Math 760 2-20-2013

Incompressible Surface Basics

Loop Theorem: Let M be a compact 3-manifold and let S be a connected surface in ∂M.
Let $K = \ker(\pi_1(S) \hookrightarrow \pi_1(M)) \neq \{1\}$.
Then a non-trivial element of K can be represented by a s.c.c. in S.

Dehn's lemma: Suppose M is a 3-manifold and $f: D \to M$ is a map s.t. $f|_{\partial D}$ is an embedding and ~~$f^{-1}(f(\partial D)) = \partial D$~~ the singularities of f do not meet ∂D, then there exists an embedding $g: D \to M$ s.t. $g|_{\partial D} = f|_{\partial D}$.

Cor | ~~An~~ ~~Ansum~~ embedded surface F in a compact 3-manifold M is incomp. iff it is π_1-injective.

Pf| ~~Suppose F is incompressible and $\pi_1(F) \hookrightarrow \pi_1(M)$ is not injective to derive a contradicti~~

If $\pi_1(F) \hookrightarrow \pi_1(M)$ is not injective then F is compressible.
Cut M along F, apply the Loop theorem and then ~~the~~ Dehn's lemma.

~~If $\pi_1(F) \to \pi_1(M)$ is injective then F is incompressible.~~

If F is compressible then clearly $\pi_1(F) \hookrightarrow \pi_1(M)$ is not injective.

Th^m (Hartshorn)
Let F be a closed incompressible embedded surface F in a closed orientable 3-manifold M. Let Σ be any Heegaard Surface for M, Then $d(\Sigma) \leq 2g(F)$.

Pf/ Let $M = H_1 \underset{\Sigma}{\cup} H_2$.

Let Γ_i be a spine for H_i

Since $M - (\Gamma_1 \cup \Gamma_2) \cong \Sigma \times (-1, 1)$, we define a height function on M & given by $h: M \to [-1, 1]$ s.t.

- $h^{-1}(-1) = \Gamma_2$
- $h^{-1}(1) = \Gamma_1$
- $h^{-1}(r) \cong \Sigma$ for $r \in (-1, 1)$.

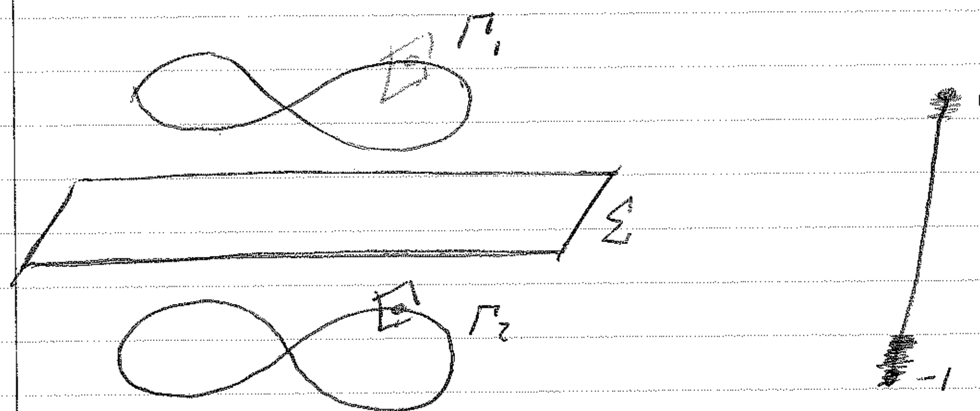
Claim 1: There are no closed, incomp. surfaces in a Handle body

Pf/ $\pi_1(\text{Handlebody}) \cong F_g$
Since ever subgroup of a free group is free and no surface group is free, then, by the Loop th^m and Dehn's lemma ever closed surface embedded in a handle body is compressible.

Hence $F \cap \Gamma_1 \neq \emptyset$ and $F \cap \Gamma_2 \neq \emptyset$.

By the Haken Lemma, we can assume $g(F) \geq 1$ ~~and~~, $d(\Sigma) \geq 2$ and M is irreducible

Also, we assume $h|_F$ is Morse.



$x \in [-1,1]$ is Blue if a curve of $h^{-1}(x) \cap F$ is essential in $h^{-1}(x)$ and bounds a comp. disk for $h^{-1}(x)$ below

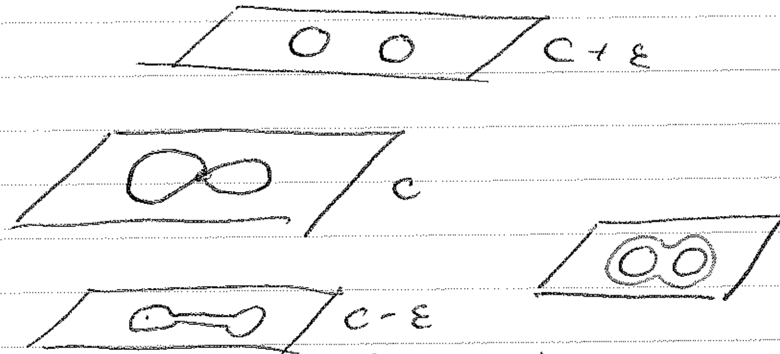
$x \in [-1,1]$ is Red if . . .

. . . . above

Claim: There is an interval $[u,v] \subset [-1,1]$ colored neither Blue nor Red s.t. $u - \varepsilon$ is blue and $v + \varepsilon$ is Red for small ~~epsilon~~, ε.

Pf] No regular point $r \in [-1,1]$ is labled both blue and red since $d(\Sigma) \geq 2$

Suppose there exists a critical point of $h|_F$
$c \in [-1, 1]$ s.t. $c-\varepsilon$ is blue and $c+\varepsilon$ is red



Then $d(\Sigma) \leq 1$, a contradiction.

Hence some non-trivial interval $[u, v]$ is labled neither red nor blue, the claim follows. $\square$

Claim: The curves of $h^{-1}(r) \cap F$ are ess. in both $h^{-1}(r)$ and F for any regular value of $h|_F$ in $[u, v]$.

or inessential in both $h^{-1}(r)$ and F

<u>Pf</u> ~~① All curves of $h^{-1}(r) \cap F$ are essential in F. Suppose not, then we can find~~

Suppose $\alpha \in F \cap h^{-1}(r)$ is essential in $h^{-1}(r)$ then α must not be inessential in F, otherwise r would be red or blue.

Suppose $\alpha \in F \cap h^{-1}(r)$ is inessential in in $h^{-1}(r)$, the α must be inessential in F as F is incompressible. $\square$