555 Lec 3

Announcements
- Class cancelled on Thursday
- HW up on web, due Tuesday.

Review

Def: If $U \subset \mathbb{R}^n$ is open, $f: U \to \mathbb{R}^m$ is smooth if $f$ has continuous partial derivatives of all orders. (If $U$ is not open, $f$ is smooth if it can be extended to an open subset of $\mathbb{R}^n$.)

Def: Given $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, $f: X \to Y$ is a diffeomorphism if

1. $f$ is smooth
2. $f$ is a bijection
3. $f^{-1}$ is smooth.
**Def.** Let $X \subset \mathbb{R}^n$. $X$ is a \textbf{(smooth) $k$-manifold} if for every $x \in X$ there is an open nbh, $U_x$, of $x$ in $X$ and a diffeomorphism $f: U_x \rightarrow V$ where $V$ is an open subset of $\mathbb{R}^k$.

**Thm.** If $X \subset \mathbb{R}^n$ is a smooth $k$-manifold, then $X$ with the subspace topology is a \textbf{(topological) manifold}.

**Pf.** Recall, we are considering $\mathbb{R}^n$ with the standard topology and $X \subset \mathbb{R}^n$ with the subspace topology $\mathcal{T}_X$. From 550, $\mathbb{R}^n$ is both 2nd-countable and Hausdorff. From last time, this implies $X$ is both 2nd-countable and Hausdorff.

To show $X$ is locally Euclidean let $x \in X$. By def. of smooth $k$-manifold $\exists U_x \in X$ and open set containing $x$ s.t. $U_x$ is diffeomorphic to $V$ an open subset of $\mathbb{R}^k$. However, diffeomorphic implies homeomorphic, so $U_x$ is homeomorphic to $V$. Hence, $X$ is locally Euclidean. $\square$
Coordinates

Let \( X \subset \mathbb{R}^n \) be a \( k \)-smooth manifold.

\( \forall x \in X \ \exists \ V_x \subset X \ \text{s.t.} \ V_x \text{ is an nbhd of} \ x \ \text{in} \ X \ \text{and there is a diffeomorphism from} \ f : U_x \to V_x \ \text{for some open set} \ U_x \ \text{in} \ \mathbb{R}^k. \)

\( f \) is a parametrization of \( V_x \)

\( f^{-1} \) is a coordinate system on \( V_x \)

Note \( f^{-1} : V_x \to U_x \subset \mathbb{R}^k \)

So, \( f^{-1} = \langle x_1, x_2, \ldots, x_k \rangle \) where \( x_i : V_x \to \mathbb{R} \)

Each \( x_i \) is a coordinate function.

We can implicitly identify \( V_x \) with \( U_x \) by identifying \( v \in V_x \) with \( \langle x_1(v), x_2(v), \ldots, x_k(v) \rangle \).
Notation | If $X$ is a $k$-dim smooth manifold, we say $\dim(X) = k$.

Thm | If $X$ and $Y$ are smooth manifolds, then $X \times Y$ is a smooth manifold and $\dim(X \times Y) = \dim(X) + \dim(Y)$

Pf | Suppose $X$ is a $k$-dim manifold in $\mathbb{R}^N$ and $Y$ is a $l$-dim manifold in $\mathbb{R}^m$. $X \times Y$ is a subset of $\mathbb{R}^{N+m} = \mathbb{R}$.

Let $(x, y) \in X \times Y$.

Since $x \in X$ exists a local parametrization $\phi : \mathcal{W} \rightarrow X$ around $x$ (where $\mathcal{W} \subset \mathbb{R}^k$ is open) and $y \in Y$ exists a local parametrization $\psi : \mathcal{U} \rightarrow Y$ around $y$ (where $\mathcal{U} \subset \mathbb{R}^l$ is open).

Define $\phi \times \psi : \mathcal{W} \times \mathcal{U} \rightarrow X \times Y$ s.t.

$\phi \times \psi(w, u) = (\phi(w), \psi(u))$

Note $\mathcal{W} \times \mathcal{U}$ is open in $\mathbb{R}^{k+l}$.

Since the product of smooth functions is smooth, $\phi \times \psi$ is smooth.

Since $\phi$ and $\psi$ are invertible, then $\phi \times \psi$ is invertible with inverse $\phi^{-1} \times \psi^{-1}(x, y) = (\phi^{-1}(x), \psi^{-1}(y))$.

Showing $\phi^{-1} \times \psi^{-1}$ is smooth is a bit more subtle, but still true.

Hence $X \times Y$ is a smooth manifold in $\mathbb{R}^{N+m}$ and $\dim(X \times Y) = \dim(X) + \dim(Y)$. 
Def: Suppose $X$ and $\mathcal{Z}$ are smooth manifolds in $\mathbb{R}^n$ s.t. $\mathcal{Z} \subset X$, then we say $\mathcal{Z}$ is a submanifold of $X$.

Note: Any open subset of a manifold is a manifold.

**Derivatives & Tangents**

Given a smooth map $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$, denote the derivative of $f$ at $x$ by $df_x$.

**Chain rule**

$$d(gof)_x = dg_{f(x)} \circ df_x$$

In other words, from the commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{f} & V \\
\downarrow & & \downarrow \\
\mathbb{R}^e & \xrightarrow{g} & \mathbb{R}^m \\
\end{array}
$$

we get

$$
\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{df_x} & \mathbb{R}^m \\
\downarrow & & \downarrow \\
\mathbb{R}^e & \xrightarrow{dg_{f(x)}} & \mathbb{R}^x \\
\end{array}
$$

$$d(gof)_x$$
Let $X$ be a smooth manifold in $\mathbb{R}^N$ and let $\phi: U \rightarrow X$ be a local parametrization at the point $x$. For convenience, assume $\phi(0) = x$.

**Def.** The tangent space of $X$ at $x$ is the image of the map $d\phi_0: \mathbb{R}^k \rightarrow \mathbb{R}^N$, and we denote it by $T_x(X)$.

**Note.** The best $k$-dim flat approximation to $X$ at $x$ is $x + T_x(X) \subset \mathbb{R}^N$.

**Pic.**

Note $v \in T_x(X)$ is called a tangent vector.

**Question:** Is tangent vector space well-defined?