**Differential Topology Lec. 1**

Outline

- Syllabus
- Quick Review of 550
- Topological Manifolds

**First Big Goal of Course**: Define calculus on Manifolds

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**Review of 550**

**Def**: A topology on a set $X$ is a collection of subsets $\tau$ of $X$ satisfying

1) Any arbitrary union of elements of $\tau$ is again an element of $\tau$
2) Any finite intersection of elements of $\tau$ is an element of $\tau$
3) $\emptyset, X \in \tau$

**Examples**

- Indiscrete $\rightarrow$ 1) $\{\emptyset, X\}$ is a topology for any set $X$
- Discrete $\rightarrow$ 2) $\mathcal{P}(X)$ is a topology

**Def**: Given a top. space $(X, \tau)$, $U \subseteq X$ is open if $U \in \tau$. Additionally $U \subseteq X$ is closed if $X - U$ is open.

Note: Clopen sets exist!
Def: A function \( f : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y) \) is **continuous** if for every \( U \subseteq \mathcal{T}_Y \), 
\[ f^{-1}(U) \subseteq \mathcal{T}_X. \]

Def: A function \( f : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y) \) is a **homeomorphism** if all of the following hold:
1) \( f \) is continuous
2) \( f \) is a bijection
3) \( f^{-1} \) is continuous.

Example that we need 3):
Let \( f : (\mathbb{R}, \mathcal{T}_{\text{std}}) \rightarrow (\mathbb{R}, \mathcal{T}_{\text{discrete}}) \)

s.t. \( f(x) = x \).

Def: Given a top. space \((X, \mathcal{T}_X)\), \( \mathcal{B} \subseteq \mathcal{T}_X \) is a **basis** if every element of \( \mathcal{T}_X \) is the union of elements in \( \mathcal{B} \).

Ex: \( \mathbb{R} \) with the standard topology has basis \( \mathcal{B} = \{ (a, b) | a < b \} \).

Def: Given a top. space \((X, \mathcal{T})\) and \( Y \subseteq X \),
\[ \forall a, b \in A \exists c \in \mathbb{C} \text{ is an open cover for } Y \text{ if } Y \subseteq \bigcup_{a \in A} U_a. \]

**Def** \((X, \mathcal{T})\) is **compact** if every open cover of \(X\) has a finite subcover.

**Examples**

\(S^1\) **compact**

\((\mathbb{R}, \text{std})\) **not compact**

\((\mathbb{R}, \text{discrete})\) **not compact**

\((\mathbb{R}, \text{indiscr})\) **compact**.

**Def** \((X, \mathcal{T})\) is **Lindelöf** if every open cover has a countable sub cover.

**Ex** \((\mathbb{R}, \text{std})\) is **Lindelöf**.

**Def** \((X, \mathcal{T})\) is **2nd-countable** if \((X, \mathcal{T})\) has a countable basis.

**Thm** If \((X, \mathcal{T})\) is 2nd-countable, then \((X, \mathcal{T})\) is Lindelöf.

**Ex** \(B = \{ (a, b) \mid a < b \text{ and } a, b \in \mathbb{Q} \} \) is a basis for \((\mathbb{R}, \text{std})\).

**Def** \((X, \mathcal{T})\) is **Hausdorff** if \(\forall x, y \in X \text{ s.t. } x \neq y, \text{ there exist } U, V \in \mathcal{T} \text{ s.t. } x \in U, y \in V \text{ and } U \cap V = \emptyset.\)
Let \((X, \tau_x)\) be a top. space and \(Y \subset X\). The subspace topology on \(Y\) is 
\(\tau_Y = \{Y \cap U | U \in \tau_x\}\).

Let \((Y, \tau_Y)\) be a subspace of \((X, \tau_x)\).

1. If \((X, \tau_x)\) is 2nd-countable, then \((Y, \tau_Y)\) is second 2nd-countable, and \(Y\) is closed.
2. If \((X, \tau_x)\) is Lindelöf, then \((Y, \tau_Y)\) is Lindelöf.
3. If \((X, \tau_x)\) is compact and \(Y\) is closed, then \((Y, \tau_Y)\) is compact.
4. If \((X, \tau_x)\) is Hausdorff, \((Y, \tau_Y)\) is Hausdorff.

Example: Technically, we put a topology on \(S^1\) by viewing it as a subset of \(\mathbb{R}^2\) with the standard topology.

\[ U \text{ open in } \mathbb{R}^2 \]
\[ U \cap S^1 \text{ open in } S^1 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}. \]

Topological Manifolds

Big idea: Spaces that locally look like Euclidean space.
**Def.** A top. space \((X, \mathcal{E})\) is **locally Euclidean** if there exists \(n \in \mathbb{Z}^+\) s.t. for every \(x \in X\) there exists \(U \in \mathcal{E}\) s.t. \(x \in U\) and \(U\) is homeomorphic to \((\mathbb{R}^n, \text{std})\).

**Ex.** \(S^1\) is locally Euclidean with \(n = 1\).

![Diagram](image)

**Def.** A top. space \((X, \mathcal{E})\) is a **\(k\)-manifold** if \((X, \mathcal{E})\) is hausdorff, 2nd-countable and locally Euclidean with \(n = k\).

**Examples**
- Connected 1-manifolds: \(S^1, \mathbb{R}\)

  ![Diagram](image)

  This is all of them!

- Connected 2-manifolds: \(S^2, S^1 \times S^1 = \text{torus}, \ldots\)

  ![Diagrams](image)

  klein bottle, ...  
  These are classified!
Connected 3-manifolds: $S^3$, $S^1 \times S^1 \times S^1$, $S^2 \times S^1$, $\bigcirc \bigcirc \bigcirc \times S^1$, ... and many many more.

Very much not classified, you win a Fields medal for getting close.