Announcements
- H.W. Due Thursday (Friday Morning)
- Want to move exam? Current March 10th, Thurs.
Recall from last time

Def: Given a smooth map $f: X \to Y$, $y \in Y$ is a critical value of $f$ if $y$ is not a regular value. Moreover, $x \in X$ is a critical point if $df_x$ is not onto and is a regular point if $df_x$ is onto.

Thm: (Sard's Theorem) If $f: X \to Y$ is a smooth map, then the set of critical values of $f$ have measure 0 in $Y$.

(Important! It is not true that the set of critical points have measure 0 in $X$)

Ex: $f: \mathbb{R}^2 \to \mathbb{R}$ via $f(x, y) = x^2 - y^2$.

Every non-zero value of $\mathbb{R}$ is vacuously a regular value of $f$. 0 is a critical value. All points in $\mathbb{R}^2$ are critical points.
We want to study the local behavior of smooth maps $f: X \to \mathbb{R}$.

First consider $f: \mathbb{R}^k \to \mathbb{R}$.

Suppose $x \in \mathbb{R}^k$ is a critical point (i.e. $df_x = 0$).

Define the Hessian matrix at $x$ to be:

$$H = \begin{bmatrix}
\frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_k} \\
\frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_k} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial x_k \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_k \partial x_k}
\end{bmatrix}$$

If $H$ is non-singular at $x$, we say $x$ is a non-degenerate critical point.

Lemma: If $x \in \mathbb{R}^k$ is a non-degenerate critical point of a smooth map $f: \mathbb{R}^k \to \mathbb{R}$, then there exists an open nbh of $x$ in $\mathbb{R}^k$ s.t. $f$ has a unique critical point at $x$. 
**PF** Define \( g: \mathbb{R}^k \to \mathbb{R}^k \) via
\[
g(x) = \left( \frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_k}(x) \right)
\]
Since \( df_x = g(x) = \left[ \frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_k}(x) \right] \)
then \( df_x = 0 \) iff \( g(x) = (0, \ldots, 0) \).
\[
dg_x = \begin{bmatrix}
\vdots \\
1
\end{bmatrix} = H
\]
Since \( x \) is a non-degenerate critical point, then \( dg_x \) is a vector space isomorphism.
By the inverse function theorem, since \( dg_x \) is a vector space isomorphism, then \( g \) is a local diffeomorphism at \( x \).
Hence, there exists an open nbhd \( U \) of \( x \) in \( \mathbb{R}^k \) s.t.
\[
g\big|_U (y) = 0 \Rightarrow y = x.
\]
Equivalently, \( f|_U \) has a unique critical point at \( x \). \( \square \)

**Thm (Morse Lemma)** Suppose \( a \in \mathbb{R}^k \) is a non-degenerate critical point of \( f: \mathbb{R}^k \to \mathbb{R} \) and let
\[
H = (h_{ij}) \text{ be the Hessian of } f \text{ at } a.
\]
Then there exists a local coordinate system around \( a \) s.t.
\[
f = f(a) + \sum h_{ij} x_i x_j \text{ near } a.
\]
Moreover, since $H$ is a real, symmetric, invertible $k \times k$ matrix, $H$ will have $p$ positive eigenvalues and $n$ negative eigenvalues so that $p + n = k$. Real symmetric matrices are diagonalizable.

From linear algebra, there exists yet another coordinate system $y_1, \ldots, y_k$ so that

$$f = f(x) + \sum_{i=1}^{p} y_i^2 - \sum_{i=p+1}^{n} y_i^2$$

Let’s investigate the Morse lemma for functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Suppose $f(0,0) = 0$

Then

$$f((x, y)) = f(0,0) + \sum_{i=p} \frac{\partial^2 f}{\partial x_i^2} x^2 + \frac{\partial^2 f}{\partial y^2} y^2$$

$$= 0 + x^2 + y^2 \quad \text{min}$$

$$+ x^2 - y^2 \quad \text{saddle}$$

or

$$-x^2 - y^2 \quad \text{max}$$

Ex1

\[ \text{Diagram of a doughnut} \]
non-degenerate critical points for smooth maps

\[ f : X \to \mathbb{R} \]

\[ x \xrightarrow{f} \mathbb{R} \]

\[ \phi(0) = x \xrightarrow{\phi} \mathbb{R} \]

\[ \phi_{(0)} = x \xrightarrow{\phi} \mathbb{R} \]

\[ x \text{ is a critical point of } f \text{ if } f \circ \phi = 0 \]

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Say \( x \) is a non-degenerate critical point of \( f : X \to \mathbb{R} \) if

\[ x \text{ is a non-degenerate critical point of } f \circ \phi. \]

Problem: Suppose \( \gamma : V \to X \) is another parameterization about \( x \in X \) s.t. \( \gamma(0) = x \).

After restricting the domain & range of \( \phi \) and \( \gamma \) we can assume \( \phi \circ \gamma \) and \( \gamma \circ \phi \) are diffeomorphisms.

Since \( f \circ \gamma = f \circ \phi \circ (\phi^{-1} \circ \gamma) \), then we must show...

Lemma: Suppose \( f : \mathbb{R}^k \to \mathbb{R} \) has a non-degenerate critical point at \( 0 \) and let \( g : \mathbb{R}^k \to \mathbb{R}^k \) be a diffeomorphism with \( g(0) = 0 \). Then \( f \circ g \) also has a non-degenerate critical value at \( 0 \).

Pf: Terrible