**Math 760**

Def: A Heegaard splitting of a closed orientable manifold $M$ is a triple $(H_1, H_2, f)$ s.t. $H_1$ and $H_2$ are handlebodies, $f: \partial H_1 \to \partial H_2$ is a homeomorphism and $M = H_1 \cup_{f} H_2 / \sim$

where $x \sim y$ if $x \in \partial H_1$ or $y \in \partial H_2$ and $f(x) = y$.

Def: A bridge splitting for $(S^3, K)$ is a triple $(T_1, T_2, f)$ s.t. $T_1$ and $T_2$ are trivial tangles, $f: \partial T_1 \to \partial T_2$ is a homeo. of the 2n-punctured spheres s.t. cond $(S^3, K) = T_1 \cup_{f} T_2 / \sim$

**Ex.**

Let $L^1_0 \to \bigcirc \circ \bigcirc \downarrow f$

\[2\text{-fold branched cover}\]

\[\bigcirc \circ \bigcirc \downarrow f\]

Th. Every closed orientable Heegaard splitting 3-manifold has a Heegaard splitting.

**If** By Moise, every 3-manifold has a triangulation $\mathcal{T}$. Let $T$ be a 3-simplex in $\mathcal{T}$.

Let $H_1 = \eta(T^1)$
Let $(\Sigma^1)^*$ be the dual 1-skeleton for $\Sigma$.

\[ I \quad \text{Let } H_2 = \eta((\Sigma^1)^*) \]

\[ M = H_2 \cup H_2 \]

Hence $M$ has a Heegaard splitting. \qed

**Theorem** Let $K$ be a knot in $S^3$. $(S^3, K)$ has a bridge splitting.

**Proof** Let $h: S^3 \to [-1, 1]$ be the natural height function on $S^3$.

\[ S^3 \cong S^2 \times [-1, 1] / \sim \]

\[ \sim: \text{Identify } S^2 \times \{1\} \text{ to a point} \]

\[ \} \text{ Identify } S^2 \times \{-1\} \text{ to a point} \]

\[ h: S^2 \times [-1, 1] / \sim \to [-1, 1] \]

by $h(x, t) = t$.

By Morse theory we can assume $h|K$ has finitely many isolated critical points.

There is an isotopy of $K$ supported in a n.h. of $\Sigma$.

Since after the isotopy all maxima of $h|K$ lie above all minima.

Any level sphere between the lowest max and highest min is a bridge sphere. \qed
Are Bridge Heegaard splittings of
Bridge splittings unique? No.

Stabilization of a Heegaard Surface

This has the effect of adding a handle
to one handle body and drilling out an
unknot boundary parallel arc from the other.

Hence every 3-manifold has \( \infty \)-many
distinct bridge Heegaard Splittings.

Stabilization of a Bridge Splittings

Ex! Justify that stabilizations of
Bridge surfaces lift to stabilizations
of Heegaard Splittings in \( 2 \)-fold
branched cover.
What do Heegaard Surfaces and Bridge Surfaces have in common?

They are bicompressible!

**Def**: A compressing disk for a surface \( F \subset M \) is an embedded disk s.t.
\[ \partial D \cap F = \partial D \] and \( \partial D \) is essential in \( F \).

**Ex.1**

\( T^2 \times \mathbb{R}^3 \)

**Def**: If \( F \subset M \) has no compressing disks, then \( F \) is bicompressible.

In comp in \( S^3 \) - trefoil

**Def**: If \( F \subset M \) has a compressing disk to each side, then \( F \) is bicompressible.

- Heegaard Splittings
- Bridge Splittings
How are Heegaard Splittings related to topology and geometry of the ambient 3-manifold?

**Def:** A Heegaard spl: A bicompressible surface $F \subset M$ is reducible if there exists an essential curve in $F$ that bounds comp disks to both sides.

**Thm:** Any Heegaard splitting surface for a reducible 3-manifold is reducible.

**Pf:** Next time.

**Def:** A bicompressible surface $F \subset M$ is weakly reducible if there exist disjoint essential curves $\gamma_1$ and $\gamma_2$ s.t. $\gamma_1$ bounds a comp disk to one side and $\gamma_2$ bounds a comp disk to the other side, and $F$ is not reducible.

**Thm (Casson & Gordon)**
If $M$ is a closed orientable 3-manifold with a weakly reducible Heegaard surface, then $M$ contains an incompressible surface.