Announcements
- H.W. Due by tomorrow morning

Outline
- Transversality of maps $\Phi$
- Generalizations of the pre image $Thm$

Recall

**Thm** If $f: x \to y$ is a smooth map and $y \in Y$ is a regular point, then $f^{-1}(y)$ is a sub manifold.

Goal: Generalize the pre image theorem to give a sufficient condition for $f^{-1}(Z)$ where $Z \subseteq Y$ is a sub manifold, to be a manifold.

**Def** Let $f: x \to y$ be a smooth map and let $Z \subseteq Y$ be a sub manifold. We say $f$ is transversal to $Z$, denoted $f \pitchfork Z$ if for every $x \in f^{-1}(Z)$

$$Im(\text{d}f_x) + T_y(Z) = T_y(Y)$$
(i.e. every vector in $T_y(Y)$ can be written as a linear combination of a vector in $\text{Im}(df_x)$ and $\mathbf{z}$, a vector in $T_y(Z)$).

**Thm** If $f: X \to Y$ is a smooth map and $\mathbf{z}$ is transversal to $Z \subseteq Y$, then $f^{-1}(Z)$ is a submanifold of $X$. Moreover, the codimension of $f^{-1}(Z)$ in $X$ is equal to the codimension of $Z$ in $Y$.

**Def** If $Z$ is a submanifold of $Y$, the codimension of $Z$ in $Y$ is $\dim(Y) - \dim(Z)$.

**Pf** Let $x \in X$. First, $f^{-1}(Z) \subseteq X \subseteq \mathbb{R}^m$.

Let $x \in f^{-1}(Z)$. Let $y = f(x) \in Z$.

From H.W., if $Z$ is an $l$-dim.$\lambda$ submanifold of the $k$-dim.$\lambda$ manifold $Y$, then there exists a local coordinate system $\{x_1, \ldots, x_k\}$ defined in a nbh $U$ of $y$ in $Y$ s.t. $Z \cap U = \{v \in U | x_{l+1}(v) = x_{l+2}(v) = \ldots = x_k(v) = 0\}$

Recall: $x_i: U \to \mathbb{R}$ is a smooth function.
The coordinate systems are linearly independent on every point in their domain. Hence \( g : U \rightarrow \mathbb{R}^d \) given by \( g(v) = (x_1(v), \ldots, x_d(v)) \) is a submersion on its domain and \( g^{-1}(0) = Z \cap U \).

Moreover \( (g \circ f)^{-1}(0) = f^{-1}(Z) \cap V \) for some suitable nbhd \( V \) of \( x \) in \( X \).

We want to show \( 0 \) is a regular value of \( g \circ f \).

Examine \( d(g \circ f)_x = dg_y \circ df_x \)

\( d(g \circ f)_x : T_x(X) \rightarrow \mathbb{R}^d \)

\( d(g \circ f)_x \) is onto iff \( dg_y \) carries \( \text{Im}(df_x) \) onto \( \mathbb{R}^d \).

However \( dg_y \) is onto with kernel \( T_y(Z) \).

Hence, by linear algebra, \( d(g \circ f)_x \) is onto iff \( \text{Im}(df_x) \) together with \( T_y(Z) \) span all of \( T_y(Y) \).

However, by def of transversal, this holds for all \( x \in f^{-1}(Z) \).
Thus, \( d(g \circ f)_x \) is onto for all \( x \in f^{-1}(z) \cap V \).

So, \( (g \circ f)^{-1}(\emptyset) \) is a submanifold of \( V \).

of dimension \( \dim(X) - 1 = \dim(X) - (\dim(Y) - \dim(z)) \).

It easily follows that \( f^{-1}(z) \) is a submanifold of \( X \) of dimension co-dimension the same as the co-dimension of \( z \) in \( Y \). \( \square \)