(Cor 5) Every link complement contains an unknot $\mathcal{U}$ s.t. $S^3 - (L \cup \mathcal{U})$ has a complete hyperbolic structure.

<u>Thm</u> (Menasco) If L is a non-split, prime, alternating link which is not a $(2,q)$ torus link then $S^3 - L$ has a complete hyperbolic structure.

Proof of Cor 5 summerized:

Show every link L has a diagram D s.t. there exists an augmenting component $\mathcal{U}$ in $S^3 - L$ that projects to a simple closed curve in D so that $L \cup \mathcal{U}$ is a non-split, prime alternating link that is **not a** $(2,q)$-torus link. In particular, we show G is an alternating diagram.

<u>Defining a new knot invariant</u>

Given a link L, let $\mathcal{U}$ be an unknot embedded in $S^3 - L$ s.t. $L \cup \mathcal{U}$ is an alternating hyperbolic link. By cor. 5, $\mathcal{U}$ always exists.

Define $\quad \mathrm{Altvol}(L) = \inf_{\mathcal{U}} \mathrm{vol}(L \cup \mathcal{U})$.

Fact: Any collection of hyperbolic volumes always ~~has a minimum~~ atains its minimum. (Thurston)

So, there exists a unknot $U_s \subset S^3 - L$ s.t.

$$\text{Altvol}(L) = \text{Vol}(S^3 - (L \cup U_s)). \leftarrow \text{Key} **$$

Thm (Lackenby) Let V_3 = volume of a regular ideal hyperbolic tetrahedron. Let D be a prime alternating diagram of a hyperbolic link K in S^3. Then

$$V_3(t(D)-2)/2 \leq \text{vol}(S^3 - K) < V_3(16t(D) - 16).$$

Def A diagram D is prime, if it is connected and if there do not exist any simple closed curves in S^2 that meet D in exactly two points in edges of D s.t. this simple closed curve seperates vertices of D.

If L is any link in S^3, Let G be a prime alternating diagram for $L \cup U_s$. By Lackenby

$$V_3(t(G)-2)/2 \leq \underset{\text{Altvol}(L)}{\overset{\|}{\text{Vol}(S^3 - (L \cup U_s))}} \leq V_3(16t(G) - 16)$$

Technically need to show the following

① $t(L) \leq t(G)$

a prime alternating diagram of $L \cup K_s$.

②