Lec. 11

Announcements

HW due a week from today

Outline

- Stability
- Sard's Theorem

Recall from last time

Smooth maps $f_0 : X \to Y$ and $f_1 : X \to Y$ are homotopic if there exists a smooth map $F: X \times I \to Y$ s.t.

$$F(x,0) = f_0(x) \text{ and } F_1(x,1) = f_1(x).$$

Def A property of smooth maps is <u>stable</u> if whenever $f_0 : X \to Y$ has the property and $F: X \times I \to Y$ is a homotopy s.t. $F(x,0) = f_0(x)$, then there exists $\varepsilon > 0$ s.t. for all $t < \varepsilon$ $F(x,t)$ also has the property.

Examples Smooth maps from $\mathbb{R}$ into $\mathbb{R}^2$.

Properties

- ~~Intersecting~~ graph intersecting the orgin (unstable)
- graph intersecting x-axis
- graph intersecting x-axis transversly.

Stability Th^m The following classes of smooth maps of a compact manifold X into a manifold Y are stable classes.

a) local diffeomorphisms

b) immersions

c) submersions

d) maps transversal to a submanifold $Z \subset Y$

e) embeddings

f) diffeomorphisms

Pf b) Claim: It suffices to show $\forall (x_0, 0)$ there in a nbh U_{x_0} of $(x_0, 0)$ in $X \times I$ s.t. $(df_t)_x$ is injective for all $(x,t) \in U_{x_0}$

Pf $\bigcup_{x_0 \in X} U_{x_0}$ is an open cover of $X \times \{0\} \subset X \times I$ by the "Tube lemma" from 550, There exists $\varepsilon > 0$ s.t. $X \times [0, \varepsilon] \subset \bigcup_{x_0 \in X} U_{x_0}$.

Equivalently $\exists\, \varepsilon > 0$ s.t. $f_t : X \to Y$ is an immersion when ever $0 \leq t \leq \varepsilon$.

Claim: $\exists$ a nbh U_{x_0} of $(x_0, 0)$ in ~~$X \times \{0\}$~~ $X \times I$ s.t. $(df_t)_x$ is injective.

Since this is a local property, we can assume ~~$f: \mathbb{R}^k \to$~~ $f: U$ open $\mathbb{R}^k$ to V open $\mathbb{R}^l$

Since f is an imersion

$(df_0)_{x_0}$ ~~is an immersion~~ is 1-1

$$(df_0)_{x_0} = \left(\cancel{\frac{\partial f_1}{\partial x_1}} \ \frac{\partial (f_0)_i}{\partial x_j}(x_0) \right)_{l \times k}$$ The jacobian

Since the jacobian is injective, then

It contains a $k \times k$ submatrix A s.t. the determinant of that matrix is non zero

However, each partial is continuous as a function on $X \times I$.

Hence, $\det A$ is continuous since it is the composition of continuous functions.

Thus A is non-singular for all points in some nbh of $(x_0, 0)$ in $X \times I$. $\square$

Others are done similarly.

Sard's Theorem If $f: X \to Y$ is any smooth map, then almost every point in Y is a regular point of f.

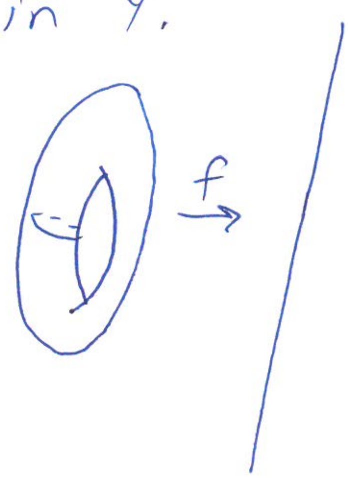
Def A set $A \subset \mathbb{R}^l$ has measure zero if it can be covered by a countable number of rectangular solids with arbitrarily small finite volume.

A set $C \subset Y$ (a smooth manifold) if for every local parameterization $\phi: U \to Y$, $\phi^{-1}(C)$ has measure zero.

Def $y \in Y$ is a critical value of $f: X \to Y$ if y is not a reg. value

Sard's thm The set of critical values of a smooth map $f: X \to Y$ has measure zero in Y.

Ex.



$f: \mathbb{R}^2 \to \mathbb{R}$
$f(x,y) = 0.$

<u>Def</u> If $f: X \to Y$ is smooth, $x \in X$ is a regular point if df_x is onto, Otherwise x is a critical point.

<u>Common error</u> Sards Thm does <u>not</u> say that the set of critical points are measure zero!