Lec. 7

From last time:

**Def:** A smooth map $f: x \to y$ is an immersion at $x$ if $df_x$ is one-to-one.

**Examples**

$g: \mathbb{R} \to S^1$ by $g(t) = (\cos(2\pi t), \sin(2\pi t))$

$g 	imes g: \mathbb{R}^2 \to S^1 \times S^1$

Let $L_m$ be a line of slope $M$ in $\mathbb{R}^2$.

$g 	imes g|_{L_m}: \mathbb{R} \to S^1 \times S^1$ is probably an immersion.

If $M \in \mathbb{R} - \mathbb{Q}$, then $\text{im}(g \times g|_{L_m})$ is nice.

If $M \in \mathbb{R} - \mathbb{Q}$, then $\text{im}(g \times g|_{L_m})$ is dense in $S^1 \times S^1$ (very bad).

It would be nice if the image of immersions were submanifolds, but they are not always.
**Def.** $f: X \to Y$ is proper if the preimage of every compact set in $Y$ is compact in $X$.

**Def.** If $f: X \to Y$ is an injective, proper immersion, then $f$ is an embedding.

**Ex.** Knot theory is the study of embeddings of $S^1$ into $\mathbb{R}^3$ (or $S^3$).

**Thm.** An embedding $f: X \to Y$ maps $X$ diffeomorphically onto a submanifold of $Y$. 

**Pf.**
**Def.** A smooth map $f: X \to Y$ is a submersion at $x$ if $df_x$ is onto.

(Best "local condition" we can hope for if $\dim(Y) \leq \dim(X)$)

**Ex.** $f: \mathbb{R}^k \to \mathbb{R}^k$ if $k \leq k$

via $f((x_1, x_2, \ldots, x_k)) = (x_1, \ldots, x_k)$.

is the "canonical submersion"

**Thm.** Suppose $f: X \to Y$ is a submersion at $x \in X$

and $f(x) = y$. Then there exists local coordinates around $x$ and $y$ s.t.

$f((x_1, \ldots, x_k)) = (x_1, \ldots, x_k)$.

(i.e., locally, $f$ is equivalent to the canonical submersion)

**Pf.** (Similar to last time)

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\phi} & & \uparrow{\psi} \\
\mathbb{R}^k & \xrightarrow{g} & \mathbb{R}^k
\end{array}
\]

s.t. $\phi(0) = \text{proj}_x X$ and $\psi(0) = Y = f(x)$.

Since $df_x = dg_0 \circ dg_0(\text{proj}_x)^{-1}$ and $\text{proj}_x$ is onto, then $dg_0$ must be onto.
So $d\phi_0 : \mathbb{R}^k \to \mathbb{R}^k$ is an onto linear map. Hence we can choose a basis for $\mathbb{R}^k$ s.t.

$$d\phi_0 = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$

Define $G : U \to \mathbb{R}^k$ by

$$G((x_1, \ldots, x_k)) = (g((x_1, \ldots, x_k), x_{k+1}, \ldots, x_k))$$

Then

$$dG_0 = \begin{bmatrix} I_k & 0 \\ 0 & I_{k-1} \end{bmatrix} = I_k$$

$k \times k$

By Inverse Function Theorem $G$ is a local diffeomorphism.

After carefully choosing $U'$, $\phi \circ G^{-1}$ is a local parameterization at $x \in X$.

and $f((x_1, \ldots, x_k)) = (x_1, \ldots, x_k)$. \qed
Def: Given a smooth map \( f : X \to Y \), \( y \in Y \) is a regular value if, for every \( x \in f^{-1}(y) \), \( df_x \) is surjective.

Thm (Big!): If \( y \) is a regular value of a smooth map \( f : X \to Y \), then \( f^{-1}(y) \) is a submanifold of \( X \) with
dim \( (f^{-1}(y)) = \dim (X) - \dim (Y) \).

Ex: \( f : \mathbb{R}^k \to \mathbb{R} \)
\[
f((x_1, \ldots , x_k)) = x_1^2 + x_2^2 + \cdots + x_n^2
\]
df_{(x_1, \ldots , x_n)} = \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 & \ldots \end{bmatrix}_{1 \times k}
So \( df_{(x_1, \ldots , x_n)} \) is onto when even
\[
(x_1, \ldots , x_n) \neq (0, \ldots , 0).
\]
In particular \( 1 \in \mathbb{R} \) is a regular value.
So \( S^{k-1} \) is a smooth \( k-1 \) manifold.