Math 760: Introduction to 3-mfld

Day 1

4C4

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OH: By Request

Course Goals:

① Learn the fundamental theorems in 3 manifolds
② Learn the major proof techniques in 3-manifolds
③ Applications of high distance Surfaces (last $1/4$-$1/3$)

Major Topics Will include

- Prime decomposition Theorem
- Heegaard Splittings
- Incompressible Surfaces
- Dehn Surgery
- High distance Surfaces *

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Theme "Understanding 3-manifolds by studying the surfaces they contain"

Any Questions?

Take attendance

What is your favorite 3-manifold?

- If you are here to figure that out, that's OK.
- Figure 8 knot complement.

Def: A topological space M is an n-manifold if

M is ① Hausdorff

② M has a countable basis

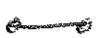
③ $\forall x \in M$ there is a nbh of x homeomorphic to an open subset of $\mathbb{R}^n_+ = \{(x,y,z) \mid z \geq 0\}$

M is a <u>closed</u> n-manifold if ① and ② hold and $\forall x \in M$ there is a nbh of x homeomorphic to an open subset of $\mathbb{R}^n$ (In other words, M has no boundary)

– Examples of compact manifolds

1-manifolds Classified

S^1  closed

D^1 

2-manifolds Classified

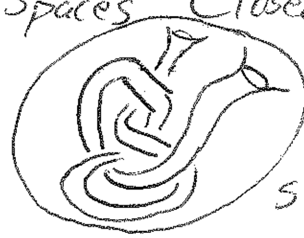
S^2, $S^1 \times S^1$, genus n surface — closed orientable

D^2, $S^1 \times S^1$ - disk, S^2 - disks — orientable

3-manifolds far from Classified.

S^3, $S^2 \times S^1$, surface $\times S^1$, Lens Spaces — Closed 3-manifolds

D^3, $D^2 \times S^1$, genus n - handlebody



$S^3 - \mathring{\eta}$ (fig 8)

A difficulty in the study of 3-manifolds

Thm No closed compact 3-manifold embedds in $\mathbb{R}^3$.

Sketch of Proof Let M be a closed compact 3-manifold.

Suppose $h: M \to \mathbb{R}^3$ is an embedding.
Since M is cpt. $h(M)$ is compact.
Let S_r be ~~family of~~ the 2-sphere of radius r centered at the orgin.
For r large M is contained inside S_r.
Let r_0 be the largest r s.t. $S_r \cap M \neq \emptyset$.
Claim: Any point of $S_{r_0} \cap M$ can not have an open nbh in M homeomorphic to $\mathbb{R}^3$.

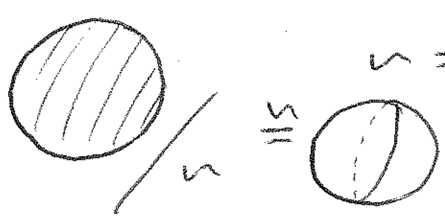
~~So, we can~~ So, we can not visualized closed, cpt. 3 manifolds, yet they are often the most interesting to study.

Central Theme: Understand 3-manifolds by decomposing them into simple peices, ~~and the~~ and understanding how the peices are glued together.

Quotient topology:

Let (X, τ_x) be a topological space with equivalence relation $\sim$. The quotient space $Y = X/\sim$ is the set of equivalence classes of X with the topology where open sets are the sets of equivalence classes whose unions are open sets in X.

Ex



$\sim \; = \; x \sim y$ iff $|x| = |y| = 1$

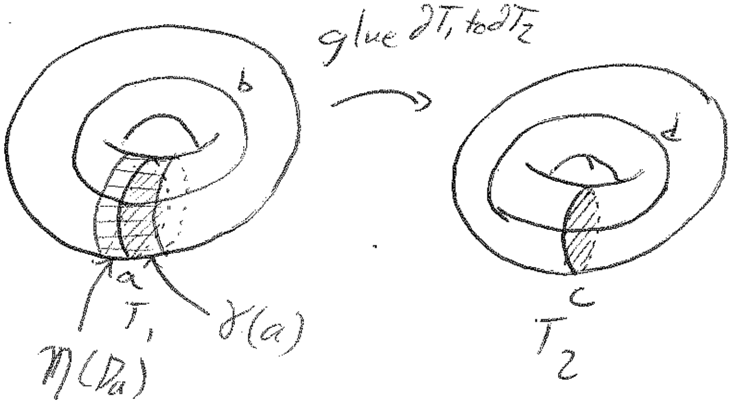
Lets use the quocient topology and the notion of understanding 3-manifolds by cutting them up into smaller peices to study a prototypical class of 3-manifolds

Lens Space

Given $T_1 = D^2 \times S^1$, ~~and~~ $T_2 = D^2 \times S^1$, and $h: \partial T_1 \to \partial T_2$ a homeomorphism, a lens space is any 3-manifold of the form

$$T_1 \cup T_2 / \sim$$

where $x \sim y$ if $x \in \partial T_1$, $y \in \partial T_2$ and $h(x) = y$.



Claim: The isotopy class of $h(a)$ determines the homeomorphism class of the lens space M.

Reorganize M as a different quotient space.

Let $\eta(D_a)$ be the closed regular nbh of the disk bounded by a

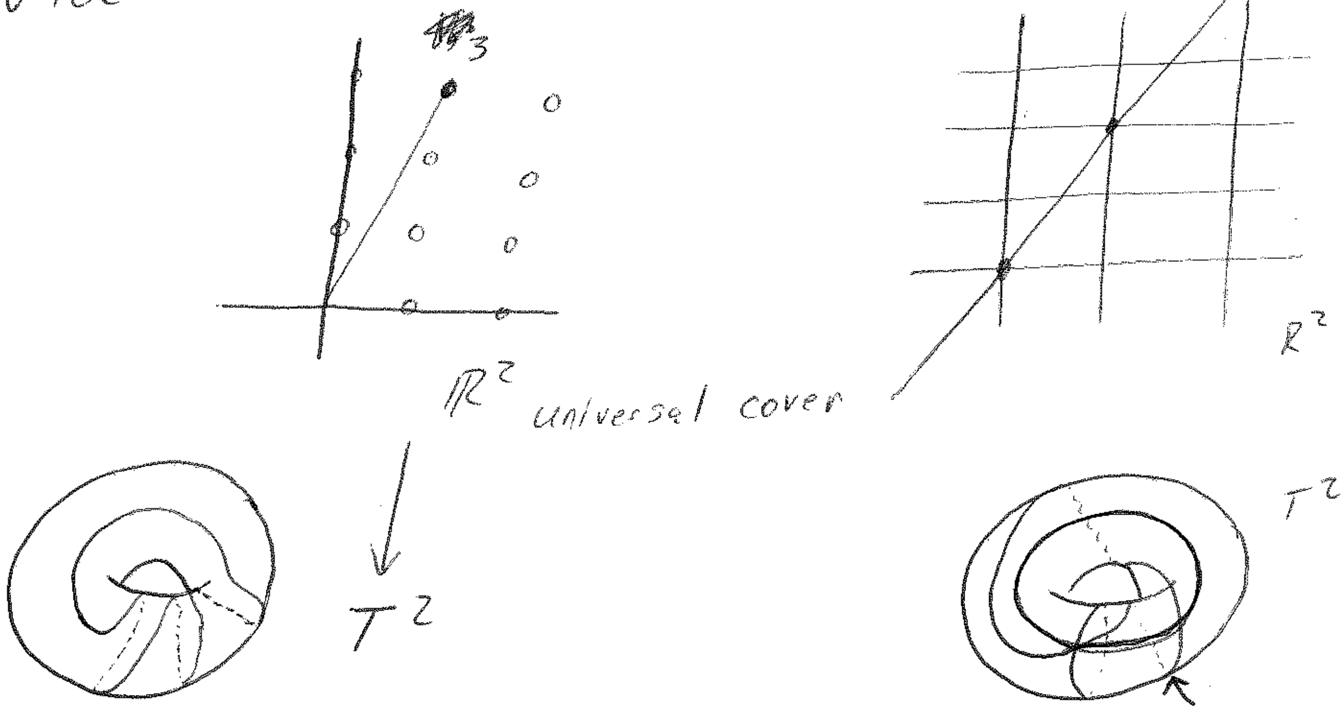
Let N be the manifold with boundary obtained by identifying $\gamma(a)$ ~~with a~~ $h(\gamma(a))$.

$M =$  $\cup N/\sim$

Since the group of automorphisms of S^2 is $\mathbb{Z}/2\mathbb{Z}$ there is a unique way of gluing a 3-ball to the ∂N. $\square$

Recall: Isotopy classes of simple closed curves in T^2 are in 1-1 correspondence with $\mathbb{Q} \cup \infty$

via



So, $L_{p/q}$ is the lens space where $h(a)$ has slope p/q.

<u>Ex</u>| Use Van Kampen to calculate
$$\pi_1(L_{p/q}).$$

<u>Harder Ex</u>| Determine when $L_{p/q} \cong L_{p'/q'}$