MATH 590: GRADUATE KNOT THEORY, HOMEWORK 2

KNOT DIAGRAMS AND P-COLORABILITY

Due in my office by midnight, Tuesday, 9/20

Problems (to turn in).

(1) Let $P \subset \mathbb{R}^2$ be a regular knot projection. Describe how to construct an unknot with projection P. (Hint: use the fact that the only knot with bridge number equal to one is the unknot.)

(2) Carefully show that 3-colorability is preserved under the R3 Reidemeister move. (Hint: a clever organization of the cases of the proof will save you time.)

(3) Let $\mathbb{Z}_p$ denote the set of integers mod p where p is a prime number. Recall that $\mathbb{Z}_p$ forms and abelian group under addition mod p. Moreover, $\mathbb{Z}_p$ is a field with multiplication defined to be multiplication mod p.

A p-coloring of a knot diagram D, where p is an odd prime, is a labeling of the arcs of D by elements of $\mathbb{Z}_p$ so that at every crossing

$$2x - y - z = 0 \ mod \ p$$

where x is the label of the over arc, y is the label of one under arc and z is the label of the other under arc. (Note that labeling all the arcs the same value is always a valid p-coloring under this definition.)

A) Given a knot diagram D, show that the number of distinct p-colorings of D is a power of p. (Hint: Show that set of p-colorings is in one-to-one correspondence to the kernel of a linear map from V to W where V and W are finite dimensional vectorspaces over the field $\mathbb{Z}_p$.)

B) Using only four illustrations and only four complete sentences, convince me that the number of p-colorings of a knot diagram is a knot invariant.

C) Find all odd prime numbers p such that the figure eight knot has a non-trivial p-coloring (i.e. one that involves at least two distinct labels). (Hint: Think about how Gaussian elimination works when your entries are elements in the field $\mathbb{Z}_p$)