**Whitney Embedding Thm 2**  Lec. 15

**Thm 2**  Every $k$-dimensional smooth manifold admits a one-to-one immersion into $\mathbb{R}^{2k+1}$.

**Pf**  Let $X$ be a smooth $k$-manifold and let $f: X \rightarrow \mathbb{R}^N$ be an embedding for $N > 2k+1$.

Define: $h: X \times X \times \mathbb{R} \rightarrow \mathbb{R}^N$ by $h(x, y, t) = t(f(x) - f(y))$.

Define: $g: T(x) \rightarrow \mathbb{R}^N$ by $g((x, v)) = df_x(v)$.

Claim: $h$ and $g$ are smooth.

**Pf**  Exercise.

Claim: If $f: X \rightarrow Y$ is a smooth map and $\dim(X) < \dim(Y)$, then $f(X)$ has measure zero.

**Pf**  Last time

By claim, both $\text{Im}(h)$ and $\text{Im}(g)$ have measure zero in $\mathbb{R}^N$. Hence, we can choose $\alpha \in \mathbb{R}^N \setminus \mathbb{R}^3$ s.t. $\alpha \not\in \text{Im}(h)$ and $\alpha \not\in \text{Im}(g)$.

Define $\pi: \mathbb{R}^N \rightarrow H$ be the projection map where $H$ is the $(N-1)$-dim subspace of $\mathbb{R}^N$ that is perpendicular to $\alpha$. 
Claim: \( \Pi \circ f: \mathcal{X} \rightarrow \mathcal{H} \) is one-to-one.

\[ \text{Pf: } \supseteq \text{ Suppose not. Let } x, y \in \mathcal{X} \text{ s.t. } x \neq y \text{ and } \Pi \circ f(x) = \Pi \circ f(y). \]

Since \( f \) is an embedding, \( f(x) \neq f(y) \).

Hence, \( \exists t \in \mathbb{R} \setminus \{0\} \) s.t. \( f(x) - f(y) = t \cdot a \).

or \( \frac{1}{t} (f(x) - f(y)) = a \).

or \( h(x, y, \frac{1}{t}) = a \).

However, this contradicts our choice of \( a \). \( \Box \)

Claim: \( \Pi \circ f: \mathcal{X} \rightarrow \mathcal{H} \) is an immersion.

\[ \text{Pf: } \supseteq \text{ Suppose not. There exists } x \in \mathcal{X} \text{ and } v \in T_x(\mathcal{X}) \text{ s.t. } v \neq 0 \text{ and } d_{\Pi \circ f_x}(v) = 0. \]

By chain rule, \( d_{\Pi \circ f_x} \circ df_x(v) = 0. \)

Since \( \Pi \) is linear, \( \Pi \circ df_x(v) = 0. \).

So \( df_x(v) = t \cdot a \) for some \( t \in \mathbb{R}. \)

*Note \( t \neq 0 \) since \( df_x \) is one-to-one.

\( \frac{1}{t} df_x(v) = a \)

\( df_x(\frac{1}{t} \cdot v) = a \)

\( h((x, \frac{1}{t} \cdot v)) = a \neq 0 \)

Thus \( \Pi \circ f: \mathcal{X} \rightarrow \mathcal{H} \subseteq \mathbb{R}^{n-1} \) is a one-to-one immersion. By induction \( \exists g: \mathcal{X} \rightarrow \mathbb{R}^{2k+1}, \) a one-to-one immersion. \( \Box \)
Corollary: If $X$ is a compact $k$-manifold, then $X$ can be embedded in $\mathbb{R}^{2k+1}$.

\textbf{pf:} By theorem, there exists a one-to-one immersion $g: X \rightarrow \mathbb{R}^{2k+1}$.

Need to show $g$ is proper. Let $C \subseteq \mathbb{R}^{2k+1}$ be compact. By H.B. $C$ is closed & bounded. Thus $g^{-1}(C) \subseteq X$ is closed. Since closed subsets of compact sets are compact, then $g^{-1}(C)$ is compact. \hfill \square

\underline{Partitions of unity}

\textbf{Thm:} Let $X$ be an arbitrary subset of $\mathbb{R}^N$.

For any covering of $X$ by open sets $U \times \mathbb{R}^k$ in $X$, there exists a sequence of smooth functions $\Theta_i : X \rightarrow \mathbb{R}$, called a partition of unity subordinate to $\bigcup U$, with the following properties:

1) $0 \leq \Theta_i(x) \leq 1$ for all $x \in X$ and all $i$.
2) $\forall x \in X \exists U_x$ a nbh of $x \in X$ s.t. all but finitely many $\Theta_i$ are non-zero on $U_x$.
3) Each $\Theta_i$ is zero outside of a closed subset of some $U_x$.
4) $\forall x \in X \sum_{i=1}^{\infty} \Theta_i(x) = 1$ (a finite sum!)

\textbf{pf:} In book.
Corollary 1: Given any manifold $X$, there is a proper map $\rho : X \to \mathbb{R}$.

Proof: Let $\mathcal{U}_{\alpha}$ be the collection of open sets in $X$ that have compact closures. Let $\mathcal{E}_{\Theta}$ be a subordinate partition of unity.

Define $\rho : X \to \mathbb{R}$ by $\rho(x) = \sum_{i=1}^{n} \Theta_i(x)$.

Note $\rho$ is smooth.

If $\rho(x) \leq j$, then at least one of $\Theta_1(x), \ldots, \Theta_j(x)$ is non-zero.

So; $\rho^{-1}([-j, j]) \subset \bigcup_{i=1}^{\infty} \{ x \in X | \Theta_i(x) \neq 0 \} \
\subset \bigcup_{i=1}^{\infty} U_{\alpha_i}$

But, $\bigcup_{i=1}^{\infty} U_{\alpha_i}$ is compact by assumption and $\rho^{-1}([-j, j])$ is closed by continuity.

So $\rho^{-1}([-j, j])$ is a closed subset of a compact set, so is compact.

By H.B. any compact subset of $\mathbb{R}$ is contained in $[-j, j]$ for some $j$. So, $\rho$ is proper. $\square$