Applications of Morse Theory to Knot Theory

Last time

Thm (Morse Lemma) Suppose \( x \in \mathbb{R}^k \) is a non-degenerate critical point of a smooth map \( f: \mathbb{R}^k \rightarrow \mathbb{R} \) and let \( H = (h_{ij}) \) be the Hessian of \( f \) at \( x \). Suppose \( H \) has \( p \) positive eigenvalues and \( n \) negative eigenvalues \((p + n = k)\), then there exist local coordinate systems \( s.t. \) \( x_1, \ldots, x_k \) s.t.

\[
f(x_1, \ldots, x_k) = f(x) + \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{k} x_i^2.
\]

Thm A function \( f: \mathbb{R}^k \rightarrow \mathbb{R} \) is Morse if all critical points are non-degenerate.

Thm The set of smooth Morse maps \( f: \mathbb{R}^k \rightarrow \mathbb{R} \) is open and dense in the space of smooth maps \( f: \mathbb{R}^k \rightarrow \mathbb{R} \) (i.e. non-Morse maps are a set of measure zero).
Knots
A knot is an embedding \( f : S^1 \to \mathbb{R}^3 \).

Two knots \( f_1 : S^1 \to \mathbb{R}^3 \) and \( f_2 : S^1 \to \mathbb{R}^3 \) are equivalent if there exists a smooth homotopy \( F : \mathbb{R}^3 \times I \to \mathbb{R}^3 \) s.t. \( F(x, t) \) is a diffeo morphism for all fixed \( t \) and \( F(x, 0) = \text{id}_{\mathbb{R}^3} \) and \( F(f_1(s), 1) = f_2(s) \).

We say \( f_1 \) is ambient isotopic to \( f_2 \).

Let \( p : \mathbb{R}^3 \to \mathbb{R} \) be \( p(x, y, z) = z \) be the standard projection map.

Corollary Every knot \( f : S^1 \to \mathbb{R}^3 \) is ambient isotopic to a knot \( f_2 : S^1 \to \mathbb{R}^3 \) s.t. \( p \circ f_2 : S^1 \to \mathbb{R} \) is morse.

Proof Appeal to the fact that non-morse are a set of measure zero.
Lemma: If $f: S' \to \mathbb{R}$ is a Morse function then the set of critical points in $S'$ is a finite set.

Proof: From last time, every critical point $x \in S'$ has a nbhd $U_x$ s.t. $x$ is the only critical point in $U_x$. Suppose the set of critical points is an infinite set. Since $S'$ is a compact subset of $\mathbb{R}^n$, then we can construct a convergent subsequence of these points s.t. $\lim_{n \to \infty} x_n = y \in S'$.

If $y$ is a non-degenerate critical point, then $y$ is a non-degenerate critical point. By the lemma from last time, if $U_y$ s.t. $y$ is the unique critical point in $U_y$.

If $y$ is a regular point, then $df_y$ has rank 1

$$df_y = \left[ \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right]$$

Since $\frac{\partial f}{\partial x_i}$ is continuous on an open neighborhood set $U_y$ in $x$, s.t.

$\ast$
By Morse theory, every critical point of $f:S^1 \to \mathbb{R}$
is locally modeled on $f(x) = f(a) + x^2$ or
$f(x) = f(a) - x^2$

i.e. a local min or a local max.

**Def.** The bridge number of a knot type $K = [f]$ is the minimal number of maxima of any Morse embedding $g \in [f]$. (Denoted $\beta(K)$)

**Ex.**

![Diagram of a knot with bridge number 2]

$\beta(g) = \beta([g]) = 2$

**Def.** Connected sum of knots

Given Morse embeddings $K_1$ and $K_2$

$K \# L \rightarrow K \# L \,$ well defined (up to worrying about


**Theorem (Schubert (54))**

\[ \beta(K_1 \# K_2) = \beta(K_1) + \beta(K_2) - 1 \]

\[ \leq \text{easy} \]

\[ \geq \text{hard!} \]

**Width**

Let \( f: S^1 \to \mathbb{R}^3 \) be a knot such that \( pof: S^1 \to \mathbb{R} \) is Morse with isolated critical values.

\[ c_1 \leq \ldots \leq c_n \]

Let \( c_i \leq r_i \leq c_{i+1} \) be representative regular values.

\[ w(g) = \sum_{i=1}^{n-1} \left| \frac{\partial f}{\partial t} \right| |(pof)^{-1}(r_i)| \]

Define \( w([g]) = \min_{g \in \mathcal{G}} w(g) \).

\( w([g]) \) is known as the width of a knot.

**Question:** \( w(K_1 \# K_2) = w(K_1) + w(K_2) - 2 \)?

\[ \leq \text{easy} \]

\[ \geq \text{known for large classes of knots} \]

\[ \geq \max \left( w(K_1), w(K_2) \right) \text{ Scharemann & Schultens.} \]