Diffential Topology Day 2

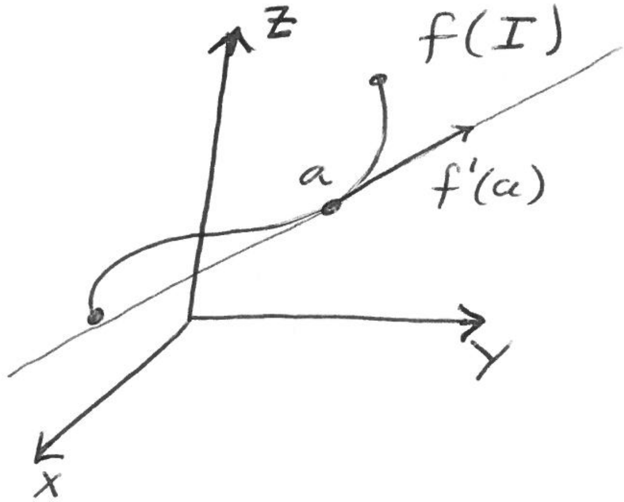
Outline

- Basics of Multivariable Calculus.
- Smooth Manifolds.

An important example from Calc. 3

Let $f:[0,1]\to\mathbb{R}^3$ a parameterized curve.
If f is differentiable $f(t)=\langle f_1(t), f_2(t), f_3(t)\rangle$ (where $f_i:[0,1]\to\mathbb{R}$) and

$$f'(t)=\left\langle \frac{df_1}{dt}(t),\ \frac{df_2}{dt}(t),\ \frac{df_3}{dt}(t)\right\rangle$$



The affine function best approximating $f(t)$ at a is $t\cdot f'(a)+f(a)$.

- We want a notion of derivative for a function $f:\mathbb{R}^n\to\mathbb{R}^m$.

Partial Derivatives

Given $f: \mathbb{R}^n \to \mathbb{R}$ by $f(x_1, \dots, x_n)$

define
$$\frac{\partial f}{\partial x_i}(a_1, \dots, a_n) = \lim_{h \to 0} \frac{f(a_1, \dots, a_i + h, \dots a_n) - f(a_1, \dots a_n)}{h}$$
the <u>partial derivative</u> of f at $(a_1, \dots, a_n)$ in the dirrection of x_i.

Def A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is <u>differentiable</u> at $\vec{x}_0$ if there exists a linear map $J: \mathbb{R}^n \to \mathbb{R}^m$ s.t.
$$\lim_{\vec{h} \to \vec{0}} \frac{\| f(\vec{x}_0 + \vec{h}) - J(\vec{h}) \|_{\mathbb{R}^n}}{\| \vec{h} \|_{\mathbb{R}^m}} = 0$$

Additionally, if $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{x}_0$ and $f(x_1, \dots, x_n) = \langle f_1(x_1, \dots, x_m), \dots, f_m(x_1, \dots, x_n) \rangle$ then J is given by the <u>Jacobian Matrix</u>

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

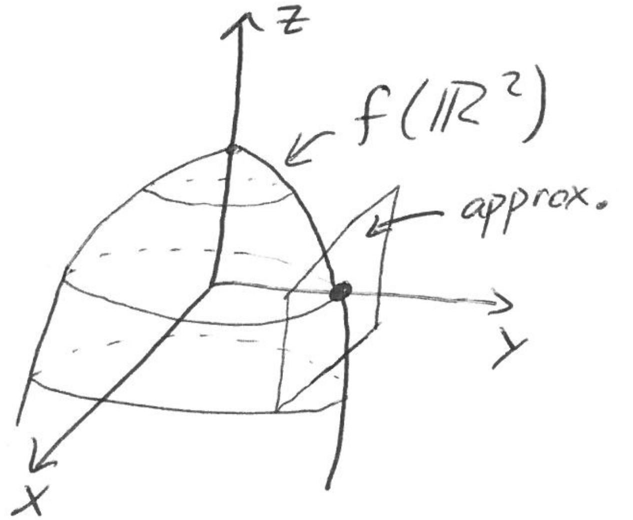
Example $f: \mathbb{R}^2 \to \mathbb{R}^3$

$$f(x,y) = \langle x, y, 1-x^2-y^2 \rangle$$

$f_1(x,y) = x$

$f_2(x,y) = y$

$f_3(x,y) = 1-x^2-y^2$



$$J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2x & -2y \end{bmatrix}$$

$$J_{(0,1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -2 \end{bmatrix}$$

$\text{Image}(J_{(0,1)}) = \text{span}(\langle 1,0,0 \rangle, \langle 0,1,-2 \rangle)$

So, the tangent plane approximation of $f(\mathbb{R}^2)$ at $\langle 0,1 \rangle$ is the plane spanned by $\langle 1,0,0 \rangle$ and $\langle 0,1,-2 \rangle$ and translated by $f(\langle 0,1 \rangle) = \langle 0,1 \rangle$.

Important Given $f: \mathbb{R}^n \to \mathbb{R}^m$, the existance of all partial derivatives $\frac{\partial f_i}{\partial x_j}$ at a point $\vec{x}_0$, does not garantee that f is differentiable.

Ex

$$f(x,y) = \begin{cases} x & \text{if } y \neq x^2 \\ 0 & \text{if } y = x^2 \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = 0 \qquad \frac{\partial f}{\partial y}(0,0) = 0$$

But, f is <u>not</u> differentiable at $(0,0)$.

Smooth Manifolds

Def Let U be an open set in $\mathbb{R}^n$.
$f: U \to \mathbb{R}^m$ is <u>smooth</u> if f has continuous partial derivatives of all orders.
(i.e. $\frac{\partial^2 f_2}{\partial x_3 \partial x_4}$ is continuos, $\frac{\partial^2 f_3}{\partial x_1^2}$ is continuous)

First, we want to define "smooth" for maps whose domain are <u>not</u> open sets.

Def $f: X \to \mathbb{R}^m$ is <u>smooth</u> if $\forall x \in X$ there is an open nbh $U \subset \mathbb{R}^n$ and a smooth map $F: U \to \mathbb{R}^n$ s.t.

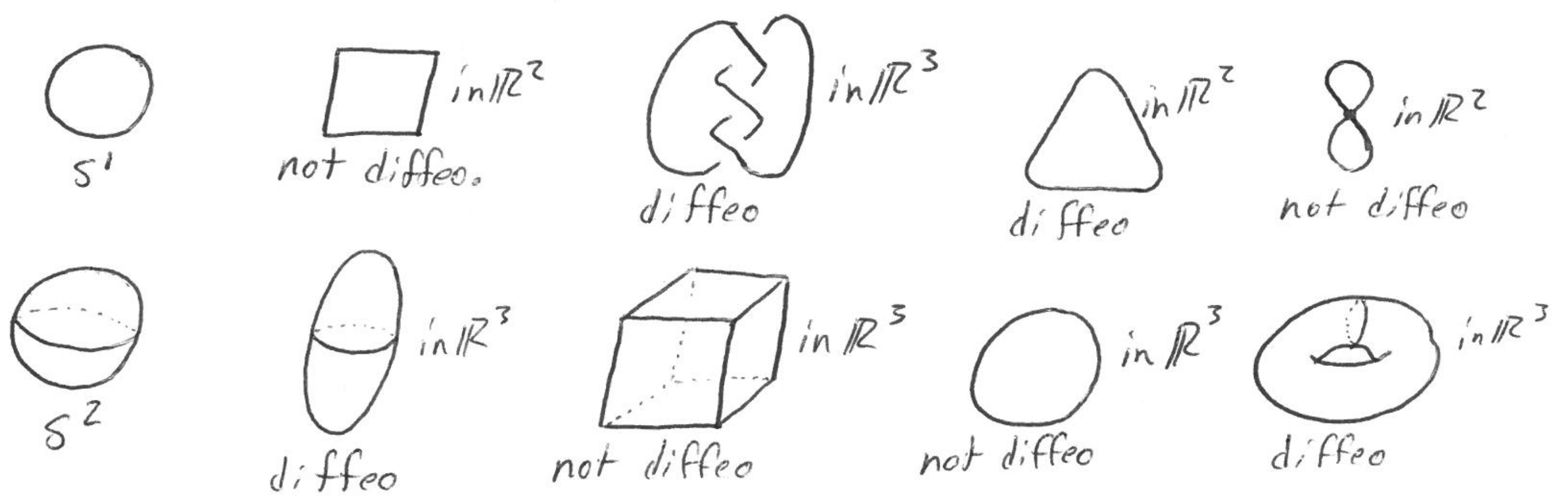
$$F|_{X \cap U} = f.$$

Def Given $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, a smooth map $f: X \to Y$ is a <u>diffeomorphism</u> if it is a bijection and f^{-1} is smooth.

If such a map exists, we say X and Y are <u>diffeomorphic</u>.

Important! Diffeomorphic will be the notion of "sameness" for this class.

Lets develop our intuition for diffeomorphic with some examples



Note: If X and Y are diffeomorphic, then X and Y are homeomorphic.

Def Let $X \subset \mathbb{R}^N$. X is a (smooth) <u>k-manifold</u> if for every $x \in X$ there is an open nbh, U_x, of x in X and a diffeomorphism $f: U_x \to V$ where V is an open subset of $\mathbb{R}^k$.

Thm If $X \subset \mathbb{R}^N$ is a smooth k-manifold, then X with the subspace topology is a (topological) manifold.

Pf Recall, we are considering $\mathbb{R}^N$ with the standard topology and $X \subset \mathbb{R}^N$ with the subspace topology τ_X. From 550, $\mathbb{R}^N$ is both 2nd-countable and Hausdorff. From last time, this implies X is both 2nd-countable and Hausdorff.

To show X is locally Euclidean let $x \in X$. By def. of smooth k-manifold $\exists\ U_x \subset X$ and open set containing x s.t. U_x is diffeomorphic to V, an open subset of $\mathbb{R}^k$. However, diffeomorphic implies homeomorphic, so U_x is homeomorphic to V. Hence, X is locally Euclidean. $\square$