two founding members of the family are the brothers Jakob (or Jacques or James) and Johann (or Jean or John).

**James Bernoulli** (1654-1705) taught at the University of Basel, the same city where Euler is going to be born. Although his mathematical interests were very wide (typical of the age), his most lasting influence was in the field of probability and statistics—indeed he is considered one of the first statisticians in history. In 1713, eight years after his death, his nephew Nicholas (son of John) published, posthumously, James’ masterpiece: *Ars Conjectandi*, an important book, the heir to Huygens’, and the predecessor to Laplace’s. In it, he tackles the theorem known sometimes as the (weak) Law of Large Numbers, or as the Law of Averages, or sometimes as Bernoulli’s Theorem.

But it is his younger brother’s name most closely associated with the Brachistochrone. The name of the younger John Bernoulli (1667-1748) would be even better known (than already is) if the theorem he provided L’Hôpital to include in the latter's calculus book had received the proper reference of the creator of the theorem as opposed to the writer of the textbook.

**John** will follow his brother at the University where he will teach the young Euler, who
will befriend John’s children. As with his brother, John’s interests were very wide, but we concentrate on a topic close to his heart (and his brother’s): free fall:

*Consider a particle falling from one point to another on a prescribed path, ignoring friction and all other forces except gravity. How much time will it take?*

**Bernoulli** was interested in finding the path that would take the shortest time, and this is known as the **Brachistochrone Problem** (minimum time). Amazingly, the answer to the Brachistochrone problem turns out to be the fascinating cycloid.

In John’s own words:

*With justice we admire Huygens because he first discovered that a heavy particle falls on a cycloid in the same time always, no matter what the starting point may be. But you will be petrified with astonishment when I say that exactly this same cycloid, the tautochrone of Huygens, is the Brachistochrone which we are seeking.*

To prove that the cycloid is the shortest path among all possible paths is a very different kind of problem, and new machinery, called the *calculus of variations*, was developed to tackle this kind of problem.

It is an indicator of the poor state of the nature of argumentation at that time in mathematics that when John gave an argument that the cycloid was indeed the solution to the Brachistochrone problem, James would not accept it, nor could he convince John as what was wrong with the argument, and the dispute led to the eventual alienation of the two brothers. A shame indeed.

We end this interesting period with **Wallis**

**John Wallis** (1616-1703) wrote the *Arithmetica Infinitorum* (The Arithmetic of the Infinite), a book that Isaac Newton paid close attention to, and was influenced by. The book reflects Wallis taste for patterns, their recognition, and his, perhaps too aggressively, ability to extend them.

We saw earlier in the chapter that the integration formula

\[ \int_0^1 x^n dx = \frac{1}{n+1} \]

where \( n \) is a positive integer, was available at the time. Certainly, Wallis was one of the contributors to the development of such a formula. More importantly, he extended it to other powers, especially fractions. The first observation he made was simple and well
He wanted to compute \( \int_0^1 x^q \, dx \) where \( q \) is a positive integer. He did an easy, yet creative step. What does this integral represent? The area under the curve \( y = x^q \) in the interval \([0, 1]\). It is the darkened area in the picture. But we know the area of the square, 1 and what is the white area in the picture? By integrating on the \( y \)-axis, we see that the white area is

\[
\int_0^1 y^q \, dy = \frac{1}{q+1},
\]

and thus

\[
\int_0^1 x^q \, dx = 1 - \frac{1}{q+1} = \frac{q}{q+1},
\]

and recognizing the pattern that held for both positive integers and their reciprocals, Wallis aggressively, but correctly, conjectured

\[
\int_0^1 x^q \, dx = \frac{1}{\frac{p+1}{q}} = \frac{q}{p+q}
\]

for any positive integers \( p \) and \( q \).

We give another example of Wallis talent, and taste. It again reflects his desire to extend a pattern from the positive integers to fractions, and he is later proven correct, however, at the time of his life, he was criticized for extending patterns without justification.

Here is another pattern to be recognized:

\[
\int_0^1 (x - x^2)^1 \, dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},
\]

\[
\int_0^1 (x - x^2)^2 \, dx = \frac{1}{3} - \frac{2}{4} + \frac{1}{5} = \frac{40 - 60 + 24}{1 \times 2 \times 3 \times 4 \times 5} = \frac{4}{5!} = \frac{2^2}{5!} = \frac{(2!)^2}{5!},
\]

\[
\int_0^1 (x - x^2)^3 \, dx = \frac{1}{4} - \frac{3}{5} + \frac{3}{6} - \frac{1}{7} = \frac{1260 - 3024 + 2520 - 720}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} = \frac{36}{7!} = \frac{6^2}{7!} = \frac{(3!)^2}{7!},
\]

\[
\int_0^1 (x - x^2)^4 \, dx = \frac{576}{9!} = \frac{24^2}{9!} = \frac{(4!)^2}{9!}.
\]

So we can speculate that

\[
\int_0^1 (x - x^2)^n \, dx = \frac{(n!)^2}{(2n+1)!}
\]

for any positive integer \( n \).

Let’s extend the pattern to \( n = \frac{1}{2} \). Then we would have
\[ \int_0^1 (x-x^2)^{\frac{1}{2}} \, dx = \int_0^1 \sqrt{x-x^2} \, dx = \left( \frac{1}{2} \right)! = \frac{1}{2}. \]

But this integral is known, since \( \sqrt{x-x^2} \) is a semicircle:

with center \( \left( \frac{1}{2}, 0 \right) \), and radius \( r = \frac{1}{2} \).

Thus \( \int_0^1 \sqrt{x-x^2} \, dx = \frac{\pi}{8} \). From which he concluded, correctly, but daringly, that

\[ \left( \frac{1}{2} \right)! = \frac{\sqrt{\pi}}{2}. \]

With a similar extension of another pattern of integrals from whole numbers to fractions, Wallis derived his formula:

\[ \frac{\pi}{2} = \frac{2 \times 2 \times 4 \times 4 \times 6 \times \cdots}{1 \times 3 \times 3 \times 5 \times 5 \times \cdots}. \]
Chapter 14
The Birth of Calculus

He who can digest a second or third fluxion, a second or third difference, need not, methinks, be squeamish about any point in Divinity. And what are these fluxions? The velocities of evanescent increments. And what are these evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them ghosts of departed quantities?—Berkeley

It is almost universally agreed upon that the two characters we encounter in this chapter, Isaac Newton (1642-1727) and Gottfried Leibniz (1646-1716) are the discoverers—creators of calculus. But what do we mean by this? We have seen in the previous chapter that derivatives were already known and so was their connection with tangents and with the extremal values of functions. In addition, the areas under curves of varied complexity had been computed by basically doing Riemann sums integration. Finally, the connection between the two processes of integration and differentiation had been foreseen, and Newton had been exposed to it from Barrow’s lectures. One could say that Newton and Leibniz did understand thoroughly the fundamental theorem of calculus (as we call it today), and also both appreciated the power and range of the subject. Certainly, Newton used Calculus-type thinking to push the frontiers of mechanics and physics.

European History Highlights

<table>
<thead>
<tr>
<th>Date</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>1660</td>
<td>Restoration Age—Charles II becomes king of England.</td>
</tr>
<tr>
<td>1662</td>
<td>John Graunt publishes Bills of Mortality.</td>
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<tr>
<td>1663</td>
<td>Seven Year War ends—Canada is ruled by Britain.</td>
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<tr>
<td>1665</td>
<td>Great Plague of London.</td>
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<tr>
<td>1666</td>
<td>National Observatory of Paris is founded. Great Fire of London.</td>
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<tr>
<td>1675</td>
<td>Greenwich Observatory is established.</td>
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<tr>
<td>1682</td>
<td>Peter, the Great begins his reign in Russia.</td>
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<tr>
<td>1685</td>
<td>Huguenots are persecuted in France. J. S. Bach is born.</td>
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<tr>
<td>1686</td>
<td>Fahrenheit is born.</td>
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<tr>
<td>1688</td>
<td>Glorious Revolution in England—William and Mary become monarchs.</td>
</tr>
<tr>
<td>1694</td>
<td>Voltaire is born.</td>
</tr>
<tr>
<td>1706</td>
<td>Benjamin Franklin is born.</td>
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</tbody>
</table>

What is the Fundamental Theorem of Calculus? Most everyone would agree that essentially it is about the connection between the tangents of the graph of one function and the area under the curve of another. This connection will be clearer after we discuss Leibniz’s work.
Like Fermat, Gottfried Leibniz, was not a mathematician by trade. He was a diplomat who traveled widely, and as such came to meet and discuss mathematics with all the best-known mathematicians and scientists of his time, including Huygens and Newton.

It is, in a sense, unfortunate that Leibniz met and corresponded with Newton, since he will, many years after the meeting, be accused of plagiarizing his ideas on calculus from Newton. A long, scandalous dispute followed, and although his name was eventually cleared, the dispute left a bitter taste in the soul of mathematicians on both sides of the English Channel. This led to a partial isolation of English mathematicians from those in the Continent, where calculus, and its consequent disciplines such as differential equations, will explode into a massive and powerful discipline. Leibniz would die unbeknownst to the world and in relative poverty.

Leibniz developed much of our modern notation for calculus such as \( \frac{dy}{dx} \) and \( \int \) and it is this notation (as opposed to Newton's fluxion notation) that will be adopted in the rest of Europe.

It is fortunate, however, that Leibniz met Huygens, since it is a question posed to him by Huygens that possibly stimulated Leibniz's discovery of the connection between integration and differentiation.

Huygens asked what the sum of the reciprocal of the triangular numbers added to:

\[
1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \cdots = ?
\]

Recall Oresme. As it turned the answer was already known, but unknown to Leibniz, he plunged ahead into the problem.

He understood that from a given sequence: \( \alpha : a_1, a_2, a_3, a_4, \ldots \), one could obtain two other ones, the difference and the sum. The first one of these: the difference, \( \Delta(\alpha) \), is defined as follows, \( \Delta(\alpha) : b_1, b_2, b_3, \ldots \) where: \( b_1 = a_2 - a_1 \), \( b_2 = a_3 - a_2 \), \( b_3 = a_4 - a_3 \), etcetera. Thus, for example, if \( \alpha \) is the sequence of triangular numbers, \( \alpha : 1, 3, 6, 10, 15, \ldots \) then \( \Delta(\alpha) \) is 2,3,4,5,\ldots.
The sum (or series), $\Sigma(\alpha)$, is defined as follows: $\alpha$, $a_1 + a_2$, $a_1 + a_2 + a_3$, $a_1 + a_2 + a_3 + a_4$, etc. For example, if we start with the simple sequence 1, 1, 1, 1, ... and take consecutive sums of it, we obtain the following configuration:

$$
\begin{array}{cccccccccccc}
\alpha: & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\Sigma(\alpha): & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\Sigma^2(\alpha): & 1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45 & 55 & 66 & 78 \\
\Sigma^3(\alpha): & 1 & 4 & 10 & 20 & 35 & 56 & 84 & 120 & 165 & 220 & 286 & 364 \\
\Sigma^4(\alpha): & 1 & 5 & 15 & 35 & 70 & 126 & 210 & 330 & 495 & 715 & 1001 & 1365 \\
\end{array}
$$

If we think of this as a matrix with the rows and columns labeled by 0, 1, 2, ..., then we see that we are dealing with Pascal's triangle since the $(i, j)$ entry is nothing but $\binom{i+j}{i}$ (it is in this form that Pascal originally wrote his triangle).

In the context of sequences, the relation between the difference and the sum is easily understood. If $\alpha: a_1, a_2, a_3, a_4, ...$. Then

$$
\Delta(\Sigma(\alpha)) = a_2, a_3, a_4, ...
$$

which is almost $\alpha$ (all we would need to recover $\alpha$ is attach $a_1$ at the beginning. Also

$$
\Sigma(\Delta(\alpha)) = a_2 - a_1, a_3 - a_1, a_4 - a_1, ...
$$

and again $\alpha$ is easily recoverable from this sequence, once $a_1$ is given.

Leibniz set out to a parallel construction to the triangle above. But instead of consecutive sums he took consecutive differences, but since he was starting with a decreasing sequence, we need to modify the difference to mean:

$$
b_1 = a_1 - a_2, b_1 = a_1 - a_2, b_2 = a_2 - a_3, b_3 = a_3 - a_4, ... \text{ et cetera.}
$$

His first (row) sequence was the sequence of harmonic numbers: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$

$$
\begin{array}{cccccccccccc}
\alpha: & 1 & 1/2 & 1/3 & 1/4 & 1/5 & 1/6 & 1/7 & 1/8 & 1/9 & 1/10 & 1/11 & 1/12 \\
\Delta(\alpha): & 1/2 & 1/6 & 1/12 & 1/20 & 1/30 & 1/42 & 1/56 & 1/72 & 1/90 & 1/110 & 1/132 & 1/156 \\
\Delta^2(\alpha): & 1/3 & 1/12 & 1/30 & 1/60 & 1/105 & 1/168 & 1/252 & 1/360 & 1/495 & 1/660 & 1/858 & 1/1092 \\
\Delta^3(\alpha): & 1/4 & 1/20 & 1/60 & 1/140 & 1/280 & 1/504 & 1/840 & 1/1320 & 1/1980 & 1/2860 & 1/4004 & 1/5460 \\
\Delta^4(\alpha): & 1/5 & 1/30 & 1/105 & 1/280 & 1/630 & 1/1260 & 1/2310 & 1/3960 & 1/6435 & 1/10010 & 1/15015 & 1/21840 \\
\end{array}
$$

Then he easily observed that the second row consisted of the halves of the reciprocals of the triangular numbers. Hence if we let $\alpha: a_1, a_2, a_3, a_4, ...$ and $\Delta(\alpha) := b_1, b_2, b_3, ...$, then we know $b_1 = a_1 - a_2, b_2 = a_2 - a_3, b_3 = a_3 - a_2, ...$. So, in a similar fashion to the discussion above, $\Sigma(\Delta(\alpha)) := a_1 - a_2, a_1 - a_3, a_1 - a_4, ...$. So the sum of all the reciprocals of the triangular numbers, which is

$$
\lim_{n \to \infty} \Sigma(\beta) = \lim_{n \to \infty} \Sigma(\Delta(\alpha)) = 2a_1 = 2
$$
since \( a_n \to 0 \) as \( n \to \infty \).

This was very exciting to him, since he realized he could adequately add one sequence by simply taking differences of another. Although certainly that reminds one of the basic ideas behind the fundamental theorem of calculus, it did not become real calculus until he pushed it further, and this is what we look at next.

In the 1670's, Leibniz discovered a general principle or technique to evaluate areas, which he referred to as transmutation of areas. A technique basically equivalent to the Fundamental Theorem of Calculus, which we now give in some detail, and in, more or less, modern notation and ideology.

Suppose we have an interval \([a,b]\) and we have a function \( y = f(x) \) defined on this interval. We are interested in the area under the curve of this function.

![Diagram of area under curve]

Start by considering two neighboring (very close) points \( P \) and \( Q \) on the graph of this function, where \( P \) has coordinates \( x \) and \( y = f(x) \), and \( Q \)'s coordinates are \( x + dx \) and \( y + dy \), where \( dx \) is a small change in \( x \), and \( dy \) is the corresponding small change in \( y \). We will let \( O \) denote the origin.

Continuing in the language of indivisibles, we let the length of the curve from point \( P \) to point \( Q \) be denoted by \( ds \) (recall that \( s \) usually denotes length of a curve in Calculus).

Consider the tangent line to the curve at the point \( P \) and suppose it intersects the \( y - \)axis at \( T = (0,z) \).

Since the right triangle \( \Delta TUP \) is similar to the right triangle \( \Delta PRQ \), we have that

\[
\frac{y - z}{x} = \frac{dy}{dx},
\]

and solving for \( z \) we get,

\[
z = y - x \frac{dy}{dx}.
\]

We can use this expression to define a new function \( z \) of \( x \), whose graph is given by:

At the origin \( O \), draw the perpendicular to the tangent line
**TP** and let it intersect this line at point **S**, which has hypotenuse \( z \), and let \( k \) be the distance from **S** to the origin.

Since \( \angle STO + \angle PTU = 90^\circ \),

and \( \angle PTU = \angle QPR \),

we have \( \angle STO = \angle PQR \).

Hence right triangle \( \triangle OST \) is similar to right triangle \( \triangle PRQ \), and we have \( \frac{dx}{k} = \frac{ds}{z} \).

Consider now the infinitesimal triangle \( \triangle OPQ \):

Its base is \( ds \) and its height is \( k \), hence its area is \( \frac{kds}{2} \), which by the similarity above equals \( \frac{zdx}{2} \).

But as the picture on the left illustrates, \( \frac{zdx}{2} \) equals half of the area in the indicated infinitesimal rectangle under the function \( z \).

Hence we have that the area under the graph of \( z \), which is given by \( \int_a^b zdx \), is twice the area of the shape made from all the infinitesimal triangles. Hence, in the picture on the right, the area on the left is half of the area on the right.
But by cutting and pasting in the picture below, we get that the area under \( f(x) \) equals the area
\[
\text{OCD} + \text{OBC} - \text{OAD}.
\]

But by the result above, \( \text{OCD} \) equals one half the area under \( z \). Easily, \( \text{OBC} \) equals \( \frac{1}{2}bf(b) \) and \( \text{OAD} \) is \( \frac{1}{2}af(a) \), so symbolically we have:
\[
\int_a^b y \, dx = \frac{1}{2} \left( \int_a^b z \, dx + bf(b) - af(a) \right)
\]
and we have **exchanged one computation of areas for another** that may turn out to be simpler than the original as we will exemplify below.

But before we do that let's make a couple of observations.

First, \( bf(b) - af(a) \) can be simplified using standard evaluation notation to \( [xy]_a^b \).

Second, we can use the fact that \( z = y - x \frac{dy}{dx} \) to substitute in the equation above to obtain
\[
\int_a^b y \, dx = \frac{1}{2} \left( \int_a^b z \, dx + bf(b) - af(a) \right)
\]
\[
= \frac{1}{2} \left( \int_a^b \left( y - x \frac{dy}{dx} \right) \, dx + [xy]_a^b \right)
\]
\[
= \frac{1}{2} \int_a^b y \, dx - \frac{1}{2} \int_a^b x \, dy + \frac{1}{2} [xy]_a^b
\]
which after clearing and multiplying by 2 yields the **integration by parts** formula due to Leibniz:
\[
\int_a^b y \, dx = [xy]_a^b - \int_a^b f(b) \, dx,
\]
which is tantamount—in the picture—to the three shaded areas filling in the rectangle—confirming the geometric reasoning he had employed in the more sophisticated
equation 1 above.

Leibniz himself was pleased with the following application of his ideas. Consider a circle of radius 1 centered at the point (1,0) so that it is tangent to the y-axis.

Then its upper semicircle has the equation \( y = \sqrt{2x-x^2} \). Since \( y^2 = 2x - x^2 \), differentiating, we get

\[
\frac{dy}{dx} = \frac{1-x}{y},
\]

and so

\[
z = y - x \frac{1-x}{y} = \sqrt{\frac{x}{2-x}}.
\]

Solving for \( x \), we have

\[
x = \frac{2z^2}{1+z^2}.
\]

But, then we have

\[
\frac{\pi}{4} = \int_0^1 ydx
\]

\[
= \frac{1}{2} \left[ \int_0^1 zdx + [x\sqrt{2x-x^2}]_0^1 \right] \quad \text{by 1 above}
\]

\[
= \frac{1}{2} \left[ 1 - \int_0^1 xdz \right] + 1 \quad \text{by picture:}
\]

\[
= 1 - \int_0^1 \frac{z^2}{1+z^2}dz
\]

\[
= 1 - \int_0^1 z^2 \left( 1 - z^2 + z^4 - z^6 + \cdots \right)dz \quad \text{by geometric series}
\]

\[
= 1 - \left[ \frac{z^3}{3} - \frac{z^5}{5} + \frac{z^7}{7} - \frac{z^9}{9} + \cdots \right]_0^1
\]

\[
= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \quad \text{by term-wise integration}
\]

However, although one can find the expression

\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots
\]
quite beautiful, it is not very practical for doing computation—it converges too slowly. For example, if we use the fact that any alternating series toggles between exceeding and being less than the true sum, we have the not very good estimate $3.108269 < \pi < 3.173842$ even after 30 terms.

**Newton**

Born on Christmas Day, 1642, to a relatively poor widow, Isaac Newton showed promise as a student, and thus a brother of his mother agreed to support him in college. He attended Cambridge University. In 1665, during an outbreak of the plague, he was sent home, and it was during that period that he developed some of his best ideas. Soon after that, his teacher, Isaac Barrow resigned his position so that Newton can be appointed to follow him. For the next 30 years Newton was a professor at Cambridge— alas, a terrible lecturer, hardly any one would attend his lectures, but a widely known scholar. In 1693, he suffered a nervous breakdown, partially caused by the stress suffered during the dispute with Leibniz. After he recovered, he was appointed in charge of the Royal Mint where he spent the remainder of his life. When he died, he was the most famous scientist in the world, and was buried with all the glory and ceremony at Westminster Abbey.

Sir Isaac Newton is one of the most distinguished names in the history of mathematics and science. He can be considered one of the founders of modern science, and his book *Philosophiae Naturalis Principia Mathematica* (1687) (often referred simply as the *Principia*) is a major book in Western civilization.

We will be far from doing justice to Newton since we could—without much effort—spend a whole semester with him, just as with Euler or Gauss.

Newton had major impact on both mechanics and optics. We briefly glance at one of his three laws of motion.

1. **The Second Law of Motion:** $F = m \times a$

   *Force = mass × acceleration,*

   but actually Newton was even more accurate than that:

   *Force = mass × velocity,*
where the dot stood for fluxion which is the word he used for derivative, or equivalently, in Leibniz's notation $F = \frac{d(m \times v)}{dt}$, so if mass is a constant, we get the more common version of the second law. However, if mass is not constant, we get a different law—one that is valid at the very high speeds of atomic particles of modern physics.

A very easy application, when this law is linked with Galileo's conclusion that gravity is constant, is the calculation of the path followed by a projectile such as a cannon ball: Suppose a projectile is shot from the origin at angle $\theta$ with speed $v_0$. What is the path of the projectile? If we separate the force into its two components, one in the $x$-direction and one in the $y$-direction (this idea is much older than Newton), we get that $F_x = 0$ while $F_y = -g$, a constant. But then $a_x = 0$ while $a_y = \frac{-g}{m}$, a constant. But, by integrating accelerations, we get velocities, $v_x = v_0 \cos(\theta)$ and $v_y = \frac{-g}{m} t + v_0 \sin(\theta)$, if we assume time is measured so $t = 0$ is when we shot the projectile. Hence, $s_x = v_0 \cos(\theta) t$ and $s_y = \frac{-g}{2m} t^2 + v_0 \sin(\theta) t$, and if we graph this path, we get a parabola.

2  The Law of Gravitation

Newton may have independently arrived to this law:

two objects attract each other with a force proportional to their masses and inversely proportional to the square of their distance,

but others had also proposed it (Hooke for one). But he was definitely the first one to have done something with it. He used mathematics (calculus ideas) to infer Kepler's first two laws of planetary motion from the law of gravitation, and naturally this served as a major piece of evidence of support for this law.

3  The Binomial Theorem

One of his first successes, and a definite step toward calculus, was his extension of the binomial theorem to other exponents besides positive integers. What started as a technique to improve the computation of squared roots, and other roots, became a broader weapon, and made him a superb manipulator of series—, which was critical to his whole view of calculus. We recall that

$$(1 + x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i,$$

where $n$ is an arbitrary positive integer. One of the advantages of our present day notation is that we can easily write Newton's binomial theorem, where $\alpha$ is an arbitrary
number now:

\[(1 + x)^\alpha = \sum_{i=0}^{\infty} \binom{\alpha}{i} x^i,\]

where \(\binom{\alpha}{i}\) is defined by

\[\binom{\alpha}{i} = \frac{\alpha \times (\alpha - 1) \times (\alpha - 2) \times \cdots \times (\alpha - (i - 1))}{i \times (i - 1) \times (i - 2) \times \cdots \times 1}.\]

Make the following observations:

- there are \(i\) factors in the numerator and \(i\) factors in the denominator.
- if \(n\) is a positive integer, then for \(i > n\), \(\binom{n}{i} = 0\), and thus Newton’s version is a true extension of the finite case.
- Note the recursion, \(\binom{\alpha}{i+1} = \frac{\alpha - i}{i+1} \binom{\alpha}{i}\).
- \(\binom{\alpha}{i}\) is a polynomial in \(\alpha\) of degree \(i\)—with roots \(0, 1, \ldots, i-1\).

We compute a few of these polynomials (using mainly the recursion given above).

\[
\begin{align*}
\binom{\alpha}{0} &= 1, \text{ by definition or agreement}; \\
\binom{\alpha}{1} &= \alpha; \\
\binom{\alpha}{2} &= \frac{\alpha \times (\alpha - 1)}{2 \times 1} = \frac{\alpha^2 - \alpha}{2}; \\
\binom{\alpha}{3} &= \frac{\alpha \times (\alpha - 1) \times (\alpha - 2)}{3 \times 2 \times 1} = \frac{\alpha^3 - \alpha^2 + \alpha}{3}; \\
\binom{\alpha}{4} &= \frac{\alpha - 3}{4} \times \binom{\alpha}{3} = \frac{\alpha^4 - \alpha^3 + 11\alpha^2 - \alpha}{24} + \frac{\alpha}{4}; \\
\binom{\alpha}{5} &= \frac{\alpha - 4}{5} \times \binom{\alpha}{4} = \frac{\alpha^5 - \alpha^4 + 7\alpha^3 - 5\alpha^2 + \alpha}{120} + \frac{\alpha}{5}; \\
\binom{\alpha}{6} &= \frac{\alpha^6 - \alpha^5}{720} + \frac{17\alpha^4 - 5\alpha^3}{144} + \frac{137\alpha^2 - \alpha}{360}; \\
\binom{\alpha}{7} &= \frac{\alpha^7 - \alpha^6}{5040} + \frac{5\alpha^5 - 7\alpha^4}{240} + \frac{29\alpha^3 - 7\alpha^2}{90} + \frac{\alpha}{21}.
\end{align*}
\]

So, for example, if we are interested in taking square roots, we let \(\alpha = \frac{1}{2}\) and we get the following coefficients,
\[
\begin{align*}
\left(\frac{1}{0}\right) &= 1 \\
\left(\frac{1}{1}\right) &= 1 \\
\left(\frac{1}{2}\right) &= \frac{1}{2} \\
\left(\frac{1}{3}\right) &= \frac{1}{8} \\
\left(\frac{1}{4}\right) &= \frac{5}{128} \\
\left(\frac{1}{5}\right) &= \frac{7}{256} \\
\left(\frac{1}{6}\right) &= \frac{21}{1024} \\
\left(\frac{1}{7}\right) &= \frac{33}{2048}
\end{align*}
\]

So if we put them together with the binomial theorem, we get that
\[
\sqrt{1 + x} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{5x^3}{16} - \frac{7x^4}{128} + \frac{21x^5}{256} - \frac{33x^6}{1024} + \text{higher order terms},
\]
so if \( x \) is small, we should have a reasonable approximation.

For example, if we are interested in \( \sqrt{7} \), then we can handle it this way,
\[
\sqrt{7} = \sqrt{9 - 2} = 3 \sqrt{1 - \frac{2}{9}},
\]
so we let \( x = -\frac{2}{9} \), and we get \( \frac{11248487}{12754584} \) for \( \sqrt{1 - \frac{2}{9}} \), which approximates to 0.8819171993, which when multiplied by 3, gives 2.645751598, a good estimate for \( \sqrt{7} \).

Sometimes, a closed expression for the coefficients is desired (and can be found), although it may be difficult to find the pattern at first. Let us revisit the coefficients we just have computed: 1, \(\frac{1}{2}\), \(\frac{1}{8}\), \(\frac{5}{128}\), \(\frac{7}{256}\), \(\frac{21}{1024}\), \(\frac{33}{2048}\) and at first the pattern does elude us. But let us go back to the definition:
\[
\left(\frac{1}{i}\right) = \frac{\frac{1}{2} \times (\frac{1}{2} - 1) \times (\frac{1}{2} - 2) \times \cdots \times (\frac{1}{2} - (i - 1))}{i \times (i - 1) \times (i - 2) \times \cdots \times 1} = \frac{1}{2} \times \frac{-1}{2} \times \frac{-3}{2} \times \cdots \times \frac{-(2i - 3)}{2} = \frac{(-1)^{i-1} 1 \times 3 \times 5 \times \cdots \times (2i - 3)}{2^i i!}.
\]

Can we simplify this further? Perhaps simplify is the wrong word, and what we are trying to do is reduce the expression to more familiar functions. If we could reduce everything to the factorial, we would be at peace. We need to understand then the product of consecutive odds:
\[
1 \times 3 \times 5 \times \cdots \times (2i - 3) = \frac{(2i - 3)!}{2 \times 4 \times 6 \times \cdots \times (2i - 4)} = \frac{(2i - 3)!}{(2 \times 1) \times (2 \times 2) \times (2 \times 3) \times \cdots \times (2 \times (i - 2))} = \frac{(2i - 3)!}{2^{i-2} (i - 2)!},
\]
and so we conclude that
\[
\left(\frac{1}{i}\right) = \frac{(-1)^{i-1} (2i - 3)!}{2^{i-2} (i - 2)! i!},
\]
for any \( i \). Now whether this expression is computationally acceptable depends very much on our control of the factorial. But we do have what is called a closed expression.
Newton's Method

Very often, early in our mathematical career, perhaps as early as the first course in calculus, we get exposed to a powerful procedure for finding roots of equations called Newton's Method. It is more effective than the other two methods we discussed earlier in the Descartes section—the bisection method and the method of false position. Its ideal name would be the tangent method (in contrast to the secant method, a variation of the method of false position) since it finds its new guess by following the tangent line to the function at a guess.

More formally, we review the procedure:

Start with one guess \( x_0 \) (as opposed to two guesses necessary in the other methods), and then follow one's nose by using the tangent line at the point of the original guess. How do we simply find the new guess, \( x_1 \)?

We follow the tangent line at the point of the graph corresponding to our initial guess:

Geometrically then, we have an idea of where to locate our new approximation. But how do we find the point efficiently? Easily—use the slope:

\[
\text{slope of the tangent line} = \frac{\text{rise}}{\text{run}},
\]

\[
\frac{f'(x_0)}{f'(x_0)} = \frac{f(x_0) - 0}{x_0 - x_1},
\]

and solving for \( x_1 \), we get

\[
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.
\]

Once we get \( x_1 \), we can use the same expression to get \( x_2 \), and then \( x_3 \), etceteras, continuing the iteration. When do we stop? If there is not much change from one \( x \) to the next \( x \), most probably we are close to a root, and we can stop.

We give an application of the three methods to an original example of Newton's:

\[
x^3 - 2x - 5 = 0.
\]
And we can easily see that at least in this example, Newton's is an easy winner. We note, again, that Newton's has the advantage of requiring only one guess.

But we also should remark that depending on the nature of the equation, the method can be unstable, and not lead to a root at all—the curious reader may attempt to find a root of $x^{\frac{1}{3}}$ by Newton's method.

Nevertheless, the method is so useful it is programmed in most hand-held calculators.

Needless to say, the method just described has been polished through time, and we now spend some time describing what Newton originally did. He used one of the original ideas behind calculus. That idea is 

ignoring terms of higher order than 1, in other words, ignoring everything except linear terms—actually, that is what approximating a curve by the tangent line is all about. Newton was an expert at that technique.

We illustrate, analytically, his original thinking behind his method with the example above: 

Solve $x^3 - 2x - 5 = 0$.

We start with one approximation, 2, the same as above, and so we let $x = 2 + p$, and obtain $p^3 + 6p^2 + 10p - 1 = 0$—remember $x$ is supposed to be a root. Ignoring all but the linear term, we get $10p - 1 = 0$, $p = 0.1$ and we have a better approximation $x = 2.1$.

Again, we let $x = 2.1 + p$, and substituting, we get $p^3 + 6.3p^2 + 11.23p + 0.061 = 0$, and thus, by again considering only the linear term, we have that $p = -\frac{61}{11230} \approx -0.00543187$, which quickly gives us an estimate for $x = 2.1 - 0.00543187 = 2.09456813$—just as before.
The Series for the Arcsine.

Consider the circle $x^2 + y^2 = 1$ on the first quadrant. And take an arbitrary point $P$ on the $x-$axis, at distance $x$ from the origin $O$. Then we know the height of the circle at that point is $y = 1 - x^2$, which by his binomial theorem (see 3 above, and substitute $-x^2$ for $x$) Newton could expand into

$$y = 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128} - \frac{7x^{10}}{256} - \frac{21x^{12}}{1024} - \frac{33x^{14}}{2048} + \cdots.$$ 

He could then integrate this series to obtain the total shaded area $OPQR$, consisting of the triangle $\triangle OPQ$ and the section of the circle $OQR$:

$$\Delta + \angle = x - \frac{x^3}{6} - \frac{x^5}{40} - \frac{x^7}{112} - \frac{5x^9}{1152} - \frac{7x^{11}}{2816} - \frac{21x^{13}}{13312} - \frac{33x^{15}}{30720} + \cdots.$$ 

But the area of the triangle $\triangle OPQ$ is known,

$$\Delta = \frac{x\sqrt{1-x^2}}{2} = \frac{x^3}{6} - \frac{x^5}{40} - \frac{x^7}{112} - \frac{5x^9}{1152} - \frac{7x^{11}}{2816} - \frac{21x^{13}}{13312} - \frac{33x^{15}}{30720} + \cdots.$$ 

And thus the section of the circle $\angle OQR$ is given by the difference of the two series:

$$\angle = x + \frac{x^3}{2} + \frac{3x^5}{80} + \frac{5x^7}{224} + \frac{35x^9}{2304} + \frac{63x^{11}}{5632} + \frac{231x^{13}}{16384} + \frac{143x^{15}}{20480} + \cdots.$$ 

But $\angle = \frac{\theta}{2}$, and $x = \cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$, and so $\theta = \arcsin(x)$, and we have that

$$\theta = \arcsin(x) = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \frac{35x^9}{1152} + \frac{63x^{11}}{2816} + \frac{231x^{13}}{13312} + \frac{143x^{15}}{10240} + \cdots.$$ 

The Series for the Sine (and the Cosine).

Most of us take Taylor's series expansions very much for granted. Yet this most wonderful and powerful theorem could not have even been thought of without enough examples to hint at its existence.

One of the best early examples is provided by the series expansion of the sine (and also the cosine)—which Newton developed. Start with the series of the Arcsine that we have above:

$$\theta = \arcsin(x) = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \frac{35x^9}{1152} + \frac{63x^{11}}{2816} + \frac{231x^{13}}{13312} + \frac{143x^{15}}{10240} + \cdots.$$ 

The idea is to solve for $x$ in terms of $\theta$, $x = \sin(\theta)$. At all times, we will eliminate all nonlinear terms, continuing with the basic idea behind his method.
Dropping all nonlinear terms, we have as a first approximation, \( x \approx \theta \), which as the picture indicates fits the sine function for small \( x \)’s.

We are going to illustrate the method by using the first four terms of the series for the Arcsine:

\[
x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} - \theta = 0.
\]

We let now \( x = \theta + p \), and we get:

\[
(\theta + p) + \frac{(\theta + p)^3}{6} + \frac{3(\theta + p)^5}{40} + \frac{5(\theta + p)^7}{112} - \theta = 0,
\]

Thus,

\[
\frac{2800\theta^3 + 1260\theta^5 + 750\theta^7}{1680} + \left(\frac{16 + 80^2 + 60^4 + 50^6}{16}\right)p = \text{higher order terms} = 0.
\]

Ignoring the higher order terms and solving for \( p \), we get that:

\[
p = -\frac{2800\theta^3 + 1260\theta^5 + 750\theta^7}{1680 \left(\frac{16 + 80^2 + 60^4 + 50^6}{16}\right)} = -\frac{\theta^3}{6} + \text{higher order terms},
\]

and so

\[
x \approx \theta - \frac{\theta^3}{6}.
\]

Now we let \( x = \theta - \frac{\theta^3}{6} + p \), and substitute in \( \odot \), and we obtain:

\[
\left(\theta - \frac{\theta^3}{6} + p\right) + \frac{\left(\theta - \frac{\theta^3}{6} + p\right)^3}{6} + \frac{3\left(\theta - \frac{\theta^3}{6} + p\right)^5}{40} + \frac{5\left(\theta - \frac{\theta^3}{6} + p\right)^7}{112} - \theta = 0.
\]

And so when we expand,

\[
\frac{1,306,36800\theta^5 + \text{higher order terms in } \theta}{156,764,160} + \left(\frac{756,496 + \text{higher order terms in } \theta}{746,496}\right)p + \text{higher order terms in } p = 0.
\]

\(^{1}\text{In fact, the higher order terms in } \theta \text{ are } 622,080\theta^7 + 5,019,840\theta^9 - 3,538,080\theta^{11} + 1,088,640\theta^{13} - 187,488\theta^{15} + 18,900\theta^{17} - 1,050\theta^{19} + 25\theta^{21}. \text{ Similarly, for the other fraction.}\)
Simplifying, we have
\[
\frac{\theta^5 + \textbf{higher order terms in } \theta}{120} + (1 + \textbf{higher order terms in } \theta)p + \cdots = 0.
\]

When solving for \(p\) and ignoring anything but the lowest term order we get \(p = \frac{\theta^5}{120}\), and so we have
\[
x \approx \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120}.
\]

Continuing in this way, he eventually predicted the correct pattern:
\[
\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \frac{\theta^{11}}{11!} + \cdots.
\]

From this series he used the fact that
\[
\cos(\theta) = \sqrt{1 - \sin^2(\theta)},
\]

and substituted the series for the sine into the series for \(\sqrt{1 - x^2}\) — an absolutely formidable calculation.

Newton did this in order to establish
\[
\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \frac{\theta^{10}}{10!} + \cdots.
\]

Did he know that the derivative of the sine function is the cosine? Perhaps not, since he would most probably have used the derivative to find the series for the cosine instead. To be strictly accurate historically, we have given a modern representation of Newton’s actual calculations. At that time it was more customary to view the sine as corresponding to an arc more than an angle — so in fact radians were not needed.

Next we give the pictures for the next eight approximations of the sine series. And we see a steady improvement in our approximations.
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<td>MAA</td>
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<td>O. Neugebauer</td>
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<td>Griffin</td>
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<td>B. L. van der Waerden</td>
<td>Springer-Verlag</td>
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<td>Historical Development of the Calculus</td>
<td>C. H. Edwards, Jr.</td>
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<td>History of Mathematics</td>
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<td>History of Mathematics</td>
<td>C. Boyer</td>
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<td>V. J. Katz</td>
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Website
ACROSS

1. … the two straight lines, if produced indefinitely, … … …
3. The name of the Gou Gu Theorem in the West.
7. Our knowledge of the mathematics from this area stem mainly from a few papyri such as the Ahmes papyrus.
10. One of the curves studied by Apollonius.
12. The most important book in the history of mathematics.
16. ξ
17. The zero numeral comes from this region.
18. To the early Greeks, a product of three numbers represented a _ _ _ _ _ _.
19. Article.
20. Modus Operandi.
22. The first curvilinear shapes that were squared.
23. That man.
24. Ego, superego and _ _.
26. Epicycles were created as a commitment to this perfect shape in astronomy.
27. Not subtract.
28. Shade of color.
31. City in the land between the rivers.
33. The inside angle of this polygon is 108°.
35. The most fundamental constant in nature.
36. The opposite of the power of a number.
39. Neither. _ _ _ _.
40. The element associated with the Platonic solid with eight sides.
42. μ
43. The artistic center of the Greek world.
44. The author of the most important book in the history of mathematics.
45. A synonym for proportion.
48. The geometric mean of the sides of a _ _ _ _ _ _ _ _ is the side of the square with the same area.
50. α, α, α.
51. What a good one of 91 Across must eventually do.
52. The name of the inscribed polygon whose side is the radius.
54. ∧, the conjunction.
55. His formula produces the area of a triangle from the lengths of the sides.
56. The fourth astral body in the ancient ordering.
57. The arithmetic or the geometric or the harmonic.
58. What $\sqrt{-1}$ is not.
59. one- _ _ -one.
60. Thus.
61. What the side and the diagonal of a square are not.
62. The circumference of it was measured by 2 Down.
63. For example.
64. Twice, two-fold.
65. If, to Don Quijote.
67. A major work by 35 Across.
68. I _ _ _, I understand.
69. $2^4 + 1$ _ _ a prime.
70. Every.
71. What this puzzle is about.
72. The last of the Greek geometers.
73. Half-chord.
74. Procedure named after al-Khwarizmi.

DOWN
2. See 67 Across.
3. The most influential astronomer of antiquity.
4. The scientific center of the Greek world.
5. The civilization that overtook the Greek.
6. $2^2 + 1$ is _ _ _ prime.
7. _ _ _ _ _ Ratio.
8. The Egyptians most remarkable achievement was the computation of its volume.
9. It has a chord.
11. Antonym to always.
12. _ _ _ _ _ objects have the same shape.
13. Not even.
14. He performed 22 Across.
15. The Platonic solid with fewest sides.
17. The Platonic solid with the faces having the most number of sides.
18. It and the straightedge constituted the Euclidean tools.
19. The product of two numbers to a Greek.
20. You to Julius.
21. The author of Achilles and the Tortoise.
22. The most influential writer on number theory.
23. Sun God to Nefertiti.
24. The Platonic solid with the most number of sides.
25. The best scientific mind of antiquity.
26. Not the hypotenuse.
27. A very modern word for length.
28. Hypatia was killed by one.
29. Top of the deck.
30. Modern examples of prosthaphaeresis.
31. $\tau$.
32. One of the building blocks of all numbers.
33. The residual of a base twenty ancient system is reflected in French when this number is called four-twenties.
34. The smallest base in which one can divide easily by the first six numbers.
35. That is.
36. Piece of a circle.
37. What the modern symbol $\in$ means.
38. He considered circles to be the most perfect of curves.
39. “a long, long way to run”
40. $\chi$.
41. Not a science.
42. He gave us $\pi \approx \frac{355}{113}$
43. Abbreviation for $\frac{1}{2}$.
44. “a drink with jam and bread”
45. Surjective or _ _to.
46. “a drop of golden sun”

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