Differential Topology

Lec. 5

Announcements
- HW due Today

Outline
- More on tangent spaces.
- generalized Chain Rule.

Goal: If X and Y are smooth manifolds and $f: X \to Y$ is a smooth map, then we want to construct the best linear approximation to f at a point x. (i.e. the derivative of f at x).

What "should" this be?

$$df_x : T_x(X) \to T_y(Y) \quad \text{if } f(x) = y.$$

We want ① Expanded def. of derivative to match def. for $f: \mathbb{R}^n \to \mathbb{R}^m$.

② The Chain rule.

The Set up

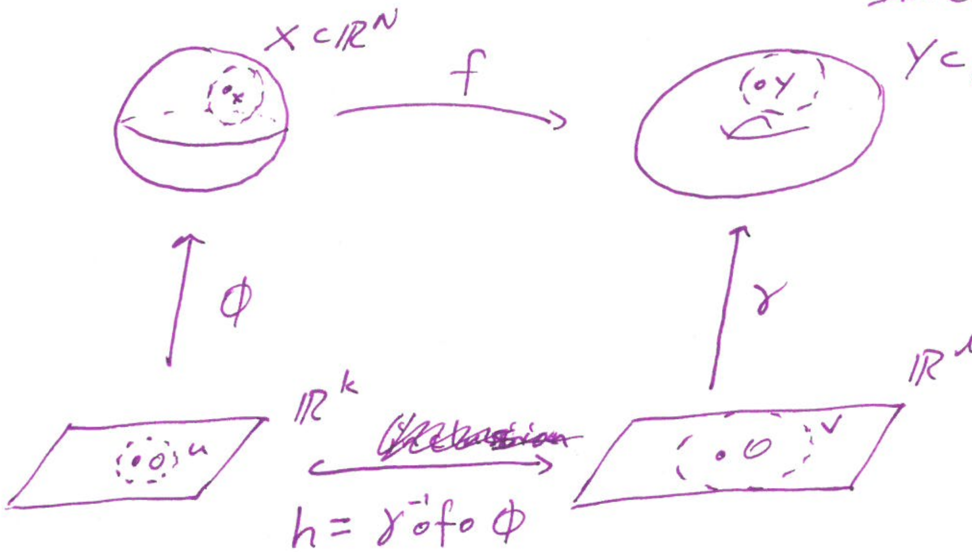
Let $\phi: U \to X$ be a parameterization about $x \in X$

Let $\psi: V \to Y$ be a parameterization of $y \in Y$.

Assume $\phi(0) = x$ and $\psi(0) = y$.

Since U and V are nbh's of 0, after shrinking U, we can assume ~~$U \subset V$ and $f(\phi(U)) \subset \gamma(V)$~~ $\mathrm{Im}(f \circ \phi) \subset \mathrm{domain}(\gamma^{-1})$.

Pic



Note: that $d\phi_0$, dh_0, $d\gamma_0$ all make sence in our previous definition of derivative. So,

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "h"'] & V \arrow[u, "\gamma"'] \end{tikzcd} \longrightarrow \begin{tikzcd} T_x(X) \arrow[r, dashed] & T_y(Y) \\ \mathbb{R}^k \arrow[u, "d\phi_0"] \arrow[r, "dh_0"'] & \mathbb{R}^d \arrow[u, "d\gamma_0"'] \end{tikzcd}$$

want this!

What is the <u>only</u> resonable deffinition of df_x? Since ϕ is a diffeo $d\phi_0$ is a vector space isomorphism. In particular, $d\phi_0^{-1}$ is a well-defined linear map. So,

Definition: $df_x = d\gamma_0 \circ dh_0 \circ d\phi_0^{-1}$!!!!

Claim The deffinition of df_x does <u>not</u> depend on the choices of ϕ and γ. (i.e. independent of parameterization).

Pf H.W.

General Chain Rule

Thm If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are smooth maps of manifolds, then $d(g \circ f)_x = dg_{f(x)} \circ df_x$.

Pf As before, let $\phi : U \longrightarrow X$ be a parameterization about $x \in X$, $\gamma : V \longrightarrow Y$ " " about $f(x)$ in Y and $\eta : W \longrightarrow Z$ be a parameterization about $g(f(x))$.

We can construct the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
U \arrow[u, "\phi"] \arrow[r, "h = \gamma^{-1} \circ f \circ \phi"'] & V \arrow[u, "\gamma"] \arrow[r, "j = \eta^{-1} \circ g \circ \gamma"'] & W \arrow[u, "\eta"]
\end{tikzcd}$$

where
$\phi(0) = x$
$\gamma(0) = f(x)$
$\eta(0) = g \circ f(x)$.

Or, forgeting the role of Y

$$\begin{tikzcd}
X \arrow[r, "g \circ f"] & Z \\
U \arrow[u, "\phi"] \arrow[r, "j \circ h"'] \arrow[ur, phantom, "\#"] & W \arrow[u, "\eta"]
\end{tikzcd}$$

By definition

$$d(g \circ f)_x = d\eta_0 \circ d(j \circ h)_0 \circ d\phi_0^{-1}$$

However, we <u>do</u> have the chain rule for $\mathcal{U} \xrightarrow{h} V \xrightarrow{j} W$.

So
$$
\begin{aligned}
d(g \circ f)_x &= d\eta_0 \, dj_0 \circ dh_0 \circ d\Phi_0 \\
&= d\eta_0 \circ dj_0 \circ (d\gamma_0)^{-1} \circ d\gamma_0 \circ dh_0 \circ d\Phi_0^{-1} \\
&= dg_{f(x)} \circ df_x \qquad \square
\end{aligned}
$$

Big Idea) The local behavior of a smooth map is determined by the derivative ~~of~~ at a point.

Q: If X and Y are smooth manifolds, how nice can a smooth map $f: X \to Y$ be locally?

Def) $f: X \to Y$ is a local diffeomorphism at x if ~~[illegible]~~ there exists a nbh $U_x \subset X$ s.t. $x \in U_x$ and a nbh $V_{f(x)}$ in Y s.t. $f|_{U_x}: U_x \to V_{f(x)}$ is a diffeomorphism.

Note: If $f: X \to Y$ is a local diffeomorphism, ~~at x~~ then df_x is a vector space isomorphism ~~for all $x \in X$~~ (Intuitively obivious, but formalized in an exercise. 4 sec. 2).

Inverse function Th^m) If $f: X \to Y$ is a smooth map of manifolds and df_x is an isomorphism, then f is a local diffeomorphism.