Differential Topology Lec. 1

Outline

- Syllabus
- Quick Review of 550
- Topological Manifolds

First Big Goal of Course: Define calculus on Manifolds

Review of 550

Def A *topology* on a set X is a collection of subsets τ of X satisfying

1) Any arbitrary union of elements of τ is again an element of τ
2) Any finite intersection of elements of τ is an element of τ
3) $\emptyset, X \in \tau$.

Examples

indescrete → 1) $\{\emptyset, X\}$ is a topology for any set X

descrete → 2) $\mathbb{P}(X)$ is a topology

Def Given a top. space (X, τ), $U \subset X$ is *open* if $U \in \tau$. Additionally $U \subset X$ is *closed* if $X - U$ is open.

Note: Clopen sets exist!

Def) A function $f:(X,\tau_X)\to(Y,\tau_Y)$ is continuous if for every $U\in\tau_Y$,

$f^{-1}(U)\in\tau_X$.

Def) A function $f:(X,\tau_X)\to(Y,\tau_Y)$ is a homeomorphism if all of the following hold

1) f is continuous
2) f is a bijection
3) f^{-1} is continuous.

Example that we need 3)

Let $f:(\mathbb{R}, \cancel{\text{std}}\ \text{discrete})\to(\mathbb{R}, \text{std.})$

s.t. $f(x)=x$.

Def) Given a top. space (X,τ_X), $\mathcal{B}\subset\tau_X$ is a basis if ~~[illegible]~~ every element of τ_X is the union of elements in $\mathcal{B}$.

Ex) $\mathbb{R}$ with the standard topology has basis $\mathcal{B}=\{(a,b)\mid a<b\}$.

Def) Given a top. space (X,τ) and $Y\subset X$

$\{\mathcal{U}_\alpha \mid \alpha \in A\} \subset \tau$ is an <u>open</u> <u>cover</u> for Y if $Y \subset \bigcup_{\alpha \in A} \mathcal{U}_\alpha$.

<u>Def</u> (X, τ) is <u>compact</u> if every open cover of X has a finite subcover.

Examples $\quad S^1$ compact

$(\mathbb{R}, \text{std})$ not compact

$(\mathbb{R}, \text{descrete})$ not compact

$(\mathbb{R}, \text{indescrete})$ compact.

<u>Def</u> (X, τ) is <u>Lindelöf</u> if every open cover has a countable sub cover.

<u>Ex</u> $(\mathbb{R}, \text{std})$ is Lindelöf.

<u>Def</u> (X, τ) is <u>2nd-countable</u> if (X, τ) has a countable basis.

<u>Thm</u> If (X, τ) is 2nd-countable, then (X, τ) is Lindelöf.

<u>Ex</u> $\mathcal{B} = \{(a,b) \mid a < b \text{ and } a, b \in \mathbb{Q}\}$ is a basis for $(\mathbb{R}, \text{std.})$

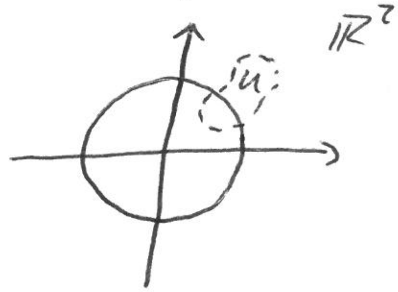
<u>Def</u> (X, τ) is <u>Hausdorff</u> if $\forall x, y \in X$ s.t. $x \neq y$, there exists $\mathcal{U}, V \in \tau$ s.t. $x \in \mathcal{U}$, $y \in V$ and $\mathcal{U} \cap V = \emptyset$.

Def Let (X, τ) be a top. space and $Y \subset X$. The <u>subspace topology</u> on Y is
$$\tau_Y = \{ Y \cap u \mid u \in \tau \}.$$

Let (Y, τ_Y) be a subspace of (X, τ_X).

1. If (X, τ_X) is 2nd-contable, then (Y, τ_Y) is ~~second~~ 2nd-countable.
2. If (X, τ_X) is Lindelöf and Y is closed, then (Y, τ_Y) is Lindelöf.
3. If (X, τ_X) is compact and Y is closed, then (Y, τ_Y) is compact.
4. If (X, τ_X) is hausdorff, (Y, τ_Y) is Hausdorff.

Example Technically, we put a topology on S^1 by viewing it as a subset of $\mathbb{R}^2$ with the standard topology.



u open in $\mathbb{R}^2$

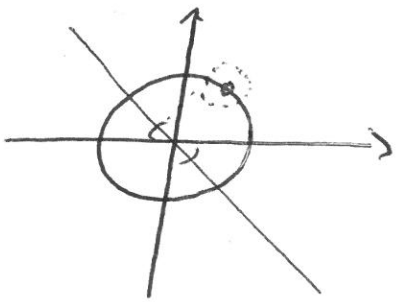
$u \cap S^1$ open in $S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$.

Topological Manifolds

Big idea: Spaces that locally look like Euclidean space.

Def A top. space $(X, \mathcal{T})$ is <u>locally Euclidean</u> if there exists $n \in \mathbb{Z}^+$ s.t. for every $x \in X$ there exists $\mathcal{U} \in \mathcal{T}$ s.t. $x \in \mathcal{U}$ and $\mathcal{U}$ is homeomorphic to $(\mathbb{R}^n, \text{std.})$.

Ex S^1 is locally Euclidean with $n = 1$.



Def A top. space (X, τ) is a <u>k-manifold</u> if $(X, \mathcal{T})$ is hausdorff, 2nd-countable and locally Euclidean with $n = k$.

Examples

- Connected 1-manifolds: S^1, $\mathbb{R}$

This is all of them!

- Connected 2-manifolds: S^2, $S^1 \times S^1 = $ torus,

klein bottle, ...

These are classified!

- Connected 3-manifolds: S^3, $S^1 \times S^1 \times S^1$, $S^2 \times S^1$,  $\times S^1$, ... and many many more.
  - Very much not classified, you win a fields medal for getting close.