Differential Topology

lec. 5

Announcements
- HW due Today

Outline
- More on tangent spaces.
- Generalized Chain Rule.

Goal: If $X$ and $Y$ are smooth manifolds and $f: X \to Y$ is a smooth map, then we want to construct the best linear approximation to $f$ at a point $x$. (i.e. the derivative of $f$ at $x$).

What "should" this be?

$$df_x: T_x(X) \to T_y(Y)$$ if $f(x) = y$.

We want:
1. Expanded def. of derivative to match def. for $f: \mathbb{R}^n \to \mathbb{R}^m$.
2. The Chain rule.

The Set up:
Let $\phi: U \to X$ be a parameterization about $x \in X$
Let $\psi: V \to Y$ be a parameterization of $y \in Y$.
Assume $\phi(0) = x$ and $\psi(0) = y$. 
Since $U$ and $V$ are nbh’s of $0$, after shrinking $U$, we can assume $\text{Im}(f \circ \Phi) \subset \text{Im}(f) \subset \text{dom}(f^{-1}).$

$X \xrightarrow{f} \mathbb{R}^n$

$\Phi \xrightarrow{\mathbb{R}^k} \mathbb{R}^n$

$\mathbb{R}^k \xrightarrow{h} \mathbb{R}^n$

$h = y^{-1} \circ f \circ \Phi$

Note that $d\phi_0$, $dh_0$, $d\gamma_0$ all make sense in our previous definition of derivative. So,

$X \xrightarrow{f} Y$

$\Phi \xrightarrow{\mathbb{R}^k} T_x(X)$

$\mathbb{R}^k \xrightarrow{d\phi_0} \mathbb{R}^k$

$U \xrightarrow{h} V$

$\gamma_0 \xrightarrow{d\gamma_0} \mathbb{R}^k$

What is the only reasonable definition of $df_x$? Since $\phi$ is a diffgeo $d\phi_0$ is a vector space isomorphism. In particular, $d\phi_0^{-1}$ is a well-defined linear map. So,

Definition: $df_x = d\phi_0 \circ dh_0 \circ d\phi_0^{-1}$!  !!!!!
Claim: The definition of $d_f \phi_x$ does not depend on the choices of $\phi$ and $x$ (i.e. independent of parameterization).

Pf: H.W.

**General Chain Rule**

**Thm:** If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are smooth maps of manifolds, then $d(gof)_x = dg_{f(x)} \circ df_x$.

**Pf:** As before, let $\phi: U \to X$ be a parameterization about $x \in X$, $\gamma: V \to Y$ about $f(x)$ in $Y$, and $\eta: W \to Z$ be a parameterization about $g(f(x))$. We can construct the following commutative diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\uparrow{\phi} & & \uparrow{\gamma} \\
U & \xrightarrow{\gamma \circ f \circ \phi} & W \\
\downarrow{k} & \downarrow{j} & \downarrow{\eta} \\
& j \circ \eta \circ \gamma \circ f \circ \phi & \\
\end{array}
$$

where

- $\phi(0) = x$
- $\gamma(0) = f(x)$
- $\eta(0) = g \circ f(x)$.

Or, forgetting the role of $Y$

$$
\begin{array}{ccc}
X & \xrightarrow{g \circ f} & Z \\
\uparrow{\phi} & & \uparrow{\eta} \\
U & \xrightarrow{j \circ \eta} & W \\
\end{array}
$$

By definition

$$
d(gof)_x = \eta_0 \circ d(j \circ \eta \circ f \circ \phi)_0 \circ \phi^{-1}_0
$$
However, we do have the chain rule for $U \xrightarrow{h} V \xrightarrow{j} W$.

So $d(gof) = d\eta \circ dj \circ dh \circ d\phi$.

\[= d\eta \circ dj \circ dh \circ d\phi^{-1}\]
\[= df_{f(x)} \circ df_x \quad \square\]
Big Idea: The local behavior of a smooth map is determined by the derivative at a point.

Q: If $X$ and $Y$ are smooth manifolds, how nice can a smooth map $f: X \to Y$ be locally?

Def: $f: X \to Y$ is a local diffeomorphism at $x$ if there exists a nbh $U_x \subset X$ s.t. $x \in U_x$ and a nbh $V_{f(x)} \subset Y$ s.t. $f|_{U_x}: U_x \to V_{f(x)}$ is a diffeomorphism.

Note: If $f: X \to Y$ is a local diffeomorphism, then $df_x$ is a vector space isomorphism. (Intuitively obvious, but formalized in an exercise 4 sec. 2).

Inverse function Thm: If $f: X \to Y$ is a smooth map of manifolds and $df_x$ is an isomorphism, then $f$ is a local diffeomorphism.