Incompressible Surface Basics

Loop Theorem: Let $M$ be a compact 3-manifold and let $S$ be a connected surface in $\partial M$. Let $K = \ker (\pi_1(M, S) \to \pi_1(M)) \neq \{\text{id}\}$. Then a non-trivial element of $K$ can be represented by a $S \subset \Sigma$ in $S$.

Dehn's lemma: Suppose $M$ is a 3-manifold and $f : D \to M$ is a map such that $f|\partial D$ is an embedding and $f^{-1}(\partial D)$ in $D$ the singularities of $f$ do not meet $\partial D$, then there exists an embedding $g : D \to M$ s.t. $g|\partial D = f|\partial D$.

Corollary: An embedded surface in a compact 3-manifold is incomp. iff it is $\pi_1$-injective.

Pf: Suppose $F$ is incompressible and $\pi_1(F) \subset \pi_1(M)$ is not injective. To derive a contradiction, if $\pi_1(F) \subset \pi_1(M)$ is not injective then $F$ is compressible. Cut $M$ along $F$, apply the Loop Theorem and then the Dehn's lemma. If $\pi_1(F) \subset \pi_1(M)$ is injective then $F$ is incompressible. If $F$ is compressible then clearly $\pi_1(F) \subset \pi_1(M)$ is not injective.
Thm (Hartshorn)

Let $F$ be a closed incompressible embedded surface in a closed orientable 3-manifold $M$. Let $\Sigma$ be any Heegaard surface for $M$. Then $d(\Sigma) \leq 2g(F)$.

Proof: Let $M = H_1 \cup H_2$.

Let $\Gamma_i$ be a spine for $H_i$.

Since $M - (\Gamma_1, \Gamma_2) \cong \mathbb{R} \times (-1, 1)$, we define a height function on $M$ given by $h: M \to [-1, 1]$ s.t.

\[ h^{-1}(-1) = \Gamma_2 \]
\[ h^{-1}(1) = \Gamma_1 \]
\[ h^{-1}(r) \cong \Sigma \text{ for } r \in (-1, 1) \]

Claim 1: There are no closed, incomp. surfaces in a handlebody.

Proof: Let $(\text{Handlebody}) \cong F_g$. Since every subgroup of a free group is free and no surface group is free, then by the Loop thm and Dehn's lemma ever closed surface embedded in a handlebody is compressible.
Hence $F \cap \Gamma_i \neq \emptyset$ and $F \cap \Gamma_2 \neq \emptyset$.

By the Hakeen Lemma, we can assume $g(F) > 1$ and $d(\mathcal{E}) \geq 2$ and $\mathcal{G}$ is irreducible.

Also, we assume $h^{-1}(x)$ is Morse.

\[ \infty \quad | \quad \Gamma_1 \]

\[ \infty \quad | \quad \Gamma_2 \quad \Rightarrow \quad 1 \]

$x \in [-1, 1]$ is Blue if a curve of $h^{-1}(x) \neq \emptyset$ is essential in $h^{-1}(x)$ and bounds a comp. disk for $h^{-1}(x)$ below.

$x \in [-1, 1]$ is Red if ... above.

Claim: There is an interval $[v, w] \subset [-1, 1]$ colored neither Blue nor Red s.t.

$u - \varepsilon$ is blue and $v + \varepsilon$ is Red for small $\varepsilon$.

PF: No regular point $v \in [-1, 1]$ is labeled both blue and red since $d(\mathcal{E}) \geq 2$. 
Suppose there exists a critical point of $h_f$ $c \in [-1,1]$ s.t. $c - \varepsilon$ is blue and $c + \varepsilon$ is red.

Then $d(c) \leq 1$, a contradiction.

Hence some non-trivial interval $[u,v]$ is labeled neither red nor blue, the claim follows. \[ \square \]

Claim: The curves of $h^{-1}(r) \cap F$ are essential in both $h^{-1}(r)$ and $F$ for any regular value of $h_f$ in $[u,v]$.

Proof: All curves of $h^{-1}(r) \cap F$ are essential in $F$.

Suppose not, then we can find

Suppose $a \in F \cap h^{-1}(r)$ is essential in $h^{-1}(r)$ then $a$ must be inessential in $F$.

Otherwise, $r$ would be red or blue.

Suppose $a \in F \cap h^{-1}(r)$ is inessential in $F$, $h^{-1}(r)$, then $a$ must be inessential in $F$, as $F$ is incompressible. \[ \square \]