Knot

Intuitive: A knot is a loop in 3D space.

Differential topology: A knot is a smooth embedding of S^1 into S^3.

Piece-wise linear topology: A knot is a simple closed polyhedral curve in $\mathbb{R}^3$.

What do the words in these def.s mean?

- $S^1 \cong \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\} \cong$ unit circle
- $S^3 = \{(x,y,z,w) \in \mathbb{R}^4 \mid x^2+y^2+z^2+w^2=1\}$

Note: S^3 is the "one point compactification" of $\mathbb{R}^3$ i.e. $S^3 \cong (\mathbb{R}^3 \cup \{pt\}$, certain topology$)$. So, we can do knot theory in S^3 or $\mathbb{R}^3$ without loosing information

- Recall from vector calc that a peramitrized curve in $\mathbb{R}^3$ given by $V(t) = \langle f(t), g(t), h(t) \rangle$ is smooth if each of $f(t)$, $g(t)$ and $h(t)$ are all ∞-ly differentiable.
- There is a similar notion of ∞-ly differentiable for maps between manifolds (i.e. $f: S^1 \to S^3$).
- A map $f: X \to Y$ is an embedding if it is continuous, 1-to-1 and onto its image.

Def. of simple closed polyhedral curve in $\mathbb{R}^3$:

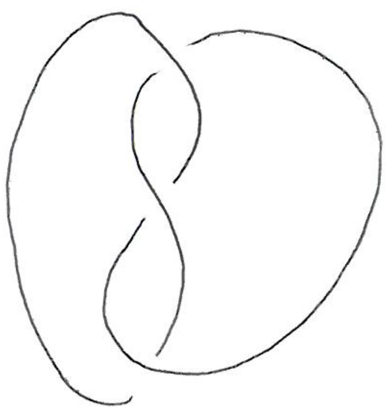
Given distinct points $p, q \in \mathbb{R}^3$, let $[p,q]$ denote the straight line segment in $\mathbb{R}^3$ that connects p to q.

Let $(P_1, \dots, P_n)$ be an ordered set of distinct points in $\mathbb{R}^3$. $[P_1, P_2] \cup [P_2, P_3] \cup \cdots \cup [P_{n-1}, P_n] \cup [P_n, P_1]$ is a <u>simple</u> <u>closed</u> <u>polyhedral</u> <u>curve</u> if

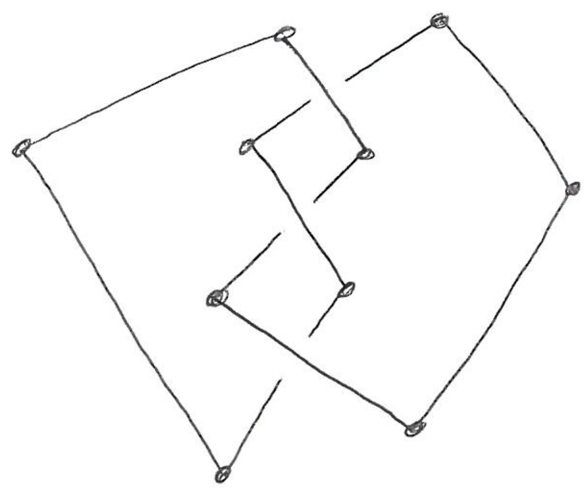
$$[P_i, P_{i+1}] \cap [P_{i+1}, P_{i+2}] = \{P_i\} \text{ for all } i \bmod n$$

and $[P_j, P_{j+1}] \cap [P_i, P_{i+1}] = \emptyset$ when ever $|i-j \bmod n| \geq 2$

<u>Smooth</u>



<u>PL</u>



When are two knots equivalent

Intuitive: When ever you can (finitely) bend, strecth tangle or un tangle one to match the other.

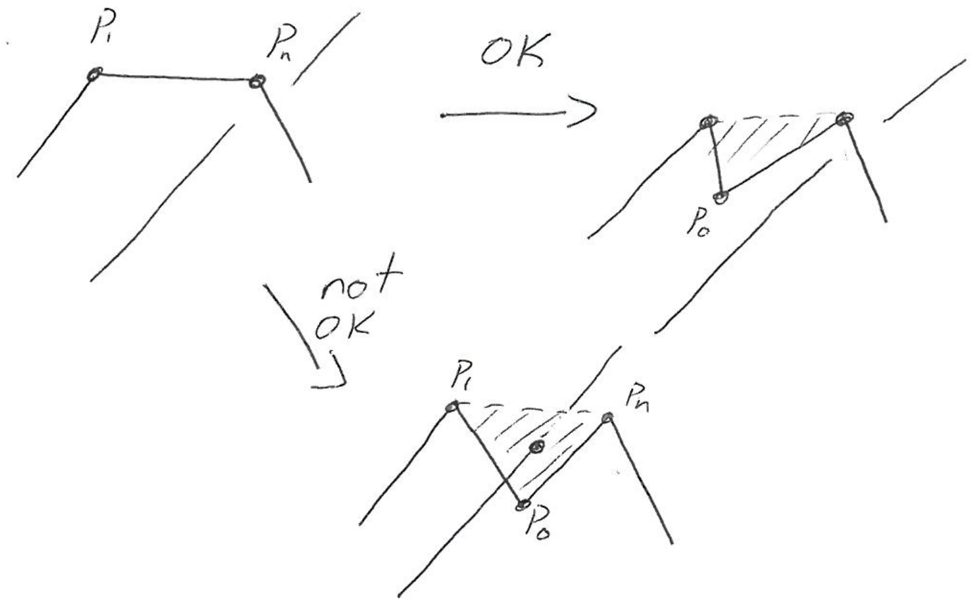
<u>Smooth</u>| Two knots $f: S^1 \to S^3$ and $g: S^1 \to S^3$ are equivalent If there exists a smooth map $H: S^3 \times I \to S^3$ s.t. $H(s,t)$ is a diffeomorphism for every fixed t and $H(f(s), 1) = g(s)$. (This is an ambient isotopy of S^3 taking f to g).

P.L. Answer:

Def | A knot J is an elementary deformation of a knot K if one of the knots J and K is determined by $(p_1, \dots, p_n)$ and the other is determined by $(p_0, p_1, \dots, p_n)$ s.t.

① p_0 is not colinear with p_1 and p_n

② The triangle spanned by p_1, p_0, p_n intersects $[p_1, p_2] \cup \cdots \cup [p_{n-1}, p_n] \cup [p_n, p_1]$ only in $[p_1, p_n]$.

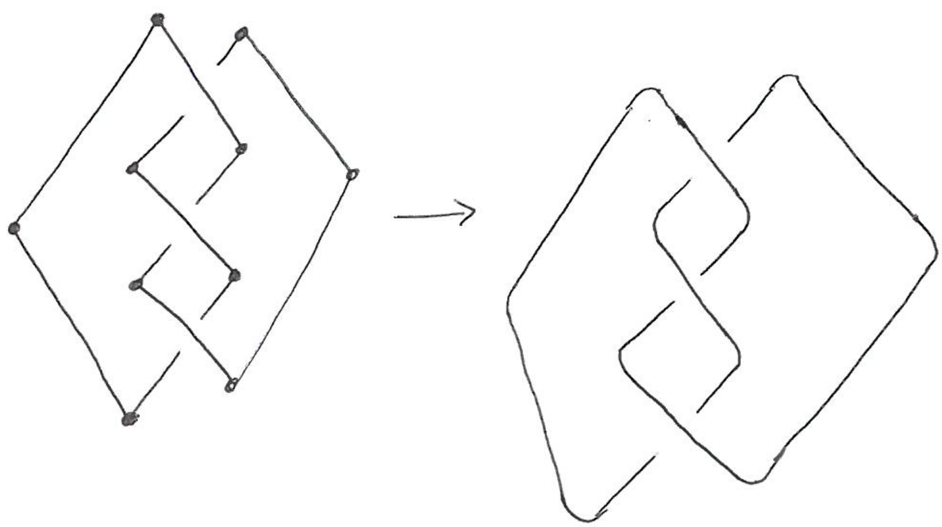


Knots J and K are equivalent if there exists a sequence of knots $K = K_0, K_1, K_2, \dots, K_n = J$ s.t. K_{i+1} is an elementary deformation of K_i for each i s.t. $0 \le i \le n-1$.

Deep, Important fact:

The equivalence classes of smooth knots up to isotopy are naturally in one-to-one correspondence with the equivalence classes of P.L. knots up to sequences of elementary deformations.

P.L. to smooth



Smooth to P.L.

* So, we can think of a knot as being smooth or PL for whatever best suits our needs. *

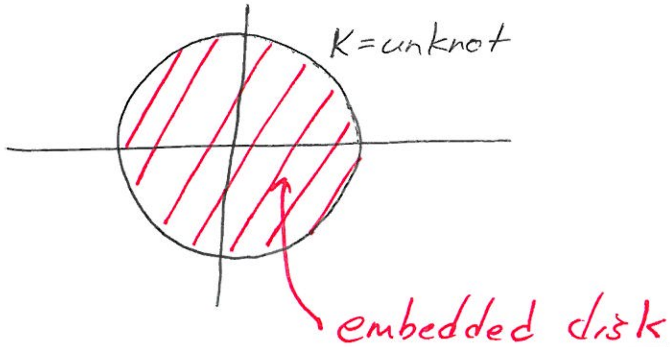
Additional definitions

A Link is an embedding of $\coprod_{i=1}^{n} S^1_i$ into S^3.

i.e. .

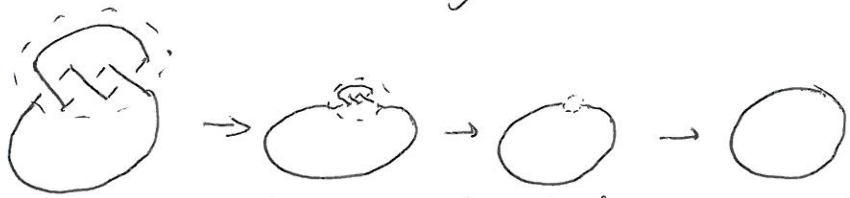
The <u>unknot</u> is any ~~an~~ knot in S^3 that bounds an embedded disk.

(i.e.)



Why are these definitions of knot and knot equivalence so complicated?

want to avoid this being a knot.

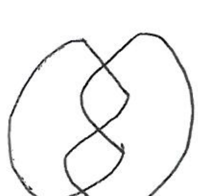


want to avoid every knot being equivalent to the unknot.

3D is hard, so why don't we make everything 2D?
Thanks to Reidermeister we can.

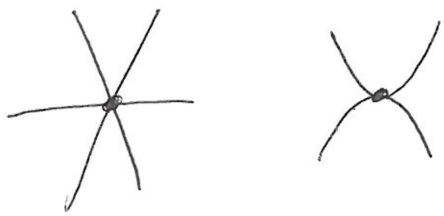
Consider the projection map $P: \mathbb{R}^3 \to \mathbb{R}^2$ given by $P(x, y, z) = (x, y)$. A projection of a knot K is the image of a knot $K \subset \mathbb{R}^3$ under the map P, (i.e. the image of $P|_K$ (P restricted to K)).

<u>Ex</u>



a projection of the trefoil.

Def| A knot projection is *regular* if no three points on the knot project to the same point in $\mathbb{R}^2$. and if no two points project to a point of tangency.

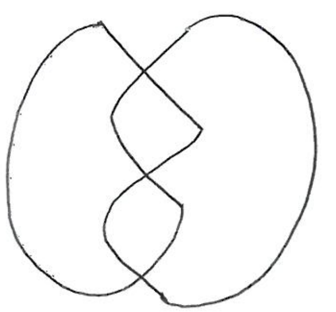
i.e. not 

 is OK

Thm| Every knot is equivalent to a knot with a regular projection.

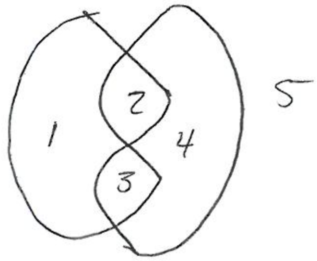
Every regular knot projection is a 4-valent graph in $\mathbb{R}^2$

- A graph in $\mathbb{R}^2$ is a collection of vertices $\mathcal{V}$ and a collection of edges $\mathcal{E}$ s.t. the end points of every edge is contained in the set $\mathcal{V}$, every edge is embedded in $\mathbb{R}^2$, two edges can only meet in their endpoints.



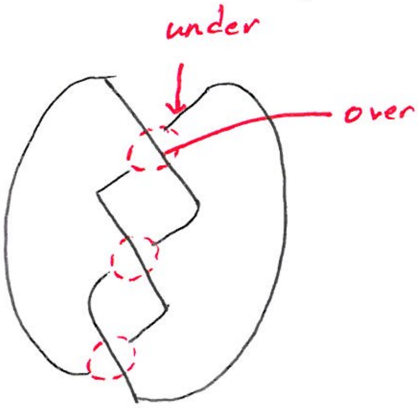
- A graph is 4-valent if every vertex is incident to exactly 4 edges (not nessisarily distinct).

Given a knot projection $P(K)$ a region of $P(K)$ is a connected component of $\mathbb{R}^2 - P(K)$

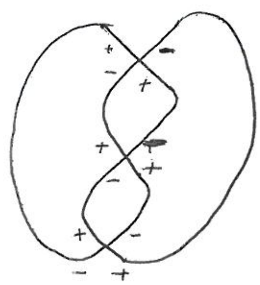


← This knot diagram has 5 regions.

Def| A knot diagram is a knot projection together with information at each crossing that indicates which strand goes over and which goes under.



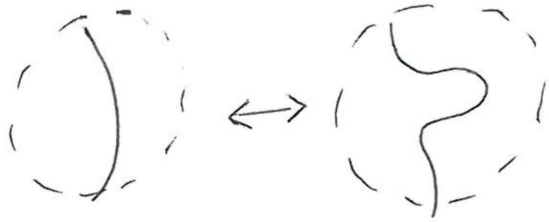
or



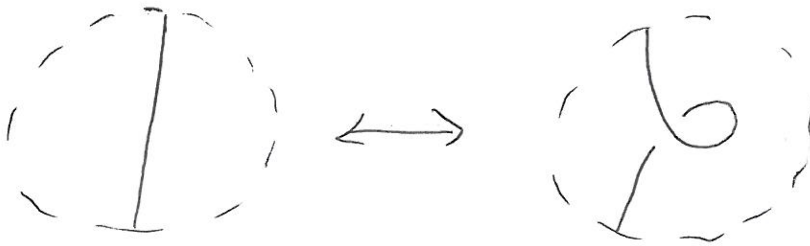
$+$ = goes over
$-$ = goes under.

Th^m (Reidermeister)| Two knots K and L are equivalent iff every diagram of K and every diagram of L can be related by planar isotopy and a sequence of Reidermeister moves RI RII and RIII.

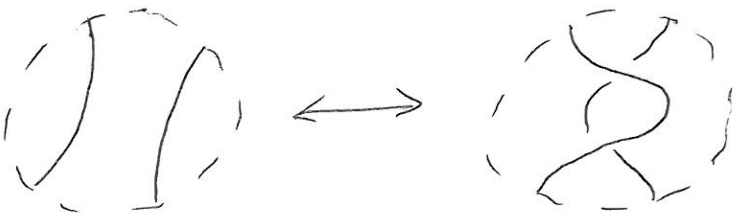
Planar isotopy:



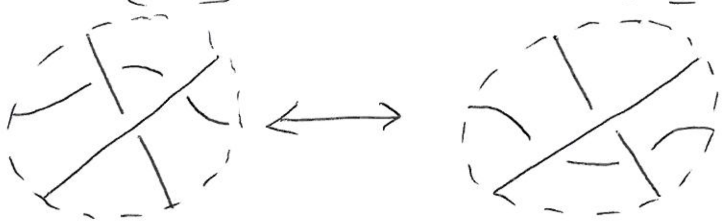
R I



R II



R III



Thus, we have made this 3D problem a 2D-problem.

Problem!

Show that the following diagream is equivalent to the unknot diagram by finding a sequence of Reidermeister moves