An important example from Calc. 3!

Let \( f: [0, 1] \rightarrow \mathbb{R}^3 \) a parameterized curve. If \( f \) is differentiable \( f(t) = \langle f_1(t), f_2(t), f_3(t) \rangle \) (where \( f_i: [0, 1] \rightarrow \mathbb{R} \) and \( f'(t) = \langle \frac{df_1}{dt}(t), \frac{df_2}{dt}(t), \frac{df_3}{dt}(t) \rangle \)

The affine function best approximating \( f(t) \) at \( a \) is \( t \cdot f'(a) + f(a) \).

- We want a notion of derivative for a function \( f: \mathbb{R}^n \rightarrow \mathbb{R}^m \).
Partial Derivatives

Given \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) defined by \( f(x_1, \ldots, x_n) \)

\[
\frac{df}{dx_i} (a_1, \ldots, a_n) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_i+h, \ldots, a_n) - f(a_1, \ldots, a_n)}{h}
\]

the partial derivative of \( f \) at \((a_1, \ldots, a_n)\) in the direction of \( x_i \).

**Def.** A function \( f: \mathbb{R}^n \rightarrow \mathbb{R}^m \) is differentiable at \( \bar{x}_0 \) if there exists a linear map \( J: \mathbb{R}^n \rightarrow \mathbb{R}^m \) such that

\[
\lim_{h \to 0} \frac{\|f(\bar{x}_0 + h) - J(h)\|_{\mathbb{R}^n}}{\|h\|_{\mathbb{R}^n}} = 0
\]

Additionally, if \( f: \mathbb{R}^n \rightarrow \mathbb{R}^m \) is differentiable at \( \bar{x}_0 \) and \( f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) \),

then \( J \) is given by the Jacobian Matrix

\[
J = \begin{bmatrix}
\frac{df_1}{dx_1} & \cdots & \frac{df_1}{dx_n} \\
\vdots & \ddots & \vdots \\
\frac{df_m}{dx_1} & \cdots & \frac{df_m}{dx_n}
\end{bmatrix}
\]
Example: $f : \mathbb{R}^2 \to \mathbb{R}^3$

$$f(x, y) = \langle x, y, 1-x^2-y^2 \rangle$$

$$f_1(x, y) = x$$

$$f_2(x, y) = y$$

$$f_3(x, y) = 1-x^2-y^2$$

$$J = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-2x & -2y & 1
\end{bmatrix}$$

$$\text{Image } (J_{(0,1)}) = \text{span}(\langle 1,0,0 \rangle, \langle 0,1,-2 \rangle)$$

So, the tangent plane approximation of $f(\mathbb{R}^2)$ at $\langle 0,1 \rangle$ is the plane spanned by $\langle 1,0,0 \rangle$ and $\langle 0,1,-2 \rangle$ and translated by $f(\langle 0,1 \rangle) = \langle 0,1 \rangle$. 
Important: Given \( f: \mathbb{R}^n \to \mathbb{R}^m \), the existence of all partial derivatives \( \frac{df}{dx_i} \) at a point \( \bar{x}_0 \) does not guarantee that \( f \) is differentiable.

Example:
\[
  f(x, y) = \begin{cases} 
    x & \text{if } y \neq x^2 \\
    0 & \text{if } y = x^2
  \end{cases}
\]

\[
  \frac{df}{dx} (0, 0) = 0 \quad \frac{df}{dy} (0, 0) = 0
\]

But, \( f \) is not differentiable at \((0,0)\).

Smooth Manifolds:

**Def:** Let \( U \) be an open set in \( \mathbb{R}^n \).

\( f: U \to \mathbb{R}^m \) is **smooth** if \( f \) has continuous partial derivatives of all orders.

(i.e. \( \frac{\partial^2 f_z}{\partial x_3 \partial x_4} \) is continuous, \( \frac{\partial^2 f_z}{\partial x_1^2} \) is continuous)

First, we want to define "smooth" for maps whose domain are not open sets.

**Def:** \( f: X \to \mathbb{R}^m \) is **smooth** if \( \forall x \in X \) there is an open nbhd \( U \subset \mathbb{R}^n \) and a smooth map \( F: U \to \mathbb{R}^n \) s.t.

\[
  F|_{x \in U} = f.
\]
Definition: Given \( X \subset \mathbb{R}^n \) and \( Y \subset \mathbb{R}^m \), a smooth map \( f: X \rightarrow Y \) is a diffeomorphism if it is a bijection and \( f^{-1} \) is smooth.

If such a map exists, we say \( X \) and \( Y \) are diffeomorphic.

Important! Diffeomorphic will be the notion of "sameness" for this class.

Let's develop our intuition for diffeomorphic with some examples:

- \( S^1 \) in \( \mathbb{R}^2 \): not diffeo.
- \( \text{not diffeo} \) in \( \mathbb{R}^3 \): diffeo.
- \( \text{not diffeo} \) in \( \mathbb{R}^2 \): not diffeo.
- \( \text{not diffeo} \) in \( \mathbb{R}^2 \): not diffeo.
- \( \text{not diffeo} \) in \( \mathbb{R}^3 \): diffeo.
- \( \text{not diffeo} \) in \( \mathbb{R}^3 \): diffeo.
- \( \text{not diffeo} \) in \( \mathbb{R}^3 \): diffeo.
- \( \text{not diffeo} \) in \( \mathbb{R}^3 \): diffeo.

Note: If \( X \) and \( Y \) are diffeomorphic, then \( X \) and \( Y \) are homeomorphic.
Def. Let $X \subset \mathbb{R}^N$. $X$ is a (smooth) $k$-manifold if for every $x \in X$ there is an open nbhd, $U_x$, of $x$ in $X$ and a diffeomorphism $f : U_x \rightarrow V$ where $V$ is an open subset of $\mathbb{R}^k$.

Thm. If $X \subset \mathbb{R}^N$ is a smooth $k$-manifold, then $X$ with the subspace topology is a (topological) manifold.

Pf. Recall, we are considering $\mathbb{R}^N$ with the standard topology and $X \subset \mathbb{R}^N$ with the subspace topology $\tau_x$. From 550, $\mathbb{R}^N$ is both 2nd-countable and Hausdorff. From last time, this implies $X$ is both 2nd-countable and Hausdorff.

To show $X$ is locally Euclidean let $x \in X$. By def. of smooth $k$-manifold $\exists U_x \subset X$ and open set containing $x$ s.t. $U_x$ is diffeomorphic to $V$, an open subset of $\mathbb{R}^k$. However, diffeomorphic implies homeomorphic, so $U_x$ is homeomorphic to $V$. Hence, $X$ is locally Euclidean. $\square$