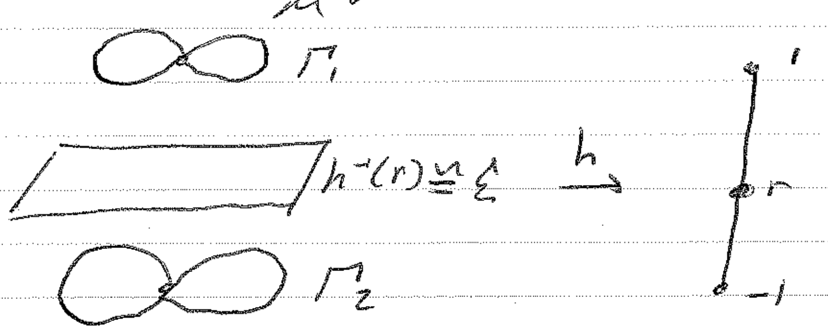
Math 760 2-25

Th^m (Hartshorn)
Let F be a closed incompressible embedded surface in a closed orientable 3 manifold M. If Σ is any Heegaard Surface for M then
$$d(\Sigma) \leq 2g(F).$$

Pf] Last time we showed:

There is a height function $h: M \to [-1,1]$



$x \in [-1,1]$ a regular value of $h|_F$ is blue if $h^{-1}(x) \cap F$ contains a curve bounding a compressing disk for $h^{-1}(x)$ bellow.

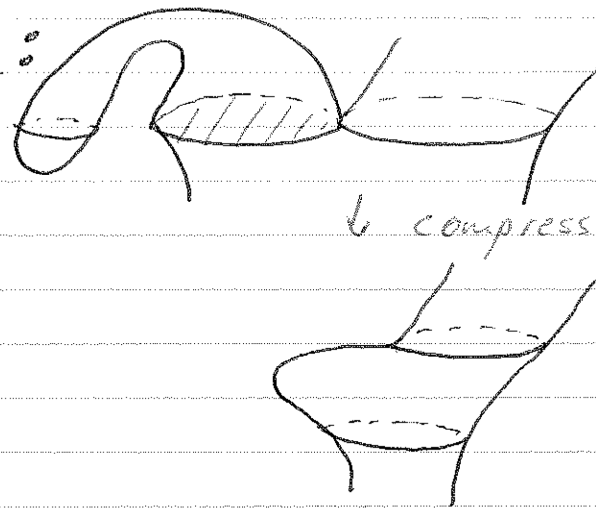
Simillarly $x \in [-1,1]$ is blue if ..
. . . . above.

We showed there is a nontrivial interval $[u,v] \subset [-1,1]$ s.t. no point in $[u,v]$ is collered blue or red and
$u-\varepsilon$ is blue
$u+\varepsilon$ is red
for ε small.

Additionally, we showed for any regular value $r \in [u,v]$ $h^{-1}(r) \cap F$ consists of curves essential in <u>both</u> $h^{-1}(r)$ and F or curves inessential in <u>both</u> $h^{-1}(r)$ and F. ; *

Claim: After surgery on F, we can assume * does not occur.

Hand Wave:



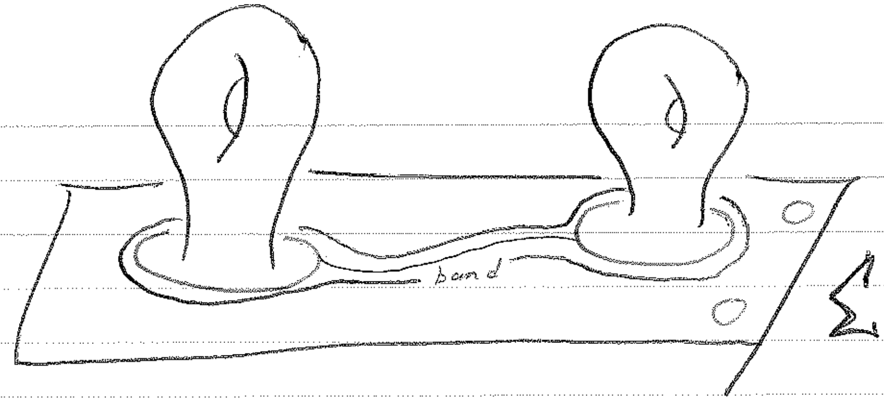
So all curves of intersection are essential in both $h^{-1}(r)$ and F

Let c be a critical ~~pt~~ value of $h|_F$ in $[u,v]$

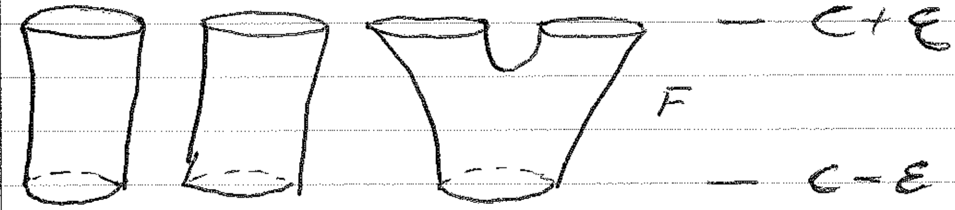
Claim: $d_{\mathcal{C}_2}(\alpha, \beta) \leq 1$ where

α is any curve in $h^{-1}(c-\varepsilon) \cap F$ and β is any curve in $h^{-1}(c+\varepsilon) \cap F$.

<u>Pf</u> Passing through a saddle corresponds to banding together curves of intersection



Claim: $\chi(F \cap h^{-1}([c+\varepsilon, v])) = \chi(F \cap h^{-1}([c-\varepsilon, v])) + 1$



So $d_{C_\Sigma}(\Sigma) \leq d_{C_\Sigma}(\alpha, \beta)$

s.t. $\alpha \in h^{-1}(u-\varepsilon) \cap F$ that bounds a comp. disk below
$\beta \in h^{-1}(v+\varepsilon) \cap F$ .. . . above

$\leq 2 + d_{C_\Sigma}(h^{-1}(u+\varepsilon) \cap F, h^{-1}(v-\varepsilon) \cap F)$

$\leq 2 - \chi(F \cap h^{-1}([u+\varepsilon, u-\varepsilon]))$

$\leq 2 - \chi(F)$

$\leq 2 - 2 + 2g(F)$

$\leq 2g(F)$ $\square$

Corollary: If M is a closed orientable 3-manifold and Σ is any Heegaard surface s.t. $d(\Sigma) \geq 3$ then M is hyperbolic of small Seifert fibered.