Homework #2

1. **Newton’s Binomial Theorem.** According to the theorem, \((1 + x)^{\frac{1}{5}} = \sum_{i=0}^{\infty} \frac{1}{i!} x^i\). Do the following:
   - Compute \(\frac{1}{5i}\) for \(i = 0, \ldots, 5\).
   - Consider the polynomial \(p(x) = \sum_{i=0}^{5} \frac{1}{i!} x^i\). What happens if you plug in \(x = 1000\)?
   - If you compute \(p(11)\), what are you computing? How close are you?
   - In order to compute \(\sqrt[5]{1000}\) we do as follows: we find an integer \(n\) such that \(n^5\) is close to 1000 and thus we write \(1000 = n^5 - (n^5 - 1000)\) and then \(\sqrt[5]{1000} = \sqrt[5]{n^5} - \sqrt[5]{(n^5 - 1000)} = n\sqrt[5]{1 - \frac{x}{n}} = np(-x)\). What value to six digits do you get if you do that?
   - What does your calculator tell you \(\sqrt[5]{1000}\) is (to six digits)?
   - Comment on the technique.

2. **On Power Series.** We all know the geometric series: \(\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots\).
   - Use the geometric series to find the power series expansion for \(\frac{1}{1+x^2}\). Hint: Use your head.
   - Use \(\odot\) to find a power series expansion for \(\frac{1+x^2}{1+x^4}\).
   - Use the differential quotient rule together with facts that \(d(sin x) = cos x dx\) and \(d(cos x) = -sin x dx\) to derive \(d(tan x)\).
   - Bonus: Use Leibniz’s method to derive \(d(sin x) = cos x dx\).
   - Use the fact that if \(y = arctan x\) then \(x = tan y\) to find \(d(arctan x)\).
   - Compute the derivative (not differential) of the function \(\left(\frac{\sqrt{2} \arctan(\sqrt{2}x + 1)}{2}\right) + \left(\frac{\sqrt{2} \arctan(\sqrt{2}x - 1)}{2}\right)\).
   - Derive and justify Newton’s result:
     \[1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots = \frac{\pi}{2\sqrt{2}}\].

3. **Newton’s Method.** Consider the function \(f(x) = x^2 - 1000\). We are to search for a root of it. Take an initial guess 20 as a solution.
① Apply Newton’s Method to get the next approximation 5 approximations.
② Apply three steps of the technique to $f(x) = x^2 + 1$ with original guess 1.
③ Consider the equation $\cos x = x$. Find a solution to at least three correct digits starting with the guess $a = 1$. 