Preliminaries:

Theorem: Every embedded 2-sphere in $\mathbb{R}^3$ bounds an embedded 3-ball.

Definition: A compact surface $F$ in a 3-manifold $M$ is 2-sided if there is an embedding $h: F \times I \to M$ such that $h(x, y_2) = x$ for all $x \in F$ and $h(F \times I) \cap \partial M = h(\partial F \times I)$.

Example:

Theorem: If $F$ is a compact, orientable, properly embedded surface in a 3-manifold $M$, then $F$ is 2-sided.
Def: If $M$ is a connected 3-manifold and $S$ is an embedded sphere s.t. $M - S$ has two components $M'$ and $M''$, and $M'$ is obtained by filling in $M$'s boundary sphere with a 3-ball, then $M$ is the connected sum $M \# M''$.

Ex: 

![Diagram of connected sum operation]

Connect sum operation is:
- Well defined (Unique way up to homeo to glue a $S^3$ to a $S^2$ boundary component)
- Commutative
- Has $S^3$ as identity

Def: A 3-manifold $M$ is prime if $M = P \# Q$ implies $P \cong S^3$ or $Q \cong S^3$.

By Alexander's thm $S^3$ is prime.

Def: A 3-manifold $M$ is irreducible if every 2-sphere $S \subset M$ bounds a 3-ball in $M$.

Irreducible $\Rightarrow$ prime.
Prime $\not\Rightarrow$ irreducible.
Theorem: The only orientable prime 3-manifold which is not irreducible is $S^1 \times S^2$.

\textbf{Pf:} If $M$ is prime and $S$ is a 2-sphere s.t. $M - S$ has 2-components then $S$ bounds a ball. Hence, we can assume $|M - S| = 1$ and $M - S$ is path connected.

Let $N$ be the submanifold consisting of a closed regular nbh of $S$ union an arc from $S^+$ to $S^-$.

$\partial N$ is a separating 2-sphere in $M$, so $\partial N$ bounds a 3-ball. Since $N$ is not a 3-ball then $M = N \cup B^3 \cong S^2 \times S^1$. $\square$

\[ M = S^2 \times I \cup D_1^2 \times I \cup D_2^2 \times I \quad \text{s.t.} \]
\[ \partial D_1^2 \times \mathbb{E}t^3 = \partial D_2^2 \times \mathbb{E}t^3 \]
It remains to show $S^1 \times S^2$ is prime.

Suppose $S^1 \times S^2 \cong V \# W$.

Then $\mathbb{Z} \cong S^1 \times S^2 \cong \pi_1(V) \ast \pi_1(W)$

WLOG $V$ is simply connected

So $V$ lifts to $\tilde{V}$ a homeomorphic copy of $\mathbb{R}^3$ in the universal cover of $S^1 \times S^2$, $\mathbb{R}^3 - \{0\}$.

$\partial \tilde{V}$ bounds a ball in $\mathbb{R}^3$ by Alexander's thm.

So $\tilde{V}$ is a ball. Thus $S^1 \times S^2$ is prime.

---

**Prime decomposition theorem**

Let $M$ be a compact, connected, orientable 3-manifold. Then there is a decomposition $M = P_1 \# \ldots \# P_n$ with each $P_i$ prime and this decomposition is unique up to insertion or deletion of $S^3$'s.
Existence:

**Step 1:** Take care of $S^2 \times S^1$ summands.

If $M$ contains non-separating spheres, these give rise to $S^2 \times S^1$ summands, as previously seen. Each $S^2 \times S^1$ summand contributes 2 to $H_1(M)$, so there must be finitely many.

**Step 2:** We may assume every 2-sphere in $M$ separates.

**Step 2:** Take care of $S^2$ boundary components

- Each such component corresponds to a $B^3$ summand and there are finitely many since $M$ is compact.

**Step 3:** We may assume every 2-sphere in $M$ separates and $M$ has no 2-sphere boundary components.

To complete the proof of existence, it suffices to show that there is an upper bound on a system of spheres $S$ satisfying

- No component of $M - S$ is a punctured $S^3$-sphere.
Note: If $S$ satisfies $\ast$ and $S_0 \subset S$ s.t. $S_i'$ and $S_i''$ are obtained by compressing one of the $S_i$ along a disk $D$, then the systems resulting from replacing $S_i$ with $S_i'$ or $S_i''$ has property $\ast$.

* If $B'$ and $B''$ are punctured spheres then $B' \cup B'' \cup P$ is a punctured sphere $\neq$

* If $B'$ is a punctured sphere, suppose $B'$ is not a punctured sphere. If $A \cup B'' \cup P$ is a punctured sphere then $A$ is a punctured sphere $\neq$.

So, one of the two new systems has property $\ast$. 