Knot

Intuitive: A knot is a loop in 3D space.

Differential topology: A knot is a smooth embedding of $S^1$ into $S^3$.

Piece-wise linear topology: A knot is a simple closed polyhedral curve in $\mathbb{R}^3$.

What do the words in these def's mean?

- $S^1 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ = unit circle
- $S^3 = \{ (x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1 \}$

Note: $S^3$ is the "one point compactification" of $\mathbb{R}^3$.

i.e. $S^3 \cong (\mathbb{R}^3 \cup \{ \text{pt} \},$ certain topology). So, we can do knot theory in $S^3$ or $\mathbb{R}^3$ without losing information.

- Recall from vector calc that a parametrized curve in $\mathbb{R}^3$ given by $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is smooth if each of $f(t)$, $g(t)$ and $h(t)$ are all $\infty$-ly differentiable.

- There is a similar notion of $\infty$-ly differentiable for maps between manifolds (i.e. $f : S^1 \to S^3$).

- A map $f : X \to Y$ is an embedding if it is continuous, 1-to-1 and onto its image.

Def. of simple closed polyhedral curve in $\mathbb{R}^3$.

Given distinct points $p, q \in \mathbb{R}^3$, let $\overline{p, q}$ denote the straight line segment in $\mathbb{R}^3$ that connects $p$ to $q$. 

\[ \]
Let \((p_1, \ldots, p_n)\) be an ordered set of distinct points in \(\mathbb{R}^3\). \([p_i, p_{i+1}]\cup[p_{i+1}, p_{i+2}]\cup \cdots \cup [p_{n-1}, p_n] \cup [p_n, p_1]\) is a simple closed polyhedral curve if

\[ [p_i, p_{i+1}] \cap [p_{i+1}, p_{i+2}] = \emptyset \text{ for all } i \mod n \]

and \([p_i, p_{i+1}] \cap [p_j, p_{j+1}] = \emptyset \text{ when ever } |i-j\mod n| \geq 2\].

Smooth

PL

When are two knots equivalent

Intuitive: Whenever you can (finitely) bend, stretch, tangle or untangle one to match the other.

Smooth: Two knots \(f: S^1 \to S^3\) and \(g: S^1 \to S^3\) are equivalent if there exists a smooth map \(H: S^3 \times I \to S^3\) s.t.

\(H(s, t)\) is a diffeomorphism for every fixed \(t\) and

\(H(f(s), t, 1) = g(s). \) (This is an ambient isotopy of \(S^3\) taking \(f\) to \(g\)).
Def: A knot $J$ is an elementary deformation of a knot $K$ if one of the knots $J$ and $K$ is determined by $(p_1, \ldots, p_n)$ and the other is determined by $(p_0, p_1, \ldots, p_n)$ s.t.

1. $p_0$ is not colinear with $p_1$ and $p_n$
2. The triangle spanned by $p_1, p_0, p_n$ intersects $[p_1, p_2] \cup \cdots \cup [p_{n-1}, p_n] \cup [p_n, p_1]$ only in $[p_1, p_n]$.

Knots $J$ and $K$ are equivalent if there exists a sequence of knots $K = K_0, K_1, K_2, \ldots, K_n = J$ s.t. $K_{i+1}$ is an elementary deformation of $K_i$ for each $i$ s.t. $0 \leq i \leq n-1$. 
Deep, important fact:

The equivalence classes of smooth knots up to isotopy are naturally in one-to-one correspondence with the equivalence classes of P.L. knots up to sequences of elementary deformations.

P.h. to smooth

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Smooth to P.h.

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∗ So, we can think of a knot as being smooth or PL for whatever best suits our needs. ∗

Additional definitions:

A link is an embedding of \( \bigoplus_{i=1}^{n} S^1 \) into \( S^3 \).

i.e. \( \bigcup \).
The **unknot** is any knot in $S^3$ that bounds an embedded disk.

(i.e.)

![Embedded disk diagram]

Why are these definitions of knot and knot equivalence so complicated?

![Knot transformation diagram]

Want to avoid this being a knot.

![Knot simplification diagram]

Want to avoid every knot being equivalent to the unknot.

3D is hard, so why don't we make everything 2D? Thanks to Reidemeister we can.

Consider the projection map $P : \mathbb{R}^3 \to \mathbb{R}^2$ given by $P(x, y, z) = (x, y)$. A projection of a knot $K$ is the image of a knot $K \subset \mathbb{R}^3$ under the map $P$, (i.e. the image of $P|_K$ (restricted to $K$)).

![Projection of trefoil diagram]

Example: a projection of the trefoil.
Def: A knot projection is regular if no three points on the knot project to the same point in \( \mathbb{R}^2 \) and if no two points project to a point of tangency.

i.e. not

\[
\begin{array}{c}
\times \\
\times \\
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\[
\times \text{ is ok}
\]

Thm: Every knot is equivalent to a knot with a regular projection.

Every regular knot projection is a \textcolor{red}{4-valent} graph in \( \mathbb{R}^2 \)

- A graph in \( \mathbb{R}^2 \) is a collection of vertices \( \mathcal{V} \) and a collection of edges \( \mathcal{E} \) s.t. the end points of every edge is contained in the set \( \mathcal{V} \), every edge is embedded in \( \mathbb{R}^2 \), two edges can only meet in their endpoints.

- A graph is \textcolor{red}{4-valent} if every vertex is incident to exactly 4 edges (not necessarily distinct).
Given a knot projection $P(K)$ a region of $P(K)$ is a connected component of $\mathbb{R}^2 - P(K)$.

This knot diagram has 5 regions.

**Def:** A knot diagram is a knot projection together with information at each crossing that indicates which strand goes over and which goes under.

\[+_+ = \text{goes over} \quad + - = \text{goes under}.\]

**Theorem (Reidemeister):** Two knots $K$ and $L$ are equivalent iff every diagram of $K$ and every diagram of $L$ can be related by planar isotopy and a sequence of Reidemeister moves RI, RII, and RIII.
Planar isotopy

Thus, we have made this 3D problem a 2D problem.

Problem: Show that the following diagram is equivalent to the unknot diagram by finding a sequence of Reidemeister moves.